TRENDS TO EQUILIBRIUM IN TOTAL VARIATION DISTANCE.

PATRICK CATTIAUX ♠ AND **ARNAUD GUILLIN** ♢

♠ Ecole Polytechnique and Université de Toulouse

♢ Ecole Centrale de Marseille and Université de Provence

Abstract. This paper presents different approaches, based on functional inequalities, to study the speed of convergence in total variation distance of ergodic diffusion processes with initial law satisfying a given integrability condition. To this end, we give a general upper bound "à la Pinsker" enabling us to study our problem firstly via usual functional inequalities (Poincaré inequality, weak Poincaré,...) and truncation procedure, and secondly through the introduction of new functional inequalities $\mathcal{I}_\psi$. These $\mathcal{I}_\psi$-inequalities are characterized through measure-capacity conditions and F-Sobolev inequalities. A direct study of the decay of Hellinger distance is also proposed. Finally we show how a dynamic approach based on reversing the role of the semi-group and the invariant measure can lead to interesting bounds.

Résumé. Nous étudions ici la vitesse de convergence, pour la distance en variation totale, de diffusions ergodiques dont la loi initiale satisfait une intégrabilité donnée. Nous présentons différentes approches basées sur l'utilisation d'inégalités fonctionnelles. La première étape consiste à donner une borne générale à la Pinsker. Cette borne permet alors d'utiliser, en les combinant à une procedure de troncature, des inégalités usuelles (telles Poincaré ou Poincaré faibles,...). Dans un deuxième temps nous introduisons de nouvelles inégalités appelées $\mathcal{I}_\psi$ que nous caractérisons à l'aide de condition de type capacité-mesure et d'inégalités de type F-Sobolev. Une étude directe de la distance de Hellinger est également proposée. Pour conclure, une approche dynamique basée sur le renversement du rôle du semigroupe de diffusion et de la mesure invariante permet d'obtenir de nouvelles bornes intéressantes.

Key words : total variation, diffusion processes, speed of convergence, Poincaré inequality, logarithmic Sobolev inequality, F-Sobolev inequality.

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1. **Introduction, framework and first results.**

We shall consider a dynamics given by a time continuous Markov process $(X_t, \mathbb{P}_x)$ admitting an (unique) ergodic invariant measure μ. We denote by L the infinitesimal generator (and $D(L)$ the extended domain of the generator), by $P_t(x,.)$ the $\mathbb{P}_x$ law of X_t and by P_t (resp. P_t^*) the associated semi-group (resp. the adjoint or dual semi-group), so that in particular for any density of probability h w.r.t. μ, $\int P_t(x,.)h(x)\mu(dx) = P_t^* h \, d\mu$ is the law of X_t with initial distribution $hd\mu$. By abuse of notation we shall denote by $P_t^*\nu$ the law of X_t with initial distribution ν.

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Our goal is to describe the rate of convergence of $P_t^*\nu$ to μ in total variation distance. Indeed, the total variation distance is one of the natural distance between probability measures. If $d\nu = hd\mu$, this convergence reduces to the $\mathbb{L}^1(\mu)$ convergence.

Trends to equilibrium is one of the most studied problem in various areas of Mathematics and Physics. For the problem we are interested in, two families of methods have been developed during the last thirty years.

The first one is based on Markov chains recurrence conditions (like the Doeblin condition) and consists in finding some Lyapunov function. We refer to the works by Meyn and Tweedie [26, 27, 20] and the more recent [21, 30, 19]. In a very recent work with D. Bakry ([3]), we have studied the relationship between this approach and the second one.

The second family of methods is using functional inequalities. It is this approach that we shall follow here, pushing forward the method up to cover the largest possible framework. This approach relies mainly on the differentiation (with respect to time) of a quantity like variance or entropy along the semigroup and a functional inequality enables then to use Gronwall's inequality to get the decay of the differentiated quantity. However, due to the non differentiability of the total variation distance, this direct method is no more possible. Let us then first give general upper bound on total variation which will lead us to the relevant functional inequalities for our study.

1.1. **A general method for studying the total variation distance.** The starting point is the following elementary extension of the so