SINGULAR DIFFUSIONS, TIME REVERSAL AND APPLICATIONS TO FOKKER-PLANCK EQUATIONS.

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Abstract. Let $A_t = \frac{1}{2}\sum_{ij} a_{ij}\partial_i\partial_j + \sum_i B_i\partial_i$ be a second order time dependent differential operator on $\mathbb{R}^d$. No smoothness nor ellipticity are required. When $B \overset{\text{def}}{=} b + a\beta$ and β satisfies a finite energy condition, we study various properties for A: existence and time reversal of an associated diffusion process; existence, uniqueness and a priori regularity for the associated Fokker-Planck equations. The key tool is the link between the finite energy condition, and some finite entropy condition on the paths space, as remarked by Föllmer ([18]) in the Brownian case. We also look at some properties of the diffusion, and its relationship with Schrödinger equation.

1. Introduction.

Let $A_t = \frac{1}{2}\sum_{ij} a_{ij}\partial_i\partial_j + \sum_i b_i\partial_i + \sum_i (a\beta)_i\partial_i$ be a second order operator with time dependent coefficients. The matrix $a : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is assumed to be symmetric and non negative, but may degenerate. σ denotes the non negative square root of a.
We shall consider A_t as a "singular" perturbation of the "regular" $L_t = A_t - \sum_i (a\beta)_i\partial_i = \frac{1}{2}\sum_{ij} a_{ij}\partial_i\partial_j + \sum_i b_i\partial_i$. Here, "regular" means that both the analytic and probabilistic properties of L_t are known, and "smooth" enough, but minimal regularity on a and b will be assumed. The kind of "singularity" for the perturbation $a\beta$ we shall focus on, is on $\mathbb{L}^2$ type, the so called "finite energy condition".

In this paper we are interested in studying various properties of A_t, in connection with its probabilistic meaning. Namely,

(i) existence of a diffusion with (time-dependent) generator A_t,
(ii) uniqueness of such a diffusion as well as absolute continuity properties,
(iii) a priori regularity and uniqueness for the Fokker-Planck equation

$$(\frac{\partial}{\partial t} - A_t)^* \nu_t = 0. \tag{1.1}$$

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An important tool will be the time reversal of the diffusion, and all the program will lie on the knowledge of a given solution ν of equation (1.1). In particular we do not solve the existence problem for such a ν in full generality. Actually as known examples show, there is no general framework for studying existence. The "finite energy condition"

$$\int_0^T \int \beta.a\beta \, d\nu_t \, dt < +\infty \tag{1.2}$$

which involves ν, furnishes one reasonable framework.
The "finite energy condition" is less strange than it seems. Indeed it appears as a natural assumption in various problems : Nelson's stochastic mechanics ([34]), large deviations with marginal constraints ([20]), hydrodynamic behaviour of a tagged particle in some interacting systems ([38]), perturbation of Dirichlet forms.

Our initial motivation in studying time reversal for singular diffusions, was an attempt to complete Nelson's program in stochastic mechanics ([34]). Indeed, in a series of papers ([11], [12], [13]), C. Łéonard and P. Cattiaux have solved the so called "stochastic quantization problem", i.e. the construction of diffusion processes of Nelson's type, extending previous celebrated results of E. Carlen ([6]; see also [7], [32], [43]). In an unpublished preprint ([15], see also [35]), we showed how to extend Föllmer's result on time reversal ([18], [19]) to general diffusions of Nelson's type. Though unpublished, these results are used in [36] which deals with the difficult case of reflected diffusions (also see [8]), and in [22] which deals with the stationary case.
New motivations lead us to complete our previous work.
Firstly the rather formidable growth of the literature in the stationary (and reversible) case, in connection with Dirichlet forms. Let us mention in particular [40], [41], [4] on one hand, and [24], [25] on the other hand.
Secondly, the paper by J. Quastel and S.R.S. Varadhan ([39]) which deals with the non stationary and time dependent case for perturbation of divergence form operators, using partial differential equations methods. This paper in particular contains a complete different motivation (the hydrodynamic behaviour of a tagged particle in some interacting system, see [38]). Existence and uniqueness of the associated diffusion process are discussed in Section 5 of [39]. Thanks to the divergence form, regularity on a can be weakened, provided additional assumptions on ν are done.
Though important, we shall not discuss the stationary case. The particular case when $A = \frac{1}{2}\triangle + \frac{\nabla p}{p}\nabla$, was already studied in [9], with our methods. Some infinite dimensional extensions are possible (see e.g. [31] (also see [5] and [25] for a Dirichlet form approach). In particular, M. Fradon and P. Cattiaux recently used time reversal and entropy

in [10], for showing that all invariant measures are Gibbs for infinite gradient systems.

Let us now summarize the contents of the present paper.
Sections 2, 3 and 4 are devoted to the general (non divergence) case.
In Section 2, we introduce the framework used in [11] and [12] and recall existence results obtained therein. We then discuss uniqueness and extremality of the solutions.
Section 3 is devoted to time reversal. Following Föllmer, we derive the (classical) duality equation between the forward and the backward drifts. Time reversal explains why the "dual finite energy condition" used by Carlen automatically holds. Let us say here that J. Picard ([37]) proved similar (and prior) results in a somewhat different setting.
Duality equation is used in Section 4 for deriving a priori regularity for the Fokker-Planck equation. The claim is that this solution is absolutely continuous and that the density ρ_t roughly satisfies $\sigma\nabla(\sqrt{\rho_t})$ belongs to $\mathbb{L}^2$, i.e. finite energy implies smoothness. Uniqueness is also discussed.
Section 5 is devoted to the case studied in [39] i.e. when $\sigma \in \mathbb{H}^1(dx)$ the usual Sobolev space (as in [39] we will restrict ourselves to the torus). Of course we cannot improve the complete analytic study which is done therein. We only intend to complete the stochastic picture. The main result here is the stochastic quantization. As in [39] some additional assumptions on ν are required. The ones we shall do are slightly less restrictive than those in [39]. We can also partly improve uniqueness results. Our proof is mainly self-contained (i.e. does not use the analytic arsenal of [39], except one point) and thus furnishes an alternate approach to the one of [39].
In section 6, we study the non attainability of the nodes, i.e. points where ρ is vanishing. This study is the starting point of the construction in the symmetric case or in [43]. Here it is obtained as a consequence of stochastic quantization and a priori regularity.
Finally, in section 7, we briefly describes the connection between Fokker-Planck and Schrödinger. In particular, large deviation theory allows to give a statistical description of the potential V. Results of this section are still not fully satisfactory, but they are a good account of what can be done in this direction.

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2. **Framework.**

Let a be a measurable flow of non negative symmetric matrices, b and β be measurable flows of vector fields. We define:

$$L(t,x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t,x)\ \partial_i\partial_j\ +\ \sum_i\ b_i(t,x)\ \partial_i, \tag{2.1}$$

and

$$A(t,x) = L(t,x) + a(t,x)\beta(t,x).\nabla_x \tag{2.2}$$

where . denotes scalar product and ∇_x is the space gradient;

$$\sigma(t,x) \text{ a measurable non negative square root of } a(t,x). \tag{2.3}$$

All functions are defined on the whole space $\mathbb{R}\times\mathbb{R}^d$, or possibly on the d-dimensional torus $\mathbb{R}\times\mathbb{T}^d$ if they are space-periodic.

We shall look at A as a perturbation of L, and so build a diffusion process associated to A as a transformation of the one associated to L by some Girsanov's like multiplicative functional. Here, the expression "diffusion process" is understood in a non rigid way which will be explained in the statement of the results. Actually, we ask for more. We want to impose the law of all time marginals of the process. This of course implies that this flow satisfies some Fokker-Planck equation; more precisely:

Definition 2.4. *Let $\nu \overset{\text{def}}{=} (\nu_s)_{s\in[0,T]}$, be a flow of Probability measures on $\mathbb{R}^d$, and Λ be a set of Borel functions defined on $\mathbb{R}\times\mathbb{R}^d$. We shall say that ν satisfies the Λ-weak forward equation on $[0,T]$ if, for every $f\in\Lambda$:*

i) *$(\frac{\partial}{\partial t}+A)f$ is defined and belongs to $\mathbb{L}^1([0,T]\times\mathbb{R}^d, ds\,d\nu_s(x))$;*

ii) *$\forall\, 0\le u\le t\le T$,*

$$\int f(t,x)\nu_t(dx) - \int f(u,x)\nu_u(dx) = \int_u^t\int(\tfrac{\partial}{\partial s}+A)f(s,x)\,ds\,d\nu_s(x).$$

In general, Λ will be a nice set, and $\mathcal{C}_0^\infty([0,T]\times\mathbb{R}^d)\subset\Lambda$.

Let us say now what we call a diffusion process.

Definition 2.5. *Let $\mathbb{Q}$ be a Probability measure on $\Omega=\mathcal{C}^0([0,T],\mathbb{R}^d)$ or on $\Omega=\mathcal{C}^0([0,T],\mathbb{T}^d)$. We say that $\mathbb{Q}$ is an A-diffusion with initial measure ν_0 if:*

i) *$\mathbb{Q}oX_0^{-1}=\nu_0$;*

ii) *$\forall\, f\in\mathcal{C}_0^\infty(\mathbb{R}\times\mathbb{R}^d)$ (resp. $\mathcal{C}_0^\infty(\mathbb{R}\times\mathbb{T}^d)$),*

$$f(t,X_t) - f(0,X_0) - \int_0^t(\tfrac{\partial}{\partial s}+A)f(s,X_s)ds$$

is a $\mathbb{Q}$-local continuous martingale up to time T, with brackets given by $\int_0^t(\nabla f.a\nabla f)(s,X_s)ds$.

Here, $t \mapsto X_t$ is the canonical process on Ω equipped with the natural right continuous and complete filtration.

The statement ii) is equivalent to a similar one replacing the full $\mathcal{C}_0^\infty$ by the coordinate functions of the process, in particular, we are in the situation of Chapter 12 of [27].
In the rest of this section, we assume the following:

(2.6) There exists a strong Markov family $(\mathbb{P}_{u,x}; (u,x) \in \mathbb{R} \times \mathbb{R}^d)$, resp. $(\mathbb{P}_{u,x}; (u,x) \in \mathbb{R} \times \mathbb{T}^d)$, such that:
 i) $\mathbb{P}_{u,x}(u_0 = u, X_0 = x) = 1$,
 ii) $u_t = u + t \quad \mathbb{P}_{u,x}$ a. s.,
 iii) $\mathbb{P}_{u,x}$ is an extremal $(\frac{\partial}{\partial u} + L)$-diffusion with initial measure $\delta_{u,x}$.

Here, the path space is $\mathcal{C}_0^\infty([0,T], \mathbb{R} \times \mathbb{R}^d)$, resp. $\mathcal{C}_0^\infty([0,T], \mathbb{R} \times \mathbb{T}^d)$, and extremal means that $\mathbb{P}_{u,x}$ is an extremal solution of the martingale problem (2.5)ii), replacing A by L.

We emphasize that (2.6) is concerned with the (now homogeneous) time-space process. Actually, (2.5) should be written in this time-space context, replacing ν_0 by $\delta_0 \otimes \nu_0$.
Of course, one can now try to build $\mathbb{Q}$ via Girsanov theory of drift transformation, i.e. by the formula:

(2.7)
$$\begin{cases} \dfrac{d\mathbb{Q}}{d\mathbb{P}_{\nu_0\,|\mathcal{F}_t}} \stackrel{\text{def}}{=} Z_t \stackrel{\text{def}}{=} \exp(\int_0^t \beta(s,X_s).dM_s - \frac{1}{2}\int_0^t \beta(s,X_s).a(s,X_s)\,\beta(s,X_s)\,ds), \\ \text{where } M_s^i = X_s^i - X_0^i - \int_0^s b_i(u_v, X_v)dv \\ \text{and } \mathbb{P}_{\nu_0} = \int \mathbb{P}_{u,x}\ \delta_0^{(u)} \otimes \nu_0(dx). \end{cases}$$

Since, Z_t is not well defined in general, the correct way to define it is the following:

(2.8) if $T_n = \inf\{t \geq 0; \int_0^t (\beta.a\beta)(s,X_s)ds \geq n\} \wedge T$, Z_t is given by (2.7) on the stochastic interval $[0,T_n]$ and $Z_t = \liminf_n Z_{T_n}$ otherwise.

We refer to [11] for more details. Also look at the erratum of [11], where a mistake about the vanishing set of Z_t is corrected.
But again, since Z is only a supermartingale, (2.7) is in general not enough to get a probability measure $\mathbb{Q}$ (in other words a conservative process). In particular, "singular" cases are those which are not covered by standard criteria like Novikov or Kazamaki. The cases of singular diffusions we are interested in, are the following

(2.9)
$$\begin{cases} \text{i) } \nu \text{ satisfies the } \Lambda\text{-weak forward equation,} \\ \text{ii) } \int_0^T \int (\beta.a\beta)(s,x)\,ds\,\nu_s(dx) < +\infty, \quad \text{(finite energy condition).} \end{cases}$$

The aim of [11], [13] was to show that, provided L is "good" enough, (2.9) is a sufficient condition for the existence of $\mathbb{Q}$. The main remark and tool for doing this is the following:
roughly speaking, the finite energy condition (2.9) ii) is equivalent to an entropy condition $H(\mathbb{Q}, \mathbb{P}_{\nu_0}) < +\infty$, where H denotes the relative entropy (or Kullback information), i.e.

$$(2.10) \qquad \begin{cases} H(\mathbb{Q}, \mathbb{P}) = \int (\log \frac{d\mathbb{Q}}{d\mathbb{P}})\, d\mathbb{Q}, \text{ if } \mathbb{Q} \ll \mathbb{P} \text{ and } \log \frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathbb{L}^1(\mathbb{Q}), \\ = +\infty \quad \text{otherwise.} \end{cases}$$

More precisely, if $\sigma\beta$ is bounded, it is not difficult to check that

$$\mathbb{Q}oX_t^{-1} = \nu_t \quad \text{and} \quad H(\mathbb{Q}, \mathbb{P}_{\nu_0}) = \frac{1}{2} \int_0^T \int (\beta.a\beta)(s,x)\, ds\, \nu_s(dx).$$

(In this case, Novikov's criterion ensures the existence of $\mathbb{Q}$ as a Probability measure.)
All the difficulty consists in showing that, approximating β by some β^k such that $\sigma\beta^k$ is bounded in the energy sense, implies a weak form of convergence in entropy of the associated $\mathbb{Q}_k$. Since the method is used in Section 5, we now only recall the main results of [11], Section 4.

Theorem 2.11 (Theorem A, see[11], Theorems 4.28 and 4.42)**.** .
Assume that σ and b are locally Hölder continuous, and (2.6) *holds. Let ν be a solution of the $\mathcal{C}_0^\infty$-weak forward equation such that the finite energy condition* (2.9)*ii) holds. Assume furthermore that*
either
i) σ and b are $\mathcal{C}^{1,2,\alpha}$, for some $\alpha > 0$;
or
ii) a is uniformly elliptic.
Then, the measure $\mathbb{Q}$ defined by (2.7) *and* (2.8) *is a Probability measure and an A-diffusion process.*
Furthermore, $\mathbb{Q}oX_t^{-1} = \nu_t$ for every $t \in [0,T]$ and

$$H(\mathbb{Q}, \mathbb{P}_{\nu_0}) = \frac{1}{2} \int_0^T \int (\beta.a\beta)(s,x)\, ds\, \nu_s(dx) < +\infty.$$

Theorem A deals with a general flow of marginals. In [11] Theorem 4.48, or [13] theorem 4.6 and Corollary 5.3, another type of result is obtained with weaker assumptions on σ and b, but stronger on ν. Here is one consequence of these results.

Theorem 2.12 (Theorem B)**.** .
Assume that σ and b are globally Lipschitz in space, uniformly in time. Assume in addition that

i) *there exists μ_0 such that if $\mu_t = \mathbb{P}_{\mu_0} oX_t^{-1}$ then $\nu_t(dx) = \rho(t,x)\, \mu_t(dx)$ for all $t \in [0,T]$ with $\rho \in \mathcal{B}_b(\mathbb{R} \times \mathbb{R}^d)$,*
ii) *ν is a solution of the $D_{e,\nu}$-weak forward equation, where*

$$D_{e,\nu} = \{ f \in \textit{extended domain of } (\tfrac{\partial}{\partial t} + L_t) \textit{ s.t.} \\ \nabla f \in \mathbb{L}^2(d\nu, dt), \quad (\tfrac{\partial}{\partial t} + L_t)\, f \in \mathbb{L}^1(d\nu, dt) \}$$

Then if the finite energy condition (2.9)*ii) is satisfied, the same conclusion as in Theorem A holds.*

Remark 2.13. i) Theorem B does not look very tractable because of condition ii). In general it is hard to describe the extended domain of the generator. The best one can hope is that approximations by $\mathcal{C}_0^\infty$ functions are possible. If one makes some regularity assumptions on ρ, allowing integration by parts, it is possible to show the density of smooth functions in $D_{e,\nu}$. It is exactly what happens in he divergence case studied in [39] and Section 5.
ii) Notice that the finite energy condition which is assumed here, only concerns β, i.e. the dual finite energy condition which is assumed for instance in [6] does not appear here. Actually, the aim of Section 3 is precisely to prove that the dual finite energy condition is automatically satisfied, thanks to the entropic interpretation.
iii) As already said, these results are connected with large deviations results (see e.g.[12] and [13]), in particular the problem of minimal elements (i.e. for a given ν, what is the β of minimal energy) which is extensively studied in these papers, is important. We shall come back later to this interpretation.

We shall now discuss uniqueness. A nice consequence of the Markovian framework (2.6) is the following uniqueness result:

Theorem 2.14. *In both Theorems A and B, if we assume that $\mathbb{P}_{u,x}$ is the unique solution to the martingale problem $\mathcal{M}(\frac{\partial}{\partial t} + L, \mathcal{C}^{1,2}, \delta_{u,x})$ for every (u, x), then $\mathbb{Q}$ is the unique A-diffusion such that*

$$\mathbb{Q}[\int_0^T (\beta.a\beta)(s, X_s)ds < +\infty] = 1.$$

In particular, $\mathbb{Q}$ is the unique A-diffusion such that $\mathbb{Q}oX_t^{-1} = \nu_t$ for every t.

Proof. The hypotheses on $\mathbb{P}_{u,x}$ imply that $\mathbb{P}_{\nu_0}$ is a solution of

$$\mathcal{M}(\frac{\partial}{\partial t} + L, \mathcal{C}^{1,2}, \nu_0)$$

which is locally unique in the sense of [27] (12.52) and (12.53), according to Theorem 12.73 of [27]. So we may apply Theorem 12.57b of [27], (notice that B_∞ therein is exactly $\int_0^T (\beta.a\beta)(s, X_s)ds$ in our case), which tells that any solution $\mathbb{Q}'$ of $\mathcal{M}(\frac{\partial}{\partial t} + A, \mathcal{C}^{1,2}, \nu_0)$ satisfies $\frac{d\mathbb{Q}'}{d\mathbb{P}_{\nu_0}} = Z$. Of course, if $\mathbb{Q}'oX_t^{-1} = \nu_t$, the condition $\int_0^T (\beta.a\beta)(s, X_s)ds < +\infty$, $\mathbb{Q}$ a.s., is a consequence of the finite energy condition. $\square$

Remark 2.15. i) The conditions on $\mathbb{P}_{u,x}$ are automatically satisfied in case A i) and B.
ii) [27] also contains information on the converse $\mathbb{P}_{\nu_0} \ll \mathbb{Q}$; see Theorem 12.48 iii) therein.

Corollary 2.16. *Under the hypotheses of Theorem 2.14, $\mathbb{Q}$ is an extremal solution to $\mathcal{M}(\frac{\partial}{\partial t} + A, \mathcal{C}^{1,2}, \nu_0)$.*

Proof. According to (2.14), any other solution $\mathbb{Q}'$ satisfies:

$$\mathbb{E}^{\mathbb{Q}'}\Big[\int_0^T (\beta.a\beta)(s, X_s)ds\,\Big] = +\infty.$$

Since $\mathbb{E}^{\mathbb{Q}}[\int_0^T (\beta.a\beta)(s, X_s)ds\,]$ is finite (due to the finite energy condition (2.9) ii), $\mathbb{Q}$ cannot be a convex combination of such $\mathbb{Q}'$. $\square$

Remark 2.17. Notice Corollary 2.16 also follows from Theorem 12.22 of [27].

In the whole section, the existence of a solution to the weak forward equation has been assumed. The above Theorems A and B allow to formulate some other existence results for the weak forward equation. This is done in Section 4. We now focus on time reversibility for such singular diffusions.

3. **Time reversal.**

Denote by R the time reversal operator on Ω, i.e.

$$R(X)\ :\ (t \mapsto X_{T-t} \stackrel{\text{def}}{=} \overline{X}_t). \tag{3.1}$$

Generally, we shall use a bar for every notation concerning the time reversed process. For instance, $\overline{\mathbb{P}}$ will be the $\mathbb{P}$ law of $\overline{X}$. The main idea of [18] and [19] is that relative entropy is preserved under time reversal, i.e.

$$H(\mathbb{Q}, \mathbb{P}) = H(\overline{\mathbb{Q}}, \overline{\mathbb{P}}). \tag{3.2}$$

Hence, if $\overline{\mathbb{P}}$ is good enough, Girsanov transformation theory furnishes a backward drift $\overline{\beta}$ of finite energy. The first point is to describe $\overline{\mathbb{P}}$.

Time reversal results for non singular diffusions are well known. We shall mainly use the ones of Hausmann-Pardoux ([26]) and Millet-Nualart-Sanz ([33]). The following is Theorem 2.3 in [33] (see also Theorem 2.1 in [26]).

Theorem 3.3. *Assume that σ and b are globally Lipschitz in space, uniformly in time. If, in addition:*

i) $\forall\, t > 0, \quad \mathbb{P}oX_t^{-1} = \mu_t(dx) = p_t(x)dx;$

ii) $div(a(t,x)\,p_t(x)) \in \mathbb{L}^1_{loc}(dt \times dx)$ *where* $div(ap)$ *is the vector field*

$$(\sum_j \partial_j(a_{ij}p))_{i=1,...,d},$$

then, $\overline{\mathbb{P}}$ *is on* $\overline{\Omega} = \mathcal{C}^0([0,T[,\mathbb{R}^d)$ *a* $\overline{L}$*-diffusion, with*

$$\overline{L}(t,x) = \frac{1}{2}\sum_{ij} \overline{a}_{ij}(t,x)\,\partial_i\partial_j \,+\, \sum_i \overline{b}_i(t,x)\,\partial_i$$

where $\overline{a}(t,x) = a(T-t,x)$,
and $\overline{b}(t,x) = -b(T-t,x) + \frac{1}{p_{T-t}(x)}\, div(a(T-t,x)\,p_{T-t}(x))\ 1_{(p_{T-t}(x)\neq 0)}$.

The global Lipschitz condition can be relaxed into a local one with some extra (intricate) hypotheses (see [33], Section 3).
Of course, it is useful to know some conditions for (3.3) i) and ii) to hold. These conditions depend on what is assumed for μ_0. Without any assumption some ellipticity or hypoellipticity is required.

Proposition 3.4. *Assume that one of the following conditions holds:*

i) σ *and* b *are* $\mathcal{C}^{0,2}$*, with bounded derivatives of first and second order, and* a *is uniformly elliptic;*

ii) $\sigma_1, ..., \sigma_d$ *are* $\mathcal{C}_b^{\beta,\infty}$ *,and* $\mathcal{L}ie\,(\sigma_1, ..., \sigma_d)(0,x)$ *is uniformly full on* $supp\,(\mu_0)$,

then, (3.3) *i) and ii) hold.*

Case i) is contained in [33], and case ii) in [14].
Once μ_0 is assumed to be absolutely continuous, much weaker conditions are allowed.

Proposition 3.5. *In addition to the Lipschitz regularity, assume that* $\mu_0 = p_0(x)\,dx$ *where* p_0 *belongs to some weighted* $\mathbb{L}^2$ *space. If one of the following conditions holds:*

i) $div(a(t,x)\,p_0(x)) \in \mathbb{L}^1_{loc}(dt \times dx)$ *and* μ_0 *is stationary;*

ii) σ *and* b *are* $\mathcal{C}^{\alpha,2}$*, with bounded derivatives up to order* 2*;*

iii) a *is uniformly elliptic;*

then, (3.3) *i) and ii) hold.*

Case i) is clear. Cases ii) and iii) are contained in [26]. Actually these authors relax the regularity on b in case ii) (which can also be obtained by using the diffeomorphism property of the associated stochastic flow, see e.g. [3] or [29]).

We now turn to the singular diffusion. Let $\mathbb{Q}$ be defined as in (2.7), and assume that $\mathbb{Q}$ is a Probability measure. Then, we know that:

$$H(\mathbb{Q}, \mathbb{P}_{\mu_0}) = H(\nu_0, \mu_0) + \frac{1}{2}\int_0^T \int (\beta.a\beta)(s, X_s)\, ds\, \nu_s(dx), \tag{3.6}$$

which is finite, thanks to the finite energy condition (2.9) provided that $H(\nu_0, \mu_0) < +\infty$. Since relative entropy is preserved under time reversal, we thus have:

$$H(\overline{\mathbb{Q}}, \overline{\mathbb{P}}_{\mu_0}) = H(\mathbb{Q}, \mathbb{P}_{\mu_0}) < +\infty,$$

so that $\overline{\mathbb{Q}} \ll \overline{\mathbb{P}}_{\mu_0}$. It follows from [27] (12.17) that $\overline{\mathbb{Q}}$ is a $\overline{A}$-diffusion, with

$$\overline{A} = \overline{L} + (\overline{a}\overline{\beta})$$

for some process $(\overline{\beta}_t)_{t\in[0,T]}$. Notice that $\overline{\beta}$ does not need to be Markovian, i.e. to be a given function $\overline{\beta}(s, X_s)$. If $\mathbb{Q}$ is Markovian, then, so does $\overline{\mathbb{Q}}$, and thus $\overline{\beta}_s = \overline{\beta}(s, X_s)$ according to Theorem 3.60 in [11]. In particular, when (2.6) holds, arguments on multiplicative functionals show that $\mathbb{Q}$ is Markovian. So under the hypotheses of Theorem A or B, $\overline{\beta}_s = \overline{\beta}(s, X_s)$. Since we do not need to be in the Markovian case for what follows, we still use $\overline{\beta}_s$.
One difficulty is now the following : if $\overline{P}_{\mu_0}$ is not an extremal $\overline{L}$-diffusion, we cannot get an explicit expression for $\frac{d\overline{\mathbb{Q}}}{d\overline{\mathbb{P}}_{\mu_0}}$. Fortunately, one can again control the energy of the backward drift thanks to the following :

Lemma 3.7. *If $H(\nu_0, \mu_0) < +\infty$,*

$$H(\nu_T, \mu_T) + \frac{1}{2}\int_0^T \mathbb{E}^{\overline{\mathbb{Q}}}[\overline{\beta}_s.\overline{a}(s, X_s)\,\overline{\beta}_s]\,ds \le H(\overline{\mathbb{Q}}, \overline{\mathbb{P}}_{\mu_0}) < +\infty.$$

In particular $\overline{\beta}$ satisfies the finite energy condition.

Proof. $\overline{M}_t = X_t - X_0 - \int_0^t \overline{b}(s, X_s)\,ds$ is a $\overline{\mathbb{P}}$-local-martingale, thus $\overline{N}_t \overset{\text{def}}{=} \overline{M}_t - \int_0^t \overline{a}(s, X_s)\,\overline{\beta}_s\,ds$ is a $\overline{\mathbb{Q}}$-local martingale (on $[0, T[$). Let t_n be a localizing sequence of stopping times. Define

$$\overline{\beta}^k = \overline{\beta}1_{|\overline{\beta}|<k}$$

and

$$Z_{k,n} = log\,(\frac{d\nu_T}{d\mu_T} \wedge k) + \int_0^{t_n} (\overline{\beta}_s^k).d\overline{M}_s - \frac{1}{2}\int_0^{t_n} (\overline{\beta}_s^k).\overline{a}(s, X_s)\,(\overline{\beta}_s^k)\,ds.$$

Then

$$\int (Z_{k,n}\wedge j)\,d\overline{\mathbb{Q}} - log \int exp\,(Z_{k,n}\wedge j)\,d\overline{\mathbb{P}}_{\mu_0} \le H(\overline{\mathbb{Q}}, \overline{\mathbb{P}}_{\mu_0}) < +\infty \tag{3.8}$$

according to the definition of relative entropy, since $(Z_{k,n}\wedge j)$ is bounded. One can take limits in j using monotone convergence in each term, and remark that, thanks to Novikov criterion $exp\,Z_{k,n}$ is the value at time

t_n of a $\overline{\mathbb{P}}_{\mu_0}$ martingale. Hence (3.8) becomes

$$\begin{aligned} H(\overline{\mathbb{Q}},\overline{\mathbb{P}}_{\mu_0}) + \int (\frac{d\nu_T}{d\mu_T}\wedge k)\, d\mu_T \quad \geq \quad & \mathbb{E}^{\overline{\mathbb{Q}}}[\int_0^{t_n} (\overline{\beta}^k_s).\overline{a}(s,X_s)\,\overline{\beta}_s\,ds] \\ & -\frac{1}{2}\mathbb{E}^{\overline{\mathbb{Q}}}[\int_0^{t_n} (\overline{\beta}^k_s).\overline{a}(s,X_s)\,(\overline{\beta}^k_s)\,ds] \\ & + \int log\,(\frac{d\nu_T}{d\mu_T}\wedge k)\,d\nu_T \end{aligned}$$

thus

$$H(\overline{\mathbb{Q}},\overline{\mathbb{P}}_{\mu_0}) \quad + \quad \int (\frac{d\nu_T}{d\mu_T}\wedge k)\, d\mu_T \geq$$

$$\frac{1}{2}\,\mathbb{E}^{\overline{\mathbb{Q}}}[\int_0^{t_n} (\overline{\beta}^k_s).\overline{a}(s,X_s)\,(\overline{\beta}^k_s)\,ds] + \int log\,(\frac{d\nu_T}{d\mu_T}\wedge k)\,d\nu_T. \tag{3.9}$$

and the result follows by taking limits first in n then in k. $\square$

Since $\mathbb{Q} \ll \mathbb{P}_{\mu_0}$, $\nu_t \ll \mu_t$, and with assumption (3.3) i), we have:

$$\text{for } t \in]0,T],\ \nu_t(dx) = \rho_t(x)dx = \gamma_t(x)\,p_t(x)dx. \tag{3.10}$$

Our aim is to describe the relationship between β, $\overline{\beta}$ and ρ. To this end, we now show that $\mathbb{Q}$ satisfies an integration by parts formula, similar to 3.14 in [18].

Proposition 3.11. *Assume that $H(\mathbb{Q},\mathbb{P}_{\nu_0})$ is finite. Then, for dt almost every $t \in]0,T]$, every f and ϕ in $\mathcal{C}_0^\infty(\mathbb{R}^d)$, we have:*

$$-\mathbb{E}^{\mathbb{Q}}[(\nabla\phi\,.\,a\,\nabla f)(t,X_t)]$$

$$= \mathbb{E}^{\mathbb{Q}}[f(X_t)\,((L_t\phi + \overline{L}_{T-t}\phi)(t,X_t) + \nabla\phi(X_t).a(t,X_t)(\beta_t + \overline{\beta}_{T-t}oR))],$$

provided that, for every $k \in I\!N$ and every $\varepsilon \in]0,T]$,

$$\int_\varepsilon^T \int_{|x|\leq k} |\frac{div(ap)}{p}|(t,x)\,dt\,\nu_t(dx) < +\infty.$$

Proof. The proof is a straightforward copy of what is done in [18] replacing $X_t - X_{t-h}$ by $\phi(X_t) - \phi(X_{t-h})$. However, we now briefly repeat it, to see how our hypotheses are used. The basic idea is to write:

$$\mathbb{E}^{\mathbb{Q}}[(\phi(X_t) - \phi(X_{t-h}))f(X_t)] = \mathbb{E}^{\overline{\mathbb{Q}}}[(\phi(X_{T-t}) - \phi(X_{T-t+h}))f(X_{T-t})],$$

and to obtain alternate expressions of both hand sides, to use It's formula. To be rigorous, one has to introduce in both hand sides, localizing sequence of stopping times, namely:

$$S_k = \inf\{s \geq 0, |X_s| \geq k \text{ or } \int_0^s \beta_u.a(u,X_u)\,\beta_u du \geq k\},$$

and

$$\overline{S}_k = \inf\{s \geq 0, |X_s| \geq k \text{ or } \int_0^s \overline{\beta}_u.a(u,X_u)\,\overline{\beta}_u du \geq k\}.$$

The fact these are localizing sequences of stopping times is proved in [11]. In what follows, one has to replace t by $t \wedge S_k$ (resp. $t \wedge \overline{S}_k$) in the left (resp. right) hand side, and then to take the limits. No problem occurs with the limiting procedure, so that we do not write the details.

$$\mathbb{E}^{\mathbb{Q}}[(\phi(X_t) - \phi(X_{t-h}))f(X_t)] = \sum_{i=1}^{5} C_i$$

with

$$C_1 = \mathbb{E}^{\mathbb{Q}}[f(X_{t-h}) \int_{t-h}^{t} (L_s + \beta_s.a(s, X_s)\nabla)\, \phi(s, X_s)ds],$$

$$C_2 = \mathbb{E}^{\mathbb{Q}}[\int_{t-h}^{t} (\nabla\phi.a\nabla f)(s, X_s)ds],$$

$$C_3 = \mathbb{E}^{\mathbb{Q}}[\{\int_{t-h}^{t} \nabla f(X_s).dN_s\}\{\int_{t-h}^{t} A_s\phi(X_s)ds\}],$$

$$C_4 = \mathbb{E}^{\mathbb{Q}}[\{\int_{t-h}^{t} \nabla\phi(X_s).dN_s\}\{\int_{t-h}^{t} A_s f(X_s)ds\}],$$

$$C_5 = \mathbb{E}^{\mathbb{Q}}[\{\int_{t-h}^{t} A_s\phi(X_s)ds\}\{\int_{t-h}^{t} A_s f(X_s)ds\}].$$

But, according to the finite energy condition (or the entropy condition on $\mathbb{Q}$), we may apply Cauchy-Schwarz inequality and the boundedness of $\nabla\phi$ in order to get:

$$\mathbb{E}^{\mathbb{Q}}[\{\int_{t-h}^{t} \nabla\phi(X_s).a(s, X_s)\, \beta_s\, ds\}^2]$$

$$\leq \mathbb{E}^{\mathbb{Q}}[\{\int_{t-h}^{t} (\nabla\phi.a\, \phi)(s, X_s)\, ds\}\{\int_{t-h}^{t} \beta_s.a(s, X_s)\, \beta_s\, ds\}] = o(h)$$

uniformly in t. The same of course holds with f instead of ϕ. Thus, it is easily seen that C_3, C_4 and C_5 are respectively $\sqrt{h}\, o(\sqrt{h})$, $\sqrt{h}\, o(\sqrt{h})$ and $o(\sqrt{h})\, o(\sqrt{h})$, i.e. is a $o(h)$ (uniformly in t, hence independently of k when replacing t by $t \wedge S_k$. Hence, taking first limits in k, dividing by h, and letting h go to 0, these last three terms disappear in the limit. To calculate the limit of the first two terms (after the same manipulation), one uses the following well-known lemma:

Lemma 3.12. *(see e.g.* [18] *Proposition 2.5).*
If for some $p \geq 1$,

$$\mathbb{E}^{\mathbb{Q}}[\int_0^T |\eta_s|^p\, ds] < +\infty,$$

then, for dt-almost all $t \in [0, T]$, we have:

$$\lim_{h \to 0} \frac{1}{h} \int_{t-h}^{t} \eta_s \, ds = \eta_t$$

where the limit both takes place in $\mathbb{L}^p(\mathbb{Q})$ and $\mathbb{Q}$ a.s.

Hence, for dt almost all t, we get :

$$\lim_{h \to 0} \frac{1}{h} \mathbb{E}^{\mathbb{Q}}[(\phi(X_t) - \phi(X_{t-h})) \, f(X_t)] \tag{3.13}$$

$= \mathbb{E}^{\mathbb{Q}}[f(X_t)\{L_t \phi(X_t) + \beta_t.a(t, X_t) \nabla \, \phi(X_t)\}] + \mathbb{E}^{\mathbb{Q}}[\nabla \phi(X_t).a(t, X_t) \, \nabla f(X_t)]$.

In the same way, we calculate

$$-\mathbb{E}^{\overline{\mathbb{Q}}}[(\phi(X_{T-t+h}) - \phi(X_{T-t})) \, f(X_{T-t})]$$

$$= -\mathbb{E}^{\overline{\mathbb{Q}}}[f(X_{T-t}) \int_{T-t}^{T-t+h} (\overline{L}_s \phi(X_s) \; + \overline{\beta}_s.\overline{a}(s, X_s) \, \nabla \phi(X_s)) ds].$$

One can apply Lemma 3.12 with $\eta_s = \overline{L}_s \phi(X_s)$ and $p = 1$ thanks to the final assumption in Proposition 3.11, and with $\eta_s = \overline{\beta}_s.\overline{a}(s, X_s) \nabla \phi(X_s)$ and $p = 2$ thanks to the finite energy condition (3.7) and Cauchy-Schwarz inequality. We thus obtain:

$$\lim_{h \to 0} -\frac{1}{h} \mathbb{E}^{\overline{\mathbb{Q}}}[(\phi(X_{T-t+h}) - \phi(X_{T-t})) \, f(X_{T-t})] \tag{3.14}$$

$= -\mathbb{E}^{\overline{\mathbb{Q}}}[f(X_{T-t}) \, \{\overline{L}_{T-t} \phi(X_{T-t}) \; + \; \overline{\beta}_{T-t}.\overline{a}(T - t, X_{T-t}) \, \nabla \phi(X_{T-t})\}]$.

It suffices now to use the separability of $\mathcal{C}_0^\infty(\mathbb{R}^d)$, and to equalize (3.13) and (3.14), to get Proposition 3.11. □

Applying 3.11 to a function $\phi \in \mathcal{C}_0^\infty$ such that $\phi(x) = e.x$ on the support of f, where e is a fixed element of $\mathbb{R}^d$, we thus get the following integration by parts formula:

Corollary 3.15. *Under the hypotheses of Proposition 3.11, for every $e \in \mathbb{R}^d$, we have:*

$$-\mathbb{E}^{\mathbb{Q}}[e.a(t, X_t) \nabla f(X_t)]$$

$$= \mathbb{E}^{\mathbb{Q}}[f(X_t) \, \{e.a(t, X_t)(\beta_t + \overline{\beta}_{T-t} oR)\}] + \mathbb{E}^{\mathbb{Q}}[f(X_t) \, \{e.\frac{div(ap)}{p}(t, X_t)\}].$$

Corollary 3.15 is what we shall call "the duality equation". Its statement is similar to the classical regular case (i.e 3.3) but cannot be deduced from 3.3 due to the lack of regularity of β.

In the next section, we shall investigate consequences of Corollary 3.15 for ρ_t or γ_t (see (3.10)).

Remark 3.16. Of course if $\mu_0 = p_0(x) \, dx$ is not only stationary but reversible the situation is even simpler.

4. Application to Fokker-Planck equations.

In this section, we discuss existence, uniqueness and (a priori) regularity. First of all, we assume that:

(4.1) $\nu = (\nu_s)_{s\in[0,T]}$ is a flow of Probability measures which satisfies the $\mathcal{C}_0^\infty([0,T]\times\mathbb{R}^d)$ resp. $\mathcal{C}_0^\infty([0,T]\times\mathbb{T}^d)$, weak forward equation.

(4.2) Hypotheses of Theorem A or Theorem B are satisfied.

Remark 4.3. When the state space is the torus $\mathbb{T}^d$, the constant function equal to one belongs to $\mathcal{C}_0^\infty(\mathbb{T}^d)$. Hence, mass is preserved, i.e. if $(\nu_s)_{s\in[0,T]}$ is a flow of non negative measures, solution of the weak forward equation, ν_s is a probability measure as soon as ν_0 is. In the non compact case however, a similar statement is not immediate. But if $(\nu_s)_{s\in[0,T]}$ is a tight family of positive measures, the same conclusion is true, just approximating 1.

Our first result is concerned with uniqueness and stability.

Theorem 4.4. *Assume that* (4.1) *and* (4.2) *hold. Then:*

i) ν *is the unique solution of the weak forward equation such that*

$$\int_0^T \int \beta.a\beta\,(s,x)\,ds\,\nu_s(dx) < +\infty, \text{ starting from } \nu_0;$$

ii) *if* $\nu_0^{'} \ll \nu_0$, *then there exists a solution of the weak forward equation starting from* $\nu_0^{'}$;

iii) *if* $\frac{d\nu_0^{'}}{d\nu_0}$ *is bounded, the previous solution satisfies*

$$\int_0^T \int \beta.a\beta\,(s,x)\,ds\,\nu_s^{'}(dx) < +\infty,$$

and is the unique solution (starting from $\nu_0^{'}$*) satisfying this condition.*

Proof. i) is a direct consequence of (4.2), since $\mathbb{Q}oX_s^{-1} = \nu_s$.
ii) Define $\mathbb{Q}' = \frac{d\nu_0^{'}}{d\nu_0}.\mathbb{Q}$. Since $\int \nu_0(dx)\,\mathbb{E}^{\mathbb{P}_x}[Z_T] = 1$, and $\mathbb{E}^{\mathbb{P}_x}[Z_T] \le 1$ for every x, it follows that $\mathbb{E}^{\mathbb{P}_x}[Z_T] = 1$, ν_0 a.s., hence $\nu_0^{'}$ a.s.. So $\mathbb{Q}'$ is a solution of $\mathcal{M}(\frac{\partial}{\partial t} + L_t, \mathcal{C}_0^\infty, \nu_0^{'})$, and the marginals $\mathbb{Q}'oX_s^{-1} = \nu_s^{'}$ are solution of the weak-forward equation.
iii) If in addition, $\frac{d\nu_0^{'}}{d\nu_0}$ is bounded, then $H(\mathbb{Q}', \mathbb{P}_{\nu_0}) < +\infty$. Indeed:

$$H(\mathbb{Q}', \mathbb{P}_{\nu_0}) = H(\nu_0^{'}, \nu_0) + \frac{1}{2}\int_0^T \mathbb{E}^{\mathbb{Q}'}[\beta.a\beta\,(s,X_s)]\,ds,$$

and the last term is equal to

$$\int \frac{d\nu_0^{'}}{d\nu_0}(x) \int_0^T \mathbb{E}^{\mathbb{Q}_x}[\beta.a\beta\,(s,X_s)]\,d\nu_0(x)\,ds,$$

where $\mathbb{Q}_x$ is a ν_0 regular desintegration of $\mathbb{Q}$. But, according to the finite energy condition, $\int_0^T \mathbb{E}^{\mathbb{Q}_x}[\beta.a\beta\,(s,X_s)]\,ds \in \mathbb{L}^1(\nu_0)$. Since $\frac{d\nu_0'}{d\nu_0} \in \mathbb{L}^\infty$, we have $H(\mathbb{Q}',\mathbb{P}_{\nu_0}) < +\infty$. We are thus in the situation of i). $\square$

Now, we want to study a priori regularity, using the results of Section 3. To this end, we assume that the hypotheses of Theorem 3.3 (i.e. e.g. of Proposition 3.4 or 3.5) are satisfied and that $H(\nu_0,\mu_0) < +\infty$.
In this case, we have:

$$\forall\, t \in]0,T], \nu_t(dx) = \gamma_t(x)\,\mu_t(dx) = \gamma_t(x)\,p_t(x)dx = \rho_t(x)dx, \tag{4.5}$$

with $\mu_t = \mathbb{P}_{\mu_0} o X_t^{-1}$.

Since $H(\nu_t,\mu_t) \le H(\mathbb{Q},\mathbb{P}_{\mu_0}) < +\infty$, it follows that

$$\gamma_t\,|log\gamma_t| \in \mathbb{L}^1(\mu_t), \text{i.e. } p_t\,\gamma_t\,|log\gamma_t| \in \mathbb{L}^1(dx). \tag{4.6}$$

But Corollary 3.15 implies stronger regularity. To describe this regularity we need the definition of a twisted derivative in $\mathcal{D}'(\mathbb{R}^d)$.

Definition 4.7. *Let u and v be two functions such that $u\,\partial_i v \in \mathbb{L}^1_{loc}$. We then define*

$$v\,\partial_i u = -\,u\,\partial_i v + \partial_i(u\,v), \quad in\ \mathcal{D}'(\mathbb{R}^d).$$

We can now state the main result of this section

Theorem 4.8. *Assume that (4.1) and (4.2) hold. Assume in addition that σ and its first order derivatives are locally bounded, that $H(\nu_0,\mu_0) < +\infty$, and that:*
i) either one of the hypotheses of Proposition 3.4 or Proposition 3.5 is satisfied;
ii) or $\mu_0 = p_0 dx$ is a reversible Probability measure of the Markov process $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$.
Assume in addition that

$$(\sigma\frac{\nabla p_t}{p_t}) \in \mathbb{L}^2_{loc}(d\nu_t(x)\,dt).$$

Then, $d\nu_t(x) = \rho_t(x)\,dx$ for every $t \in]0,T]$, and the following holds:

$$\sigma\nabla\rho_t \in \mathbb{L}^1_{loc}(dx) \quad and \qquad \int_\varepsilon^T\int_K \frac{|\sigma\nabla\rho_t|^2}{\rho_t}\,dt\,dx < +\infty,$$

for any compact subset K of $\mathbb{R}^d$ and any $\varepsilon > 0$. Furthermore when ρ is locally bounded, then $\sigma\nabla\rho_t \in \mathbb{L}^2_{loc}(dx)$. In cases ii) or i) with the hypotheses of Proposition 3.5, one may take $\varepsilon = 0$.

This theorem is an immediate consequence of the following

Lemma 4.9. *Under the hypotheses of Theorem 4.8, for all* $t \in [0,T]$, *there exists* $\psi_t \in \mathbb{L}^\infty_{loc}$ *such that*

$$\sigma \nabla \rho_t = \rho_t \sigma \{ \frac{\nabla p_t}{p_t} + (\beta(t,.) + \overline{\beta}(T-t,.))\} + \rho_t \psi_t \text{ in } \mathcal{D}'.$$

Before to prove the Lemma let us make few remarks.

Remark 4.10. i) If $\sigma \frac{\nabla p_t}{p_t} \in \mathbb{L}^2(\rho_t(x)dxdt)$, then we may suppress the localization in the conclusion of 4.8 at least when σ^{-1} exists. In general, the compensation ψ in lemma 4.9 satisfies $|\psi(x)| \leq 2\max_i |\partial_i \sigma(x)|$. Hence, if $\partial_i \sigma$ are bounded , the same holds.
ii) It is not clear at all that $\frac{1}{\sqrt{\rho}}(\sigma \nabla \rho) = 2\sigma \nabla(\sqrt{\rho})$ in $\mathcal{D}'(\mathbb{R}^d)$. However, if we define

$$H^1_{loc}(\sigma\, dx) = \{f \in \mathbb{L}^2_{loc},\ \sigma \nabla f \in \mathbb{L}^2_{loc}\},$$

one can show that the chain rule $\sigma \nabla(g \circ f) = g'.\sigma \nabla f$ holds for smooth real valued functions, provided $\mathcal{C}^\infty_0$ is dense in $H^1_{loc}(\sigma\, dx)$ (see e.g. [22]), in particular when $\sigma \in \mathcal{C}^1_b$. Taking here again some regularization of the square root, one can then show that Theorem 4.8 implies that $\sigma \nabla(\sqrt{\rho}) \in \mathbb{L}^2_{loc}(dx)$. We shall come back to this point later.
iii) It should be more convenient to look at γ_t when $\rho_t = \gamma_t\, p_t$. But γ_t may be no element of $\mathcal{D}'(\mathbb{R}^d)$. If we introduce $U_t = \{p_t > 0\}$, and assume that p_t is locally bounded from below and from above in U_t, then $\gamma_t \in \mathbb{L}^1_{loc}(U_t, dx)$. If moreover, $p_t \in \mathcal{C}^1_b$, then $\nabla \rho_t = p_t \nabla \gamma_t + \gamma_t \nabla p_t$ in $\mathcal{D}'(U_t, dx)$. Since, $\frac{\nabla p_t}{p_t}$ is locally bounded in U_t, we deduce that $\sigma \nabla \gamma_t \in \mathbb{L}^2_{loc}(U_t, p_t dx)$. In infinite dimension, where no reference (Lebesgue) measure exists, this is a more natural point of view.
iv) In particular in the reversible case, we recover and extend results in the literature, particularly for invariant measures ν (see e.g. [4]). For the Brownian case, the results of the last section of [9] (which are completely correct, contrary to what is suggested in [1]), indicate the relationship between invariant and reversible measures.
v) The assumption

$$(\sigma \frac{\nabla p_t}{p_t}) \in \mathbb{L}^2_{loc}(d\nu_t(x)\, dt)$$

is not very restrictive. For example it is satisfied when $\log p_t \in \mathcal{C}^1$.

We turn to the proof of Lemma 4.9.

Proof. (*of Lemma* 4.9)
According to a previous remark, $\mathbb{Q}$ is markov. So does $\overline{\mathbb{Q}}$ and $\overline{\beta}$ is also markov, i.e. a function of (s,x). Thus, applying Corollary 3.15 (or Remark 3.16), we get that, for dt almost every t:

$$-\int e.a(t,x)\, \nabla f(x)\, d\nu_t(x) \tag{4.11}$$

$$= \int f(x)\, e.\{\frac{div(ap)}{p}(t,x) + a(t,x)\,(\beta(t,x) + \overline{\beta}(T-t,x))\}\, d\nu_t(x)$$

(in the reversible case, $\frac{div(ap)}{p} = 2b$), i.e. the duality equation

(4.12)

$$div(a(t,.)\rho_t) = \rho_t\{a(t,.)(\beta(t,.)+\overline{\beta}(T-t,.)) + \frac{div(ap)}{p}(t,.)\} \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

With our hypotheses on σ we may apply Definition 4.7 since $\partial_j a_{ij}$ is a function and $\rho_t\, \partial_j a_{ij} \in \mathbb{L}^1_{loc}$. Note that for any test function ϕ:

$$< a_{ij}\, \partial_j \rho_t, \phi > = - < \rho_t, \partial_j(a_{ij}\phi) > .$$

In particular, with our hypotheses, $div(a(t,.)p_t) \in \mathbb{L}^1_{loc}(dx)$ and it follows (see e.g. Lemma A.2 in [33]) that:

$$\frac{a(t,.)\nabla p_t}{p_t} = [-div(a(t,.)) + \frac{div(a(t,.)p_t)}{p_t}]\, 1_{p_t \neq 0},\ dx\ a.e.$$

(in the symmetric case, just define $\frac{a\nabla p}{p} = -div(a) + 2b = 0$).
So since the coefficients a_{ij} and all their derivatives are locally bounded, equation (4.12) may be rewritten as:

$$a(t,.)\nabla \rho_t = \rho_t\,\{\frac{a(t,.)\nabla p_t}{p_t} + a(t,.)(\beta(t,.) + \overline{\beta}(T-t,.))\}, \tag{4.13}$$

in $\mathcal{D}'(\mathbb{R}^d)$ for dt almost every $t \in]0,T]$.

Since $\frac{div(ap)}{p} \in \mathbb{L}^1_{loc}(\nu_t)$, so does $\frac{a\nabla p}{p}$, and the above formula shows that $a\nabla\rho$ is a function, belonging to $\mathbb{L}^1_{loc}(dx)$ as soon as the finite energy condition holds. (We recall that, if β is of finite energy, so does $\overline{\beta}$ according to Lemma 3.7).

We want to go a little bit further and to study $\sigma\,\nabla\rho$. Of course, we cannot directly write $\sigma^{-1}(a\,\nabla\rho) = \sigma\,\nabla\rho$ when σ is invertible, because the multiplication takes place in $\mathcal{D}'$, so that it is not well a priori defined. So, we use an approximation procedure.
Consider $\sigma_{\varepsilon\eta} = (\sigma + \varepsilon Id)^{-1} * g_\eta$, where, for instance, g_η is the Gaussian kernel of variance η^2. Then, for any test function ϕ, we have:

$$\begin{aligned} <(\sigma_{\varepsilon\eta} a\nabla\rho)_j, \phi> &= \sum_k <(a\nabla\rho)_k, (\sigma_{\varepsilon\eta})_{jk}\phi> \\ &= \sum_{k,i} <a_{ki}\partial_i\rho, (\sigma_{\varepsilon\eta})_{jk}\phi> \\ &= -\sum_{k,i} <\rho, \partial_i(a_{ki}(\sigma_{\varepsilon\eta})_{jk}\phi)> \\ &= -\sum_i <\rho, \partial_i((\sigma_{\varepsilon\eta} a)_{ji}\phi)> . \end{aligned}$$

Thanks to Lebesgue's theorem, we may now take $\eta = 0$. From now on we do not write the dependence in t.

Case 1. If σ^{-1} exists, is locally bounded with bounded derivatives, we may also take $\varepsilon = 0$, which yields as expected

$$\sigma^{-1}(a\,\nabla\rho) = \sigma\,\nabla\rho = \rho\sigma\{\frac{\nabla p}{p} + (\beta + \overline{\beta})\}$$

provided that $\sigma\frac{\nabla p}{p}$ is well defined.
If furthermore, $\sigma\frac{\nabla p}{p} \in \mathbb{L}^2_{loc}(\rho_t(x)\,dx)$, then, under the finite energy condition,

$$(\sigma\,\nabla\rho) \in \mathbb{L}^1_{loc}(dx) \text{ and } (\frac{1}{\sqrt{\rho}}\,\sigma\nabla\rho) \in \mathbb{L}^2_{loc}(dx).$$

If ρ is locally bounded, then $(\sigma\,\nabla\rho) \in \mathbb{L}^2_{loc}(dx)$.

Case 2. If σ is not invertible, we have to pass to the limit when ε goes to 0. It is easy to see that $(\sigma + \varepsilon Id)^{-1}a$ goes towards σ in $\mathbb{L}^\infty_{loc}$, but in general, $\partial_j((\sigma + \varepsilon Id)^{-1}a)$ does not converge to $\partial_j\sigma$. In the one dimensional case, the limit is $(\partial\sigma)\,1_{\sigma\neq 0}$. In the multidimensional case, we can calculate:

$$\partial_j((\sigma+\varepsilon Id)^{-1}a) = (\sigma+\varepsilon Id)^{-1}\partial_j\sigma(-(\sigma+\varepsilon Id)^{-1}a+\sigma)+(\sigma+\varepsilon Id)^{-1}\sigma\partial_j\sigma.$$

It is easy to see that, on a given compact subset,

$$||(\sigma+\varepsilon Id)^{-1}||_\infty \leq \frac{1}{\varepsilon}, ||\sigma-(\sigma+\varepsilon Id)^{-1}a||_\infty \leq \varepsilon \text{ and } ||(\sigma+\varepsilon Id)^{-1}\sigma||_\infty \leq 1.$$

Hence, up to a subsequence, we may assume that both $\partial_j((\sigma+\varepsilon Id)^{-1}a)$ and $(\sigma + \varepsilon Id)^{-1}\sigma\partial_j\sigma$ are convergent for the weak topology of $\mathbb{L}^1_{loc}$. Actually, $(\sigma+\varepsilon Id)^{-1}\sigma$ strongly converges (in $\mathbb{L}^\infty_{loc}$) towards $\Pi_{R(\sigma)}$, which is the projection onto the range $R(\sigma)$ of σ. It follows that there exists some locally bounded matrix h, such that:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{\eta\to 0} <(\sigma_{\varepsilon\eta}a\nabla\rho)_j, \phi> \\
&= -\sum_i <\rho, \sigma_{ji}\,\partial_i\phi> - \sum_i <\rho, h_{ji}\phi> - \sum_i <\rho, (\Pi_{R(\sigma)}\partial_i\sigma)_{ji}\phi> \\
&= -\sum_i <\rho,\, \partial_i(\sigma_{ji}\phi)> - \sum_i <\rho, h_{ji} - (\Pi_{Ker(\sigma)}\partial_i\sigma)_{ji}\phi> \\
&= <(\sigma\nabla\rho)_j, \phi> - <\psi_j\rho, \phi>
\end{aligned}$$

for some locally bounded ψ_j. Finally:

$$\sigma\nabla\rho = \rho\sigma\{\frac{\nabla p}{p} + (\beta + \overline{\beta})\} + \rho\psi \text{ in } \mathcal{D}'.$$

and the same conclusions as in the elliptic case are available. □

5. **A less regular case.**

In this section, we assume that:

$$(5.1) \qquad \begin{cases} \text{i) the state space is the torus } \mathbb{T}^d, \\ \text{ii) } L_t = \frac{1}{2}\nabla.a(t,.)\nabla (= \frac{1}{2}\, div(a(t,.)\nabla)), \end{cases}$$

so that the normalized Lebesgue measure dx is a reversible probability measure for L_t. The basic process may be written as $\exp^{iX_t}$, where X_t is the canonical process on the whole $\mathbb{R}^d$, and L_t is periodically extended to the whole space. Compactness of $\mathbb{T}^d$ will simplify some arguments.

Remark 5.2. In their paper [39], Quastel and Varadhan write the perturbed operator A_t as $A_t = L_t + \sigma c.\nabla$. Since σ is chosen as the symmetric square root of a, its kernel and range are in orthogonal sum, so that $c = \sigma\beta + c_{ker}$ for some β with $\sigma c = a\beta$ (here again β and c_{ker} may be chosen measurable). Furthermore, $|c|^2 \geq \beta.a\beta$. Hence, the finite energy condition in [39] implies ours.

The main difference with what precedes is the following hypothesis. We assume, throughout this section (we use the notations of [39]) that:

$$(5.3) \qquad \text{Hypothesis (A.1): } \sigma \in \mathbb{H}^1(dt \otimes dx) \text{ on } [0,T] \times \mathbb{T}^d.$$

Indeed, we now have to face various difficulties: (2.6) is no more true, the right hand side of the $\mathcal{C}^\infty$ weak forward equation (2.4) ii) is not well defined for any probability measure ν_t, part of the arguments of [11] where (local) boundedness is assumed fails to hold, as well as arguments in Section 3.

However, since L_t is written in divergence form, one can hope that some arguments which are used in the stationary case (i.e. in the framework of Dirichlet forms) may be extended (see e.g. [2] or [21]).

Recall that the basic object we want to study is a solution of the $\mathcal{C}^\infty$ weak-forward equation, i.e. a flow $(\nu_s)_{s\in[0,T]}$ satisfying:

$$(5.4) \qquad \forall\, f \in \mathcal{C}^\infty([0,T]\times\mathbb{T}^d), \forall\, 0 \leq u \leq t \leq T,$$

$$\int f(t,x)\nu_t(dx) - \int f(u,x)\nu_u(dx) = \int_u^t \int (\frac{\partial}{\partial s} + A_s) f(s,x)\, ds\, \nu_s(dx)$$

Since $A_s = \frac{1}{2}\, div(a(s,.)\nabla) + a(s,.)\beta(s,.)\nabla$, and a (resp. $\partial_i a$) only belongs to $\mathbb{L}^1$, we restrict our attention to the flows $(\nu_s)_{s\in[0,T]}$ of the form

$$(5.5) \qquad \nu_s(dx) = \rho(s,x)dx \text{ with } \rho \in \mathbb{L}^\infty([0,T]\times\mathbb{T}^d), \rho \geq 0.$$

Provided the finite energy condition (2.9) ii) holds, i.e.

$$(5.6) \qquad \int_0^T \int |\sigma\beta|^2(s,x)\, \rho(s,x)\, ds\, dx < +\infty,$$

the right hand side in (5.4) makes sense.
Notice that, if $\rho(0,.)$ is normalized into a probability density, so does $\rho(t,.)$ thanks to (5.4) with $f \equiv 1$. So we may and shall assume that $(\nu_s)_{s\in[0,T]}$ is a flow of probability measures. In this case, it is easy to see that (5.4) coincides with the notion of very weak solution discussed in [39].
As we shall see in the sequel, the probabilistic tools developed in the previous sections are actually less efficient than the usual analytic ones, as developed in [39]. The main (in fact only) reason is that the stochastic counterpart of the unperturbed L_t is not well defined. Of course, perturbation methods for not well defined objects have no chance to be efficient. However, we shall complete the stochastic picture of [39]. Before to do that, let us recall one result taken from [39] concerning ν_t.

Definition 5.7. *$\mathbb{H}^1_a$ denotes the space of $\mathbb{L}^2$ functions ϕ such that $\sigma\nabla\phi \in \mathbb{L}^2$, where, as before: $\sigma_{ij}\partial_j\phi = \partial_j(\sigma_{ij}\phi) - \phi\partial_j\sigma_{ij}$ in $\mathcal{D}'$.*

Proposition 5.8. *If (A.1) holds, then $\mathbb{H}^1_a$ equipped with the norm*

$$||\phi||_2 + ||\sigma\nabla\phi||_2$$

is a Hilbert space, and contains $\mathcal{C}^\infty$ as a dense subspace.

Proof. The proof of the density of $\mathcal{C}^\infty$ is given in Section 2 of [39], provided you add a final regularization in time. Actually, the main arguments (lemma 2.3 ii) and lemma 2.5) are contained in [9], Theorem 2.7 (see (2.1) and lemma (2.10) therein). We recently learnt about some simplification of the argument in [30]. □

This result has to be linked with Theorem B. Indeed, if $\sigma\nabla\rho \in \mathbb{L}^2$, one can perform an integration by parts in the forward equation and show that smooth functions are dense in $D_{e,\nu}$. Hence in the divergence case, Theorem B can be improved, provided $\sigma\nabla\rho \in \mathbb{L}^2$. Of course global Lipschitz assumption is stronger than (A.1).

We shall now study the stochastic quantization, first for the "unperturbed" L_t, then for A_t.

5.1. Stochastic quantization for the unperturbed L_t.

We want to study the martingale problem $\mathcal{M}(\frac{\partial}{\partial t} + L_t, \mathcal{C}^\infty, \mu_0)$. Since no result is known, one can try to regularize σ in σ^ε as before. Thanks to the divergence form of L_t, the case of $\mu_0(dx) = dx$ can be handled through usual tightness criterion, namely:

Proposition 5.9. *(see* [39]*, lemmas 5.1 and 5.2)*
If (A.1) holds, the family $\mathbb{P}^\varepsilon$ of (unique) solutions of $\mathcal{M}(\frac{\partial}{\partial t} + L^\varepsilon_t, \mathcal{C}^\infty, dx)$ is tight. Any cluster point $\mathbb{P}$ is a solution of $\mathcal{M}(\frac{\partial}{\partial t} + L_t, \mathcal{C}^\infty, dx)$ and satisfies $\mathbb{P} = \overline{\mathbb{P}}$, i.e. is reversible.

Remark 5.10. One has to be careful with the meaning of

$$\mathcal{M}(\frac{\partial}{\partial t} + L_t, \mathcal{C}^\infty, dx).$$

What is shown in lemma 5.2 of [39] is that, for any smooth function g, $M_t^g \overset{\text{def}}{=} g(t, X_t) - g(0, X_0) - \int_0^t (\frac{\partial}{\partial s} + L_s) g(s, X_s) ds$ is a $\mathbb{P}$-$\mathbb{L}^1$-martingale. Note that in the proof of this lemma, one cannot a priori choose $F = f(s, X_s)$ (with the notation therein), because it is not clear that the limit $\mathbb{P}$ is still Markov, but the proof still works.
Actually, it is not difficult (but not immediate) to show that the laws of

$$(X^\varepsilon, M^{g,\varepsilon}, <M^{g,\varepsilon}>_. = \int_0^. |\sigma^\varepsilon \nabla g|^2(s, X_s^\varepsilon) ds)$$

are tight, and that, taking appropriate subsequences, the limits are $(X, M^g, <M^g>_. = \int_0^. |\sigma \nabla g|^2(s, X_s) ds)$, i.e. M^g is a $\mathbb{P}$-$\mathbb{L}^2$-martingale with brackets $<M^g>_.$.

Similar "non smooth" martingale problems have been studied (see [42] and [17]) but with weaker assumptions on σ. In these papers examples of non uniqueness are given. However all these examples are using some kind of killed processes. In particular in [17] some uniqueness holds provided marginals are absolutely continuous. We did not succeed in proving uniqueness in our case, but fortunately, as in section 2 we are able to show extremality.

Theorem 5.11. *If (A.1) holds, for any bounded initial density p_0, there exists at most one solution $\mathbb{P}_{p_0}$ to $\mathcal{M}(\frac{\partial}{\partial t} + L_t, \mathcal{C}^\infty, p_0\, dx)$ with bounded marginal densities.*
In particular there exists only one stationary solution, $\mathbb{P}$ of $\mathcal{M}(\frac{\partial}{\partial t} + L_t, \mathcal{C}^\infty, dx)$. This solution is the weak limit of $\mathbb{P}^\varepsilon$ and is reversible. Furthermore $\mathbb{P}$ is an extremal solution of $\mathcal{M}(\frac{\partial}{\partial t} + L_t, \mathcal{C}^\infty, dx)$

Proof. The first part is a consequence of the uniqueness of a bounded solution of

$$\frac{\partial u}{\partial t} = \nabla.\, a \nabla u$$

with $u(0,.) = p_0$. This uniqueness follows from [39] lemma 3.2, and time reversal.
Combined with Proposition 5.9 this yields the next statements. It only remains to prove the statement on extremality. If $\mathbb{P} = \alpha\mathbb{Q} + (1-\alpha)\mathbb{R}$, then, $\mathbb{Q} \ll \mathbb{P}$, hence, $\mathbb{Q} o X_t^{-1} = \rho(t,x)\, dx$ with $\rho(t,.) \le \frac{1}{\alpha}$ and $\rho(0,.) = 1$. It then suffices to apply the first part of the Theorem to deduce that $\mathbb{Q} = \mathbb{P}$. $\square$

5.2. **Stochastic quantization for the perturbed A_t.** Here is the main result of this subsection

Theorem 5.12. *Assume that assumption (A.1) holds. Let ρ be a bounded solution of the weak forward equation* (5.4) *satisfying the finite energy condition* (5.6)*, and such that $\sigma\nabla\rho \in \mathbb{L}^2(dt \otimes dx)$.*

Then there exists a solution $\mathbb{Q}$ of $\mathcal{M}(\frac{\partial}{\partial t} + A_t, \mathcal{C}^\infty, \rho(0,.)dx)$ such that

$$H(\mathbb{Q},\mathbb{P}) < +\infty \quad \textit{and} \quad \mathbb{Q}oX_t^{-1} = \rho(t,x)dx \textit{ for every } t$$

where $\mathbb{P}$ is defined in Theorem 5.11.
More precisely, $\mathbb{Q} = Z_T\,\mathbb{P}$, where Z_T is the Doleans exponential of $\int_0^T \beta(s,X_s).dM_s$, is a Probability measure, and such a solution. Furthermore,

$$H(\mathbb{Q},\mathbb{P}) = H(\rho(0,.)dx, dx) + \frac{1}{2}\int_0^T \int |\sigma\beta|^2(s,x)\,\rho(s,x)\,ds\,dx < +\infty.$$

As immediate consequences, we get in the same way as Theorem 4.3:

Corollary 5.13. *If (A.1) holds, suppose that ρ is a bounded solution of the weak forward equation satisfying both* (5.6) *and $\sigma\nabla\rho \in \mathbb{L}^2$. Then:*
i) ρ is the unique bounded solution which also satisfies both previous conditions;
ii) if there exists $c > 0$ such that $0 \le \psi \le c\,\rho(0,.)$, then there exists a bounded solution u of (5.4) *such that $u(0,.) = \psi$, and satisfying* (5.6).

Remark 5.14. i) Notice that according to Theorem 12.22 of [27], $\mathbb{Q}$ is an extremal solution of $\mathcal{M}(\frac{\partial}{\partial t} + A_t, \mathcal{C}^\infty, \rho(0,.)dx)$. But we do not know anything about uniqueness, even in the class of solutions with bounded marginal densities. This is due to the lack of uniqueness for the non-perturbed generator.
ii) The stochastic quantization problem is studied in [39], Theorem 5.3. Our assumptions are less restrictive than those of this Theorem, since the authors assume

$$\int_0^T \int \frac{|\sigma\nabla\rho_t|^2}{\rho_t}\,dt\,dx < +\infty,$$

which is stronger than $\sigma\nabla\rho \in \mathbb{L}^2(dt \otimes dx)$ since ρ is bounded.
We also misunderstand one point in their proof, namely how bounds for the entropy of marginals (Lemma (3.9) of [39]) allow to directly get a bound for the joint laws in formula (5.5) of [39].
iii) Of course, as in section 4, one hopes that the above stronger regularity will follow from a time reversal argument. This will be done in the next subsection.
iv) In [39] another hypothesis

Hypothesis (A.2): $\displaystyle\int_0^T \int_{\mathbb{T}^d} \sum_{i,j,k,l} \partial_i a_{jk}(a^{-1})_{kl}\partial_i a_{lj} ds\,dx < +\infty$ and $\sigma \in \mathbb{L}^2$.

which implies (A.1) (see[39]) is also introduced. It is shown in [39] Theorem 4.1 and Theorem 5.4, that (A.2) implies both uniqueness for bounded solutions of the weak forward equation and for the martingale problem $\mathcal{M}(\frac{\partial}{\partial t}+A_t, \mathcal{C}^\infty, \rho(0,.)dx)$ in the class of Probability measures with bounded marginal densities. The arguments used there are still available in our framework.

Proof of Theorem 5.12. The proof follows the lines of [11] (proof of Theorem 4.18). The additional difficulty here is to extend the weak forward equation to non smooth f. This is done in the first step. The second step is then very similar to [11]. We refer to the "outline of proof" (4.9) bis) and the erratum of [11], which will help the reader in what follows.
We first choose a family $\sigma_\varepsilon \in \mathcal{C}^\infty([0,T]\times\mathbb{T}^d)$ such that:

(5.15)
$$\begin{cases} \text{i) } \sigma_\varepsilon \text{ is positive, symmetric, definite;} \\ \text{ii) } \sigma_\varepsilon \text{ goes towards } \sigma \text{ in } \mathbb{H}^1(dt\otimes dx) \text{ as } \varepsilon \text{ goes to } 0; \\ \text{iii) } \|\sigma\sigma_\varepsilon^{-1}\|_\infty \le 2 \text{ for each } \varepsilon \text{ and } \|\sigma\sigma_\varepsilon^{-1}-\Pi_{R(\sigma)}\|_\infty \text{ goes to } 0 \text{ with } \varepsilon, \\ \text{where } \Pi_{R(\sigma)} \text{ is the projection operator onto the range } R(\sigma) \text{ of } \sigma. \end{cases}$$

Such a family is obtained by mollifying $(\sigma+\varepsilon Id)$ as in the previous section.
Next, we introduce;
(5.16)

i) χ_ε^k a family of smooth functions such that
$0\le\chi_\varepsilon^k\le 1$, $\chi_\varepsilon^k\equiv 1$ on $|\sigma_\varepsilon|\bigvee|\nabla\sigma_\varepsilon|<k$,
$\chi_\varepsilon^k\equiv 0$ on $|\sigma_\varepsilon|\bigvee|\nabla\sigma_\varepsilon|>k+\varepsilon$;

ii) β^k a sequence of smooth vector fields we shall choose later;

iii) $\mathbb{P}^\varepsilon_{s,x}$ the unique solution of $\mathcal{M}(\frac{\partial}{\partial t}+L^\varepsilon_t, \mathcal{C}^\infty, \delta_{\{s,x\}})$;

iv) $Z^{k,\varepsilon}$ the Doleans exponential of $\int_0^\cdot(\chi_\varepsilon^k\,\beta^k)(s,X_s).dM^\varepsilon_s$, where as before,
$$M^\varepsilon_s = X_s - X_0 - \int_0^s div(a^\varepsilon Id)(u,X_u)du;$$

v) $\mathbb{Q}^{k,\varepsilon}_{s,x} = Z^{k,\varepsilon}_T\,\mathbb{P}^\varepsilon_{s,x}$, $\mathbb{Q}^{k,\varepsilon}_\rho = \int \mathbb{Q}^{k,\varepsilon}_{0,x}\,\rho(0,x)\,dx$;

vi) $f\in\mathcal{C}^\infty(\mathbb{T}^d)$ and non negative,
$$f_{k,\varepsilon,t}(s,x) = \mathbb{E}^{\mathbb{Q}^{k,\varepsilon}_{s,x}}[f(X_{t-s})] \quad \text{for } t\in[0,T] \quad \text{and} s\in[0,t].$$

It follows from standard arguments on smooth diffusions that
$$f_{k,\varepsilon,t}\in\mathcal{C}^\infty([0,t]\times\mathbb{T}^d)$$
and satisfies:
$$(5.17)\qquad \begin{cases} (\frac{\partial}{\partial s}+L^\varepsilon_s)f_{k,\varepsilon,t}+\chi^k_\varepsilon\beta^k.a_\varepsilon\nabla f_{k,\varepsilon,t}\equiv 0 \text{ on } [0,t]\times\mathbb{T}^d; \\ f_{k,\varepsilon,t}(t,x)=f(x), \int f_{k,\varepsilon,t}(0,x)\,\rho(0,x)\,dx = \mathbb{E}^{\mathbb{Q}^{k,\varepsilon}_\rho}[f(X_t)], \end{cases}$$

so that we may apply the weak forward equation, in order to get

(5.18)
$$\mathbb{E}^{\mathbb{Q}_\rho^{k,\varepsilon}}[f(X_t)] = \int f(x)\rho(t,x)dx - \int_0^t \int (L_s - L_s^\varepsilon) f_{k,\varepsilon,t}(s,x)\,\rho(s,x)\,ds\,dx$$
$$- \int_0^t \int [\nabla f_{k,\varepsilon,t}.(a\beta - a_\varepsilon \beta^k \chi_\varepsilon^k)](s,x)\,\rho(s,x)\,ds\,dx.$$

The goal now, as the reader guessed, is to make ε go to 0, k to $+\infty$, with β^k going to β. Of course, we have to show that $\mathbb{Q}_\rho^{k,\varepsilon}$ goes (in some sense) to $\mathbb{Q}$, and that the limiting procedure in the right hand side of (5.18) is justified. We begin with the limit in ε, k being kept fixed. For studying the right hand side of (5.18), we use the following results, which are contained e.g. in [39], lemmas 3.4 and 3.9 - 3.11 (we drop the subscript t for simplicity).

(5.19)
$$\begin{cases} f_{k,\varepsilon} \text{ and } \sigma_\varepsilon \nabla f_{k,\varepsilon} \text{ are bounded in } \mathbb{L}^2, \\ \text{hence are weakly convergent, up to a subsequence.} \\ \text{If } f_k = \text{weak limit } f_{k,\varepsilon},\, v_k = \text{weak limit } \sigma_\varepsilon \nabla f_{k,\varepsilon}, \\ \text{then } v_k = \sigma \nabla f_k \text{ in } \mathcal{D}'. \\ \text{Thus, } \sigma \nabla f_k \in \mathbb{L}^2. \text{ Also note that } \|f_k\|_\infty \le \|f\|_\infty. \end{cases}$$

Since $\sigma_\varepsilon \beta^k \chi_k^\varepsilon$ converges to $\sigma \beta^k 1_{(|\sigma| \bigvee |\nabla\sigma| \le k)}$ for the strong $\mathbb{L}^2$-topology, it follows that $\sigma_\varepsilon \beta^k \chi_k^\varepsilon . \sigma_\varepsilon \nabla f_{k,\varepsilon}$ converges to $\sigma \beta^k . \sigma \nabla f_k 1_{(|\sigma| \bigvee |\nabla\sigma| \le k)}$ for the weak $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$ topology. So, we obtain:

Lemma 5.20. *If ρ is bounded, then, up to subsequences:*
$$\lim_{\varepsilon \to 0} \int_0^t \int [\nabla f_{k,\varepsilon,t}.(a\beta - a_\varepsilon \beta^k \chi_\varepsilon^k)](s,x)\,\rho(s,x)\,ds\,dx$$
$$= \int_0^t \int [\sigma \nabla f_{k,t}.\sigma(\beta - \beta^k 1_{(|\sigma| \bigvee |\nabla\sigma| \le k)})](s,x)\,\rho(s,x)\,ds\,dx.$$

Proof of Lemma 5.20. For $\sigma_\varepsilon \nabla f_{k,\varepsilon,t}.\sigma_\varepsilon \beta^k \chi_k^\varepsilon$, we use the above $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$ convergence, since ρ is bounded.
For $\sigma \nabla f_{k,\varepsilon,t}.\sigma\beta$, we use the following argument. First $\sqrt{\rho}\sigma\beta \in \mathbb{L}^2$, thanks to the finite energy condition, then $\sigma \nabla f_{k,\varepsilon,t} = \sigma \sigma_\varepsilon^{-1} \sigma_\varepsilon \nabla f_{k,\varepsilon,t}$ is bounded in $\mathbb{L}^2$ thanks to (5.15) iii), hence weakly convergent, up to a subsequence. The weak limit is $\Pi_{R(\sigma)} \sigma \nabla f_k$, according to (5.15) iii) and (5.19). But $\Pi_{R(\sigma)} \sigma \nabla f_k . \sigma\beta = \sigma \nabla f_k . \sigma\beta$ since $\sigma\beta \in R(\sigma)$, and again the result follows from the boundedness of ρ. □

The second term in the right hand side of (5.18) is even more complicated. First, we show:

Lemma 5.21. *If ρ is bounded, then, up to subsequences, we have:*
$$\lim_{\varepsilon \to 0} \int_0^t \int (\frac{\partial}{\partial s} + L_s^\varepsilon) f_{k,\varepsilon,t}\,\rho_s\,ds\,dx = \int_0^t \int (\frac{\partial}{\partial s} + L_s) f_{k,t}\,\rho_s\,ds\,dx$$

$$= -\int_0^t \int \sigma\nabla f_k.\sigma\beta^k 1_{(|\sigma|\bigvee|\nabla\sigma|\leq k)}\,\rho_s\,ds\,dx.$$

Proof. On one hand, since $(\frac{\partial}{\partial s} + L_s^\varepsilon)f_{k,\varepsilon,t} = -\sigma_\varepsilon\nabla f_{k,\varepsilon,t}.\sigma_\varepsilon\beta^k\chi_k^\varepsilon$, it converges to $\sigma\nabla f_k.\sigma\beta^k 1_{(|\sigma|\bigvee|\nabla\sigma|\leq k)}$ for the weak $\sigma(\mathbb{L}^1,\mathbb{L}^\infty)$ topology. On the other hand, it is not difficult to see that $(\frac{\partial}{\partial s}+L_s)f_k$ is well defined in $\mathcal{D}'$, and $(\frac{\partial}{\partial s}+L_s^\varepsilon)f_{k,\varepsilon,t}$ converges to $(\frac{\partial}{\partial s}+L_s)f_k$ in $\mathcal{D}'$. It follows that $(\frac{\partial}{\partial s}+L_s)f_k \in \mathbb{L}^1$, and that convergence holds for the weak $\sigma(\mathbb{L}^1,\mathbb{L}^\infty)$ topology. □

Next, using integration by parts, and the proof of Lemma 5.20, one gets:

(5.22) If ρ is bounded and $\sigma\nabla\rho \in \mathbb{L}^2$,

$$\lim_{\varepsilon\to 0}\int_0^t\int L_s f_{k,\varepsilon,t}\,\rho_s\,ds\,dx = -\frac{1}{2}\int_0^t\int \sigma\nabla f_k.\sigma\nabla\rho_s\,ds\,dx.$$

The most difficult part is $\frac{\partial}{\partial s}f_{k,\varepsilon,t}$. But, since ρ satisfies the weak forward equation, $(\frac{\partial}{\partial s}+A_s)^*\rho = 0$ in $\mathcal{D}'(]0,T[\times\mathbb{T}^d)$. Now notice that $\mathcal{C}_0^\infty(]0,t[\times\mathbb{T}^d)$ is dense in $\mathbb{H}_a^1([0,t]\times\mathbb{T}^d)$ for the natural norm of $\mathbb{H}_a^1$ (we have already mentioned that $\mathcal{C}^\infty$ is dense, and truncature in time direction is easily allowed). If $\phi \in \mathcal{C}_0^\infty(]0,t[\times\mathbb{T}^d)$, integration by parts yields:

(5.23) If ρ is bounded, and $\sigma\nabla\rho \in \mathbb{L}^2$,

$$<\frac{\partial}{\partial s}\rho,\phi> = <A_s^*\rho,\phi> =$$

$$= -\frac{1}{2}\int_0^t\int \sigma\nabla\phi.\sigma\nabla\rho_s\,ds\,dx + \int_0^t\int \sigma\beta.\sigma\nabla\phi\,\rho_s\,ds\,dx,$$

so that, if the finite energy condition is satisfied, $\frac{\partial}{\partial s}\rho$ extends on a continuous linear operator on $\mathbb{H}_a^1$ (restricted to $[0,t]$).
Since $f_{k,\varepsilon}$ goes to f_k weakly in $\mathbb{H}_a^1$, Mazur's theorem (see [16] page 422) says that some convex combinations of the $f_{k,\varepsilon}$ (of course with the maximal ε going to 0) strongly converge to f_k in $\mathbb{H}_a^1$. Of course, taking convex combinations in (5.20), (5.21), (5.22), does not modify the limit. We thus have proved:

Lemma 5.24. *If ρ is bounded and $\sigma\nabla\rho \in \mathbb{L}^2$, we have, up to convex combinations of subsequences:*

$$\lim_{\varepsilon\to 0}\int_0^t\int (L_s - L_s^\varepsilon)f_{k,\varepsilon,t}(s,x)\rho(s,x)\,ds\,dx$$

$$= -<\frac{\partial}{\partial s}\rho, f_k> -\frac{1}{2}\int_0^t\int \sigma\nabla f_k.\sigma\nabla\rho_s\,ds\,dx - \int_0^t\int (\frac{\partial}{\partial s}+L_s)f_k\,\rho_s\,ds\,dx$$

$$= 0.$$

The first equality summarizes (5.21) up to (5.23), and what precedes. The second one (i.e. the fact it is 0) would be clear if ρ were smooth. Here again, we may mollify ρ into ρ^ε (using convolutions as in [9] or [39]), and check that $\frac{\partial}{\partial s}\rho^\varepsilon$ is bounded in $(\mathbb{H}^1_a)^*$
(verify that $\partial_s\rho^\varepsilon = \frac{1}{2}div(a\nabla\rho)_\varepsilon - div(a\beta\rho)_\varepsilon)$.
Hence, it is weakly convergent, up to subsequences, so that we may pass to the limit.
Collecting all the previous results, we thus have shown (up to convex combinations of some subsequences):

$$\lim_{\varepsilon\to 0}\mathbb{E}^{\mathbb{Q}^{k,\varepsilon}_\rho}[f(X_t)] = \tag{5.25}$$

$$\int f(x)\rho(t,x)dx - \int_0^t\int \sigma\nabla f_k.\sigma(\beta - \beta^k 1_{(|\sigma|\bigvee|\nabla\sigma|\leq k)})\rho_s dxds$$

The final job is to calculate another explicit form of this limit.

Recall that $\mathbb{Q}^{k,\varepsilon}_\rho = \rho(0,X_0)Z^{k,\varepsilon}_T\mathbb{P}^\varepsilon$, where $\mathbb{P}^\varepsilon$ solves $\mathcal{M}(\frac{\partial}{\partial t} + L^\varepsilon_t, \mathcal{C}^\infty, dx)$. We want to show tightness, and find an explicit limit for $\mathbb{Q}^{k,\varepsilon}_\rho$. Note that we cannot use Lemma 5.1 in [39], since we do not know uniform bounds for the marginal densities of $\mathbb{Q}^{k,\varepsilon}_\rho$ ($Z^{k,\varepsilon}_T$ is not bounded). We replace this argument by deep results, mainly due to Le Cam, for which we refer to [28]. First, $Z^{k,\varepsilon}_T$ belongs to all the spaces $\mathbb{L}^p(\mathbb{P}^\varepsilon)$, for $1\leq p < +\infty$, with uniform bounds, i.e. $\mathbb{L}^p$ norms are bounded independently of ε, thanks to the truncation by χ^k_ε (it is only now that χ^k_ε plays a role). Since ρ is bounded, $\mathbb{Q}^{k,\varepsilon}_\rho \lhd \mathbb{P}^\varepsilon$, i.e. the family $(\mathbb{Q}^{k,\varepsilon}_\rho)_\varepsilon$ is contiguous to $(\mathbb{P}^\varepsilon)_\varepsilon$ (see [28], definition 1.1 page 249).
Now, it is not difficult to check, that the usual tightness criterion used in [39](5.1), applies for showing that the $\mathbb{P}^\varepsilon$ laws of $(X_., Z^{k,\varepsilon}_.)$ are tight (for $Z^{k,\varepsilon}_.$ use the associated stochastic differential equation and classical inequalities for martingales; here again, χ^k_ε plays a role both for σ and $\nabla\sigma$). So, up to subsequences, these laws are convergent. We now apply Theorem 3.3 page 564 of [28] which says that the corresponding $\mathbb{Q}^{k,\varepsilon}_\rho$ also converges in law to some $\mathbb{Q}^k$, which is absolutely continuous with respect to $\mathbb{P}$ (the weak limit of the $\mathbb{P}^\varepsilon$ is), the density of which is given by the weak limit of the $Z^{k,\varepsilon}_.$.
Furthermore, it is similar to Proposition 5.9 to check that the limit $\mathbb{Q}^k$ solves $\mathcal{M}(\frac{\partial}{\partial t} + A^k_t, \mathcal{C}^\infty, \rho(0,x)dx)$, with $A^k_t = L_t + \sigma\beta^k 1_{(|\sigma|\bigvee|\nabla\sigma|\leq k)}\nabla$.
But $\mathbb{P}$ is an <u>extremal</u> solution to $\mathcal{M}(\frac{\partial}{\partial t} + L_t, \mathcal{C}^\infty, dx)$. It follows that the density process of $\mathbb{Q}^k$ is given by the Doleans exponential $Z^k_.$ of

$$\int_0^. (\beta^k 1_{(|\sigma|\bigvee|\nabla\sigma|\leq k)})(s,X_s).dM_s.$$

Extremality is essential here. Of course, convex combinations do not alterate the result.

Finally:

$$\mathbb{E}^{\mathbb{P}}[\rho(0,X_0)\,Z_t^k\,f(X_t)] \tag{5.26}$$
$$= \int f(x)\,\rho(t,x)\,dx - \int_0^t \int \sigma\nabla f_k.\sigma(\beta - \beta^k 1_{(|\sigma|\bigvee|\nabla\sigma|\le k)})\,\rho_s\,ds\,dx$$

i.e. we are exactly in the same situation as formula (4.12) of [11] (see the very beginning of the proof in [11]).

We have now to mimic [11], getting an uniform bound for $||\sigma\nabla f_k||_{\mathbb{L}^2}$ by using the weak forward equation with $f^2_{k,\varepsilon}$ (see [11], identities (4.21) to (4.23)). Passage to the limit in ε is obtained by similar arguments as above, using that $||f_{k,\varepsilon}||_\infty \le ||f||_\infty$. We now leave the details to the reader. □

5.3. **Necessary conditions for stochastic quantization.** We saw in Section 4 that for smooth σ, not only $\sigma\nabla\rho \in \mathbb{L}^2$, but "roughly" $\sigma\nabla(\sqrt{\rho}) \in \mathbb{L}^2$. When assumption (A.1) holds, Theorem 5.12 solves the stochastic quantization problem for the bounded ρ such that $\sigma\nabla\rho \in \mathbb{L}^2$. So, methods of Section 3 yields a result like:

(5.27)
$$\left\{\begin{array}{l}\text{if } \rho \text{ is bounded and } \sigma\nabla\rho \in \mathbb{L}^2, \text{ if in addition, } \rho \text{ satisfies}\\ \text{the weak forward equation and the finite energy condition,}\\ \text{then } \sigma\nabla\sqrt{\rho} \in \mathbb{L}^2.\end{array}\right.$$

Such a statement does not look exciting, but is not trivial. However, if one looks at the proof of Proposition 3.11, one immediately sees that boundedness is used in order to control C_3, C_4, C_5.

Consequently, we only formulate here a time reversal result, without detailed proof (the courageous reader will easily derive all the details).

(5.28) Assumption (A.3):

i) for $n \in I\!N^*$, $t \in [0,T]$, denote $U_n(t) = \{x; |\sigma(t,x)| \vee_i |\partial_i\sigma(t,x)| \le n\}$;
ii) hypothesis (A.3) is said to hold, if, for every $t \in [0,T]$, there exists $h > 0$ and a sequence $(\chi_n) \in \mathcal{C}^\infty(\mathbb{T}^d)$ such that

$$\forall\, n, \forall\, s \in [t-h,t], 0 \le \chi_n \le 1, supp.\,\chi_n \subset U_n(s),$$

and the sequence χ_n is (dx a.s.) increasing to the constant function 1.

Theorem 5.29. *Assume the hypotheses of Theorem 5.12 and hypothesis (A.3) hold. Then:*
i) the duality equation (see Corollary 3.15 holds (with $p \equiv 1$*) for any smooth* f *such that* $supp.f \subset U_n(t)$ *for some* n*;*
ii) $\int_0^T \int \frac{|\sigma\nabla\rho|^2}{\rho}(s,x)\,ds\,dx < +\infty, i.e.\ \sqrt{\rho} \in \mathbb{H}^1_a.$

Proof. Proof of i) is obtained by truncature with χ_n; ii) follows by the same arguments as in Section 4, see in particular Remark 4.10 ii). □

Remark 5.30. Theorem 4.1 in [39] says that

$$\int_0^T \int \frac{|\sigma \nabla \rho|^2}{\rho}(s,x)\, ds\, dx < +\infty,$$

for any bounded non negative solution of the weak forward equation when the finite energy condition and assumption (A.2) (see Remark 5.14 iv)) hold. When (A.2) holds, this result is, as we already said, much better than Theorem 5.29 ii), where a priori regularity for ρ is assumed.

6. **Minimal diffusions, Nelson's estimate and the nodal set**

In the previous sections we have seen that stochastic quantization and time reversal yield a priori regularity for the solutions of the Fokker-Planck equation. Note that, except for the divergence case in [39], we do not know how to prove these results by direct p.d.e. techniques. Whether this kind of regularity is interesting or not from the p.d.e. point of view, we do not want to discuss here. Their infinite dimensional analogues are.
In this section we want to show that a priori regularity has some nice consequences from the probabilistic point of view. Indeed it allows to study attainability of the nodal set $\{\rho = 0\}$, through what is called Nelson's estimate. This estimate is derived e.g. in [34] for smooth diffusions. We shall closely follow Nelson's ideas, but technical difficulties are rather involved. Hence in order to make this section comprehensive, we shall first give the outline of the method in our general (non Brownian) framework. Full hypotheses and proofs of the technical results will be given in the next subsections.

6.1. **An outline of Nelson's method.**

The aim is to derive a decomposition for

$$\log \rho(t, X_t) - \log \rho(0, X_0)$$

which does not involve second order derivatives. Such a decomposition is well known in the symmetric context as the Lyons-Zheng decomposition.

Recall that we have described the law of the process and its time reversal. In particular

$$\text{for } f \in \mathcal{C}_0^\infty([0,T]\times\mathbb{R}^d), \quad \mathbb{Q} \text{ a.s. for } s \le t,$$

$$f(t, X_t) - f(s, X_s) = M_t^f - M_s^f \tag{6.1}$$
$$+ \int_s^t \left(\partial_u + \frac{1}{2}\sum_{i,j} a_{ij}\partial_i\partial_j + (b + a\beta)\,.\nabla\right) f(u, X_u)\, du,$$

where M^f_u is a $\mathbb{Q}$ martingale with brackets $< M^f >_t = \int_0^t |\sigma\nabla f|^2(u, X_u)\, du$; and $\mathbb{Q}$ a.s.

$$f(t, X_t) - f(s, X_s) = \overline{M}^f_{T-t} - \overline{M}^f_{T-s} \tag{6.2}$$

$$+ \int_s^t (\partial_u - \frac{1}{2}\sum_{i,j} a_{ij}\partial_i\partial_j + (b - a\overline{\beta} - \frac{div(ap_u)}{p_u}) .\nabla)\, f(u, X_u)\, du,$$

where $\overline{M}^f_u \circ R$ is a $\overline{\mathbb{Q}}$ martingale with brackets $< \overline{M}^f \circ R >_t = \int_0^t |\sigma\nabla f|^2(T-u, X_u)\, du$.

Here we make an abuse of notation since $\overline{\beta}$ is actually $\overline{\beta}(T-.)$. Identifying (6.1) and (6.2) furnishes.

$$\int_0^t [\sum_{ij} a_{ij}\, \partial_i\partial_j + a(\beta + \overline{\beta}) .\nabla + \frac{div(ap_u)}{p_u} .\nabla]\, f\,(u, X_u)\, du \tag{6.3}$$

$$= M^f_t - \overline{M}^f_T + \overline{M}^f_{T-t}.$$

But taking one half the sum of (6.1) and (6.2) yields for $s = 0$,

(6.4)

$$\begin{aligned} f(t, X_t) - f(0, X_0) &= \frac{1}{2}(M^f_t + \overline{M}^f_{T-t} - \overline{M}^f_T) \\ &\quad + \frac{1}{2}\int_0^t (a(\beta - \overline{\beta})) .\nabla\, f(u, X_u)\, du \\ &\quad + \int_0^t (\partial_u + (b - \frac{1}{2}\frac{div(ap_u)}{p_u}) .\nabla)\, f(u, X_u)\, du. \end{aligned}$$

This formula is the generalization of Lyons-Zheng decomposition.
The point now is that, thanks to Cauchy-Schwarz and martingale inequalities and to the finite energy condition

$$\mathbb{E}^{\mathbb{Q}}[\sup_{t\in[0,T]} |\int_0^t (a(\beta - \overline{\beta})) .\nabla\, f(u, X_u)\, du|\,] \leq C\|\sigma\nabla f\|_{\mathbb{L}^2(\rho\, dt\, dx)}, \tag{6.5}$$

$$\mathbb{E}^{\mathbb{Q}}[\sup_{t\in[0,T]} |M^f_t + \overline{M}^f_{T-t} - \overline{M}^f_T|\,] \leq C\|\sigma\nabla f\|_{\mathbb{L}^2(\rho\, dt\, dx)}. \tag{6.6}$$

Note that since $\sigma\nabla(log\rho)$ belongs to $\mathbb{L}^2(\rho\, dt\, dx)$ according to a priori regularity (provided C_0^∞ is dense in $\mathbb{H}^1_a(\rho)$), (6.5) and (6.6) will extend to $f = log\,\rho$. It thus remains to control the last term in (6.4).

The part involving the time derivative ∂_u is delicate. Taking the expectation in (6.4), one obtains that ρ satisfies the so called current equation

$$\frac{\partial\rho}{\partial t} = \frac{1}{2}\nabla.(\rho\ a(-\beta + \overline{\beta})) - \nabla.(\rho\ (b - \frac{1}{2}\frac{div(ap_u)}{p_u})) \tag{6.7}$$

in $\mathcal{D}'$ in the sense of section 4. One cannot divide (6.7) by ρ, but as before one can expect that the chain rule furnishes (formally)

$$\frac{\partial(\log\rho)}{\partial t} = \frac{1}{2}\nabla.\,(a(-\beta+\overline{\beta})) + \frac{1}{2}\,\frac{\sigma\nabla\rho}{\rho}\,.\,\sigma(-\beta+\overline{\beta}) \tag{6.8}$$

$$-\nabla\,.\,(b-\frac{1}{2}\frac{div(ap_u)}{p_u}) - \frac{\nabla\rho}{\rho}\,.\,(b-\frac{1}{2}\frac{div(ap_u)}{p_u}).$$

Next we have to control $\nabla.\,(a(-\beta+\overline{\beta}))$.

To this end we shall introduce the analogue of the least action principle of Nelson, i.e. minimization of entropy.

Indeed if ρ satisfies the Fokker-Planck equation for some β of finite energy, it still satisfies the Fokker-Planck equation for $B = \beta + B^{\perp}$ where $B^{\perp}$ is any vector field of finite energy which is orthogonal to $\mathcal{C}_0^{\infty}$ in $\mathbb{L}^2(\sigma\,\rho\,dt\,dx)$ i.e. such that

$$\int_0^T\int (B^{\perp}.\,a\nabla f)(s,x)\,\rho(s,x)\,ds\,dx = 0,$$

for all smooth f.
Among all possible β's, there is one which minimizes the energy, namely β_{min}, which is the projection of β onto the $\mathbb{L}^2(\sigma\,\rho\,dt\,dx)$ closure of the gradient of smooth functions, that is

(6.9) there exists a sequence of smooth functions S_n such that

$$\lim_{n\to+\infty}\int_0^T\int|\sigma\,(\beta_{min}-\nabla S_n)|^2(s,x)\,\rho(s,x)\,ds\,dx = 0.$$

The associated $\mathbb{Q}_{min}$ then minimizes relative entropy. We refer to [11] and [13] for details.

In the flat smooth case of [34], one can deduce that $\beta_{min} = \nabla S$ for some $\mathbb{L}^2_{loc}$ function S. This result is known in Analysis as de Rham's theorem, and can be obtained by using e.g. Poincaré inequality (other proofs using some lemmata of Peetre and Tartar are well known). A similar result seems difficult to get in our case, unless assuming that some Poincaré inequality holds or that the imbedding of $\mathbb{H}^1_a(\rho)$ into $\mathbb{L}^2(\rho)$ is compact. Hence we will have to still work with the sequence S_n and use a limiting procedure. However in the rest of this subsection (which is an outline) we will write

$$\beta_{min} = \nabla S.$$

Note that we can use a similar argument to show that

$$\overline{\beta}_{min} = \nabla\,\overline{S},$$

which also follows from the duality equation (thanks to a priori regularity). Our notation differs from [34]. S there is $\frac{1}{2}\,(S-\overline{S})$ here.

Contrary to what Nelson does in [34], we shall not work with $\mathbb{Q}_{min}$, but only use β_{min}.
The current equation for the log (6.8) then becomes

$$\frac{\partial(log\,\rho)}{\partial t} = \frac{1}{2}\nabla\,.\,(a(-\nabla S+\nabla\overline{S})) + \frac{1}{2}\,\frac{\sigma\nabla\rho}{\rho}\,.\,\sigma(-\nabla S+\nabla\overline{S}) \tag{6.10}$$
$$-\nabla\,.\,(b-\frac{1}{2}\frac{div(ap_u)}{p_u}) - \frac{\nabla\rho}{\rho}\,.\,(b-\frac{1}{2}\frac{div(ap_u)}{p_u}).$$

Now use (6.3) with $f=-S+\overline{S}$, again for $s=0$. This yields

$$\int_0^t [\sum_{ij} a_{ij}\,\partial_i\partial_j + a(\beta+\overline{\beta})\,.\nabla + \frac{div(ap_u)}{p_u}\,.\nabla]\,(-S+\overline{S})\,(u,X_u)\,du \tag{6.11}$$
$$= M_t^{S-\overline{S}} - \overline{M}_T^{S-\overline{S}} + \overline{M}_{T-t}^{S-\overline{S}}\,.$$

Combining (6.4), (6.10), (6.11) and the duality equation, we finally obtain, using r for $log\rho$,

(6.12)

$$\begin{aligned}
\log\rho(t,X_t) - \log\rho(0,X_0) \quad &= \frac{1}{2}\,(M_t^r + \overline{M}_{T-t}^r - \overline{M}_T^r) \\
&+ \frac{1}{2}(M_t^{S-\overline{S}} - \overline{M}_T^{S-\overline{S}} + \overline{M}_{T-t}^{S-\overline{S}}) \\
&+ \frac{1}{2}\int_0^t a(\beta-\overline{\beta}))\,.\nabla\, r\,(u,X_u)\,du \\
&- \int_0^t \nabla\,.\,(b-\frac{1}{2}\frac{div(ap_u)}{p_u})(u,X_u)\,du.
\end{aligned}$$

Hence, formally, one can control the $\mathbb{Q}$ expectation of

$$\sup_{t\in[0,T]} |\log\rho(t,X_t) - \log\rho(0,X_0)|$$

by the $\mathbb{L}^2\,(\rho\,dt\,dx)$ norm of $\sigma\,\nabla\log\,\rho$ and the relative entropy $H(\mathbb{Q},\mathbb{P})$. Of course (6.8), and (6.10) up to (6.12) are not justified, and all the job now will be to give a rigorous meaning to all this derivation.

6.2. **Non attainability of the nodal set: first approach.**

Looking at the formal derivation of the previous subsection, we see that we certainly will need some approximation procedure, but due to (6.11) this procedure should be well behaved when using stochastic calculus. In order to build such an approximation we shall use all the ingredients of the previous sections. So, we shall first introduce the hypotheses we will work with.

(6.13) <u>Hypothesis (C.1)</u>

(1) σ and its first order derivatives are bounded, b is locally Hölder continuous,

(2) $\mu_0 = p_0\, dx$, $H(\nu_0, \mu_0) < +\infty$,
(3) μ_0 is a reversible measure for $\mathbb{P}$,
(4) ν_t is a weak solution of the Fokker-Planck equation (see definition 2.4), such that the finite energy condition (2.9) ii) is satisfied,
(5) $(\sigma \frac{\nabla p_0}{p_0}) \in \mathbb{L}^2(d\nu_t dt)$.

When Hypothesis (C.1) is satisfied, one may use the results of sections 2, 3 and 4. In particular one knows that stochastic quantization is available for A_t and that

$$d\nu_t = \rho_t\, dx \text{ such that } \sigma\nabla\sqrt{\rho} \in \mathbb{L}^2(dt\, dx).$$

One should localize some of the hypotheses in (C.1). Note that when (C.1) holds,

$$b = \frac{1}{2}\,\frac{div(ap_0)}{p_0}.$$

Not to introduce additional technicalities, we prefer work with global assumptions. We will see later what can be improved.
The second hypothesis is related to section 5:

(6.14) Hypothesis (C.2)

(1) L_t is a divergence operator defined on the torus,
(2) either assumptions (A.1) and (A.3) are fulfilled or assumption (A.2) is (see section 5),
(3) ν_t is a weak solution of the Fokker-Planck equation (see definition 2.4), such that the finite energy condition (2.9) ii) is satisfied,
(4) $d\nu_t = \rho_t\, dx$ where ρ is bounded and satisfies $\sigma\nabla\rho \in \mathbb{L}^2(dt\, dx)$.

Thanks to the results of section 5, the same conclusions as for (C.1) remain valid.

Note that in both cases (C.1) or (C.2),

$$\sigma\sqrt{\rho} \in \mathbb{H}^1 \text{ (usual Sobolev space)}\ ,$$

since either σ or ρ is bounded. This is essential for some points in the approximation results below.

Notation 6.15. As in section 5 we define

$$\mathbb{H}^1_a = \{f \in \mathbb{L}^2(dt\, dx)\,,\, \sigma\nabla f \in \mathbb{L}^2(dt\, dx)\},$$
$$\mathbb{H}^1_a(\rho) = \{f \in \mathbb{L}^2(\rho\, dt\, dx)\,,\, \sigma\nabla f \in \mathbb{L}^2(\rho\, dt\, dx).$$

Lemma 6.16. *One can find an approximation of the identity J_ε such that, if we denote by g_ε the convolution $g * J_\varepsilon$,*

(1) *in case (C.1) holds, for any compactly supported $f \in \mathbb{H}^1_a(dt\, dx)$ (resp. $f \in \mathbb{H}^1_a(\rho\, dt\, dx)$), f_ε goes towards f in $\mathbb{H}^1_a$ (resp. $\mathbb{H}^1_a(\rho\, dt\, dx)$;*

(2) *in case (C.2) holds, for any bounded* $f \in \mathbb{H}^1_a(dt\,dx)$ *(resp.* $f \in \mathbb{H}^1_a(\rho\,dt\,dx)$*),* f_ε *goes towards* f *in* $\mathbb{H}^1_a$ *(resp.* $\mathbb{H}^1_a(\rho\,dt\,dx)$*).*

In all cases we may assume that convergence also holds a.s. Furthermore the chain rule is available in both $\mathbb{H}^1_a$ *and* $\mathbb{H}^1_a(\rho\,dt\,dx)$ *at least for bounded functions.*

Case (2) is mainly contained in [39] (see section 5), case (1) is mainly contained in [22] and [23] (also see [9]). In both cases one uses the already mentioned fact that $\sigma\sqrt{\rho}$ belongs to $\mathbb{H}^1$, i.e. one may take $\sigma\sqrt{\rho}$ instead of σ in [39] lemma 2.3 or in [23] lemma 2.4 (taking $\mu = dx$ with the notations of [23]). The chain rule is shown by using approximation and integration by parts. Actually the statement of the lemma has to be precised: indeed J_ε is here a time-space approximation of the identity, i.e. we also regularize in time, contrary to what is done in the quoted references. This does not introduce any trouble, provided one chooses a splitting approximation.

Thanks to Lemma (6.16), one can check the current equation for the *log* (6.8), i.e.

Lemma 6.17. *Assume that (C.1) or (C.2) holds. Then for all* $\alpha > 0$*:*

$$\frac{\partial}{\partial t}\,log(\rho+\alpha) = \frac{1}{2}\,\nabla.\Big(\frac{\rho\,a(-\beta+\overline{\beta})}{\rho+\alpha}\Big) + \frac{1}{2}\,\Big(\frac{\rho\,\sigma(-\beta+\overline{\beta})}{(\rho+\alpha)^2}\Big).\sigma\nabla\rho.$$

The proof is straightforward, taking convolutions and passing to the limit. Actually one only needs lemma (6.16) for checking the second term, using that $\sigma\nabla\rho_\varepsilon - (\sigma\nabla\rho)_\varepsilon$ goes to 0 in $\mathbb{L}^2$ on any relatively compact open domain.

Now, we may write the Lyons-Zheng decomposition (6.4) with

$$f_{\alpha\,\varepsilon} = (log(\rho+\alpha))_\varepsilon \text{ for some } \alpha > 0, \tag{6.18}$$

$$\begin{aligned} f_{\alpha\,\varepsilon}(t,X_t) - f_{\alpha\,\varepsilon}(0,X_0) &= \frac{1}{2}\,(M_t^{f_{\alpha\,\varepsilon}} + \overline{M}_{T-t}^{f_{\alpha\,\varepsilon}} - \overline{M}_T^{f_{\alpha\,\varepsilon}}) \\ &+\frac{1}{2}\int_0^t \Big((\sigma(\beta-\overline{\beta}))\,.\,\frac{\sigma\nabla\rho_\varepsilon}{(\rho+\alpha)_\varepsilon}\Big)\,(u,X_u)\,du \\ &+\int_0^t \partial_u\,f_{\alpha\,\varepsilon}\,(u,X_u)\,du. \end{aligned}$$

We can replace β and $\overline{\beta}$ by β_{min} and $\overline{\beta}_{min}$ in (6.17), and introduce the approximating sequences ∇S_n and $\nabla\overline{S}_n$ of β_{min} and $\overline{\beta}_{min}$. The second one is obtained by the same arguments as in [11], for the time reversed process. We thus have

$$f_{\alpha\,\varepsilon}(t,X_t) - f_{\alpha\,\varepsilon}(0,X_0) = \sum_{i=1}^{4} E_i, \tag{6.19}$$

where

$$E_1 = \frac{1}{2}\,(M_t^{f_{\alpha\,\varepsilon}} + \overline{M}_{T-t}^{f_{\alpha\,\varepsilon}} - \overline{M}_T^{f_{\alpha\,\varepsilon}}),$$

$$E_2 = \frac{1}{2}\int_0^t \Big((\sigma(\beta-\overline{\beta}))\,.\,\frac{\sigma\nabla\rho_\varepsilon}{(\rho+\alpha)_\varepsilon}\Big)\,(u,X_u)\,du,$$

$$\begin{aligned} E_3 &= \frac{1}{2}\int_0^t \Big(\nabla\,.\Big(\frac{\rho\,a(-\nabla S_n+\nabla\overline{S}_n)}{\rho+\alpha}\Big)\Big)_\varepsilon(u,X_u)\,du \\ &\quad +\frac{1}{2}\int_0^t \Big(\Big(\frac{\rho\,\sigma(-\nabla S_n+\nabla\overline{S}_n)}{(\rho+\alpha)^2}\Big)\,.\sigma\nabla\rho\Big)_\varepsilon(u,X_u)\,du, \end{aligned}$$

$$\begin{aligned} E_4 &= \frac{1}{2}\int_0^t \Big(\nabla\,.\Big(\frac{\rho\,a(\{\nabla S_n-\beta_{min}\}+\{\overline{\beta}_{min}-\nabla\overline{S}_n\})}{\rho+\alpha}\Big)\Big)_\varepsilon(u,X_u)\,du \\ &+\frac{1}{2}\int_0^t \Big(\Big(\frac{\rho\,\sigma(\{\nabla S_n-\beta_{min}\}+\{\overline{\beta}_{min}-\nabla\overline{S}_n\})}{(\rho+\alpha)^2}\Big)\,.\sigma\nabla\rho\Big)_\varepsilon(u,X_u)\,du. \end{aligned}$$

In order to study E_3 we shall use (6.3) with

$$f_{n,s,y}(.) = (\overline{S}_n - S_n)(.-s,\,.-y) \text{ i.e.}$$

(6.20)

$$\int_0^t [\sum_{ij} a_{ij}\,\partial_i\partial_j + a(\beta+\overline{\beta})\,.\nabla + \frac{div(ap_u)}{p_u}\,.\nabla]\,(\overline{S}_n - S_n)\,(u-s,X_u-y)\,du$$

$$= M_t^{f_{n,s,y}} - \overline{M}_T^{f_{n,s,y}} + \overline{M}_{T-t}^{f_{n,s,y}}.$$

which yields

$$E_3 = E_{31} + E_{32} + E_{33} \tag{6.21}$$

where

$$E_{31} = \frac{1}{2}\int \{M_t^{f_{n,s,y}} - \overline{M}_T^{f_{n,s,y}} + \overline{M}_{T-t}^{f_{n,s,y}}\}J_\varepsilon(y)\,dy,$$

$$\begin{aligned} E_{32} &= -\frac{1}{2}\int_0^t \Big((\sigma\frac{\nabla\rho}{\rho})\,.\,\sigma\nabla(\overline{S}_n - S_n)_\varepsilon\Big)\,(u,X_u)\,du \\ &\quad +\frac{1}{2}\int_0^t \Big(\Big(\frac{\rho\,\sigma(-\nabla S_n+\nabla\overline{S}_n)}{(\rho+\alpha)^2}\Big)\,.\sigma\nabla\rho\Big)_\varepsilon(u,X_u)\,du, \end{aligned}$$

$$E_{33} = \frac{1}{2}\int_0^t \Big(\nabla\,.\Big(\frac{\alpha\,a(-\nabla S_n+\nabla\overline{S}_n)}{\rho+\alpha}\Big)\Big)_\varepsilon(u,X_u)\,du.$$

We have used the duality equation for getting E_{32}.

We now have to pass to the limit. Because of E_4 and E_{33}, we have first to take the limits in α and n. This will oblige us to get an uniform control on $\frac{\sigma\nabla\rho_\varepsilon}{\rho_\varepsilon}$ in $\mathbb{L}^2(\rho\,dt\,dx)$ norm. Unfortunately we only have such a control for $\frac{\sigma\nabla\rho_\varepsilon}{\sqrt{(\rho_\varepsilon)}}$. Hence we would have to control $\frac{\rho}{\rho_\varepsilon}$. This leads to an another modification, but we have to assume some a priori continuity for ρ. We thus can state

Theorem 6.22. *Assume that*

i) *(C.1) or (C.2) holds,*
ii) *ρ is time-space continuous,*
iii) *p_0 is continuous and strictly positive, or more generally $log(p_0) \in L^1_{loc}(\rho_t\, dx)$ for all $t \leq T$.*

Then $\mathbb{Q}$ almost surely, the time-space process never hits the nodal set $\{\rho(t,x) = 0\}$.

Proof. For the proof we introduce

$$\tau_N = \inf \{u, \rho(u, X_u) \notin [\frac{1}{N}, N] \text{ or } |X_u| \geq N\} \wedge T\,,$$

and a family $\chi_N : \mathbb{R} \to \mathbb{R}$ of smooth functions with uniformly bounded first order derivative, such that,

$$\chi_N(z) = z, \text{ if } |z| \leq 2N\,; \; |\chi_N(z)| = 3N, \text{ if } |z| > 4N\,.$$

We replace $f_{\alpha\varepsilon}$ by $\chi_N(f_{\alpha\varepsilon})$. Since (6.18) up to (6.21) are true $\mathbb{Q}$ almost surely for all t, we can replace t by $t \wedge \tau_N$. Notice that in all terms, except the backward martingale terms, if ε is small enough (just depending of N and the modulus of continuity of ρ on $[0,T] \times \{|x| \leq 2N\}$), χ_N is unnecessary and its derivative vanishes, provided $log(N) \leq N$. Furthermore ρ_ε is uniformly (in ε) bounded from below up to time τ_N. We can thus take limits first when α goes to 0, then when n goes to ∞ and finally when ε goes to 0, except for the backward martingale terms.
Let us look at each term. In the statements below the convergence may be chosen both $\mathbb{Q}$ a.s., and in $\mathbb{L}^1(\mathbb{Q})$:

(1) $\chi_N((log(\rho+\alpha))_\varepsilon)(t \wedge \tau_N, X_{t\wedge\tau_N}) - \chi_N((log(\rho+\alpha))_\varepsilon)(0, X_0)$ goes to $\chi_N(log(\rho))(t \wedge \tau_N, X_{t\wedge\tau_N}) - \chi_N(log(\rho))(0, X_0)$;
(2) E_2 (up to time τ_N) goes to
$$\frac{1}{2} \int_0^{t\wedge\tau_N} \left((\sigma(\beta - \overline{\beta}))\,.\,\frac{\sigma\nabla\rho}{\rho}\right)\,(u, X_u)\,du;$$
(3) E_4 goes to 0;
(4) E_{32} goes to 0;
(5) E_{33} goes to 0.

For studying E_{31}, one can use standard estimates and B.D.G. inequalities in order to get

$$\mathbb{E}^{\mathbb{Q}}[\sup_{s\in[0,T]}\;\sup_{t\in[0,T]} |\int \{M_t^{f_{n,s,y}} - \overline{M}_T^{f_{n,s,y}} + \overline{M}_{T-t}^{f_{n,s,y}}\} J_\varepsilon(y)\,dy\,|] \leq K,$$

for all n and ε, where K is a constant, thanks to the finite energy condition and the definition of S_n and $\overline{S}_n$.

It remains to look at E_1, but in its new form i.e.,

$$E_1 = \frac{1}{2}\left(M^{\chi_N(f_{\alpha\,\varepsilon})}_{t\wedge\tau_N} + \overline{M}^{\chi_N(f_{\alpha\,\varepsilon})}_{T-(t\wedge\tau_N)} - \overline{M}^{\chi_N(f_{\alpha\,\varepsilon})}_{T}\right).$$

In order to get a similar bound as for E_{31}, we have to control the brackets

$$\int_0^T \left((\chi'_N(log(\rho+\alpha)_\varepsilon))^2 \, |\frac{\sigma\,\nabla\rho_\varepsilon}{(\rho+\alpha)_\varepsilon}|^2\right)(u, X_u)\,du, \tag{6.23}$$

both for $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ (replacing X_u by X_{T-u}). A quick look at (6.23) easily convinces that again, thanks to a priori regularity

$$\lim_{\varepsilon\to 0}\lim_{\alpha\to 0} \mathbb{E}^{\mathbb{Q}}[\sup_{t\in[0,T]} |E_1|] \,\le\, K' \int_0^T \int |\frac{\sigma\nabla\rho}{\rho}|^2 \rho(t,x)\,dx\,dt,$$

for all N, where K' is a bound for all χ'_N.

It follows that

$$\mathbb{E}^{\mathbb{Q}}[|\chi_N(log\rho)(t\wedge\tau_N, X_{t\wedge\tau_N}) - \chi_N(log\rho)(0, X_0)|] \tag{6.24}$$

is bounded, independently of N.
Finally since $H(\mathbb{Q},\mathbb{P})$ is finite, so is $H(\nu_t,\mu_0)$ (recall that $\mu_t = \mu_0$). Hence $\rho_t\, log(\frac{\rho_t}{p_0})$ belongs to $\mathbb{L}^1(dx)$. Since $log(p_0) \in L^1_{loc}(\rho_t\,dx)$, we deduce that τ_N is greater than the exit time of the ball $|x| \le k$, for $k \le N$. Hence we are done. □

Remark 6.25. i) If we assume stronger a priori regularity for ρ, namely $\frac{1}{2}$-Hölder in time, and Lipschitz in space, there is a much more simple proof in section 4 of [43].
ii) Non attainability of the nodal set is a key point in the classical construction of singular diffusions in the symmetric case. Here we obtain it as a byproduct of the stochastic quantization.
iii) Assumption iii) in Theorem 6.22 is natural. Indeed if for the underlying $\mathbb{P}$ the process reaches $\{p_0 = 0\}$, then it will reach the nodal set for $\mathbb{Q}$.
iv) Theorem 6.22 is of course frustrating. Not only we have to assume that ρ is continuous, but the underlying $\mathbb{P}$ is supposed to be symmetric. One can thus try to perform another approximation procedure, less natural from the analytic point of view, but better suited from the probabilistic one. This will however require additional assumptions on a, and will be the aim of the next subsection.

6.3. **Non attainability of the nodal set: second approach.**
In this subsection we shall assume the following

(6.26) Hypothesis (C.3)

(1) σ and b are $\mathcal{C}_b^{1,2}$,
(2) $\mu_0 = p_0\, dx$, $H(\nu_0, \mu_0) < +\infty$,
(3) a is uniformly elliptic,
(4) ν_t is a weak solution of the Fokker-Planck equation (see definition 2.4), such that the finite energy condition (2.9) ii) is satisfied,
(5) $(\sigma \frac{\nabla p_t}{p_t}) \in \mathbb{L}^2(d\nu_t dt)$,
(6) $\nabla \, . \, \frac{div(ap_t)}{p_t} \in \mathbb{L}^1(d\nu_t dt)$.

Again when (C.3) holds one can use all the results of the sections 3 and 4.
We shall prove the following

Theorem 6.27. *If (C.3) holds, one can find a version of ρ such that $\mathbb{Q}$ a.s. the process does not hit the set $\{\rho = 0\}$.*

Proof. Introduce as before the approximating sequence S_n of β_{min}. We can then build via usual Girsanov theory, the Probability measure $\mathbb{Q}_n$ associated to

$$L_t + a\nabla S_n \, . \, \nabla,$$

with initial law ν_0. Thanks to (1) and (3) in (C.3), the marginals of $\mathbb{Q}_n$ are absolutely continuous with at least $\mathcal{C}_b^{1,2}$ densities (for $t > 0$) denoted by ρ_n . Furthermore

$$\rho_n > 0, \text{ for } t > 0 \, .$$

One can of course use the results of sections 3 and 4 for $\mathbb{Q}_n$. In particular, $\overline{\mathbb{Q}}_n$ is a diffusion process, which drift $\overline{\beta}_n$ satisfies the duality equation

$$\sigma \frac{\nabla \rho_n}{\rho_n}(t,.) = \sigma \frac{\nabla p_t}{p_t} + \sigma \, \nabla S_n(t,.) + \sigma \, \overline{\beta}_n(T-t,.). \tag{6.28}$$

It follows that

$$\overline{\beta}_n = \nabla \overline{S}_n$$

for some $\mathcal{C}^{1,2}$ function $\overline{S}_n$. Notice that $\overline{S}_n$ is <u>not</u> the same one as in the previous subsection.

The key point is that

$$H(\mathbb{Q}, \mathbb{Q}_n) = \frac{1}{2} \int_0^T \int |\sigma\beta - \sigma \nabla S_n|^2(t,x) \, \rho(t,x) \, dt \, dx \; < +\infty. \tag{6.29}$$

Using the invariance of relative entropy by time reversal we also have

$$H(\mathbb{Q}, \mathbb{Q}_n) = H(\overline{\mathbb{Q}}, \overline{\mathbb{Q}}_n) = \tag{6.30}$$

$= H(\nu_T, \rho_n(T,.)\,dx) + \dfrac{1}{2}\displaystyle\int_0^T \int |\sigma\overline{\beta} - \sigma\nabla\overline{S}_n|^{\,2}(T-t,x)\,\rho(t,x)\,dt\,dx \;< +\infty.$

Moreover,

$$H(\mathbb{Q}_{min}, \mathbb{Q}_n) \text{ goes to } 0,$$

which implies that

$$H(\overline{\mathbb{Q}}_{min}, \overline{\mathbb{Q}_n}) \text{ goes to } 0 \text{ too.}$$

Hence, actually we are in a similar situation for $\overline{S}_n$ as in the previous subsection.

Applying the Lyons-Zheng decomposition with $f = \chi_N(log\rho_n)$, where χ_N is defined as in the proof of theorem 6.22, and the current equation satisfied by ρ_n up to the exit time τ_N of the ball of radius N, we obtain

$$\chi_N(\log\rho_n)(t\wedge\tau_N, X_{t\wedge\tau_N}) - \chi_N(\log\rho_n)(0, X_0) = \tag{6.31}$$

$$\begin{aligned}
&= \frac{1}{2}\,(M^r_{t\wedge\tau_N} + \overline{M}^r_{T-(t\wedge\tau_N)} - \overline{M}^r_T) \\
&+ \frac{1}{2}(M^{S_n-\overline{S}_n}_{t\wedge\tau_N} - \overline{M}^{S_n-\overline{S}_n}_T + \overline{M}^{S_n-\overline{S}_n}_{T-(t\wedge\tau_N)}) \\
&+ \frac{1}{2}\int_0^{t\wedge\tau_N} a(\beta-\overline{\beta}))\,.\nabla\, r\,(u, X_u)\,du \\
&- \int_0^{t\wedge\tau_N} \nabla\,.\,(b - \frac{1}{2}\,\frac{div(ap_u)}{p_u})(u, X_u)\,du.
\end{aligned}$$

where $r = \chi_N(log\rho_n)$. Actually (6.31) is not completely correct. Indeed, since $\overline{S}_n$ and its derivatives are not bounded, we have to replace it by a truncated version which vanishes outside the ball of radius $2N$ for instance. This modification does not change the current equation for $log\rho_n$ inside the ball of radius N and the proof goes on. $\overline{S}_n$ in (6.31) denotes this truncated version.
Thanks to (C.3) (5), (6.28), (6.29) and (6.30) we know that

$$\int_0^T \int |\sigma\frac{\nabla\rho_n}{\rho_n}|^{\,2}\,\rho\,dt\,dx$$

is bounded, independently of n. It follows as in the proof of theorem 6.22 that

$$\mathbb{E}^{\mathbb{Q}}[|\chi_N(log\rho_n)(t\wedge\tau_N, X_{t\wedge\tau_N}) - \chi_N(log\rho_n)(0, X_0)|] \tag{6.32}$$

is bounded, independently of n and N.

The final job is to get some convergence in the left hand side of (6.31). To this end, first remark that taking the difference between (6.31) at time t and (6.31) at time s, one easily sees that the $\mathbb{Q}$ laws of the processes

$$t \to \chi_N(\log\rho_n)(t\wedge\tau_N, X_{t\wedge\tau_N})$$

are tight. Next one uses the well known Pinsker inequality

$$||\nu - \mu|| \leq \Big(2\, H(\nu, \mu)\Big)^{\frac{1}{2}},$$

where $||.||$ denotes the variation norm. It implies, thanks to the convergence of $H(\mathbb{Q}_{min}, \mathbb{Q}_n)$ to 0, that ρ_n goes to ρ in $\mathbb{L}^1(dt\, dx)$. One easily deduces that any cluster point is the law of a modification of

$$\chi_N(\log \rho)(t \wedge \tau_N, X_{t\wedge\tau_N}).$$

Of course one can choose the modification so as it is a projective system in N. The rest of the proof is straightforward. $\square$

Remark 6.33. i) Note that the version of ρ which is obtained in the previous proof, is $\mathbb{Q}$ almost surely continuous along the paths of the process. It corresponds to the quasi continuous version which is chosen in the symmetric case.
ii) It would be more natural to study $\gamma = \frac{\rho}{p}$, since the underlying measure is $p_t\, dx$ and not dx. Actually, with additional work, one can extend the previous proof to γ and relax the elliptic hypothesis. But we did not succeed in recovering the general framework of Theorem A (see Theorem 2.11).

7. From Fokker-Planck to Schrödinger.

The goal of Nelson's program was to give a probabilistic interpretation of Schrödinger equation, close to the Lagrangian formalism, hence without using Feynman path integral. Since we have now (rather completely) completed the preliminaries of this program, it should be interesting to see what can be done in this direction.

Not to introduce disturbing technicalities (both previous sections are certainly intricate enough), we shall work with simplified hypotheses, namely the classical flat case of Brownian motion i.e.

$$L_t = \frac{1}{2}\, \triangle.$$

Let us first consider the smooth case.
Take $\beta = \nabla S$ for some smooth S, and an initial law $\nu_0 = \rho_0\, dx$. Hence ρ is $\mathcal{C}_b^{1,2}$ and strictly positive. The duality equation

$$\nabla log\rho_t = \nabla S(t,.) + \overline{\beta}(T-t,.) = \nabla S(t,.) + \nabla \overline{S}(t,.)$$

holds and we may choose

$$\overline{S}(t,.) = log\rho_t - S(t,.).$$

We have inverted time in order to get easier notations in what follows. We also have the current equation

$$\begin{aligned}\partial_t \, log\rho_t &= \frac{1}{2}\,\nabla.\,(-\nabla S + \nabla\overline{S}) \\ &\quad +\frac{1}{2}\,\nabla log\rho_t\,.\,(-\nabla S + \nabla\overline{S}),\end{aligned}$$

which can be rewritten as

$$\partial_t \, log\rho_t = \frac{1}{2} \triangle \theta_t + \frac{1}{2}\,\nabla log\rho_t\,.\,\nabla\theta_t, \tag{7.1}$$

where $\theta_t(.) = -S(t,.) + \overline{S}(t,.)$.

The current equation is actually the imaginary part of some Schrödinger equation.

Indeed define the wave function

$$\psi_t = \rho_t^{\frac{1}{2}}\, e^{-\frac{1}{2}\, i\theta_t}. \tag{7.2}$$

An easy calculation shows that

$$i\,\partial_t\,\psi_t = -\,\frac{1}{2}\triangle\psi_t + V\,\psi_t, \tag{7.3}$$

where

$$2\,V(t,.) = \partial_t\theta_t - \frac{1}{4}\,|\nabla\theta_t|^2 + \frac{1}{4}\,|\nabla\, log\rho_t|^2 + \frac{1}{2}\triangle\, log\rho_t. \tag{7.4}$$

Of course we have one degree of freedom in the choice of the wave function. Indeed we may add to θ_t any function α which depends only on t. This will only modify V, adding $\partial_t\,\alpha$.

Let us consider the case of a general flow in the framework of this paper. One can approximate β_{min} by ∇S_n as in section 6.3, and then define $\overline{S}_n$. What happens when taking limits ? To get an account of what is happening, we shall assume that

(7.5) ρ is time-space continuous and strictly positive on $[0,T]\times\overline{B}$ where $\overline{B}$ is the closure of an open ball B.

Remark 7.6. In the previous section, we have remarked that ρ can be chosen $\mathbb{Q}$ almost surely continuous along the paths. It would thus be natural to introduce the "fine topology" induced by the process, under $\mathbb{Q}_{min}$. Thanks to the results of [11], $\mathbb{Q}$ and $\mathbb{Q}_{min}$ are equivalent, hence this topology would only depends on the flow ρ. Of course the word "fine" is abusive, since it is only well defined in the framework of Markov processes, while we are working here with a single $\mathbb{Q}$.

Since ρ and $\frac{1}{\rho}$ are bounded on $[0,T]\times B$, one may use the Poincaré inequality

$$\int_0^T\int_B |f - \int_B f(t,z)\,dz|^2\,\rho(t,x)\,dt\,dx \leq C\int_0^T\int_B |\nabla f|^2\,\rho(t,x)\,dt\,dx,$$

where C only depends on the radius of B and bounds for ρ and $\frac{1}{\rho}$. It immediately follows that

$$S_n - \int_B S_n(t,z)\,dz$$

is a Cauchy sequence in $\mathbb{L}^2([0,T] \times \mathbb{H}^1(B))$, hence converges to some S in $\mathbb{L}^2([0,T] \times \mathbb{H}^1(B))$ with

$$\nabla S = \beta_{min}. \tag{7.7}$$

One similarly obtains $\overline{S}$, thus θ and finally the wave function

$$\psi_t = \rho_t^{\frac{1}{2}}\, e^{-\frac{1}{2}\, i\theta_t},$$

which belongs to $\mathbb{L}^2([0,T] \times \mathbb{H}^1(B))$ thanks to a priori regularity.
Using the boundedness of $\frac{1}{\rho}$, one obtains that the potentials V_n are converging to some V in $\mathbb{H}^{-1}([0,T] \times B)$. This is not satisfactory. Actually we would like that

$$V \in \mathbb{L}^2([0,T] \times \mathbb{H}^{-1}(B))$$

in order to $V\psi$ be well defined as an operator.

Notice that the choice we have made, i.e.

$$\alpha_n(t) = \int_B S_n(t,z)\,dz,$$

is the one which minimizes the $\mathbb{H}^{-1}$ norm of V_n (up to the addition of such an α, see above). One should think it is thus the optimal choice.

Though the situation is not fully satisfactory, the derivation above indicates how one can build the potential V starting from the statistical observation of a particles system. Indeed, relative entropy is the rate function for the large deviations of the empirical mean of the positions of Brownian particles, and $\mathbb{Q}_{min}$ is thus the most probable paths-law when one observes the flow of marginals ρ (see [12] and [13]). Finally, when one starts with the Schrödinger equation for regular enough V, $\mathbb{Q}_{min}$ has drift $\mathcal{R}e\frac{\nabla\psi}{\psi} + \mathcal{I}m\frac{\nabla\psi}{\psi}$, and one can build V starting from the diffusion. We refer to [7] section 2, for a precise study of the Schrödinger equation and precise hypotheses which have to be made on V.

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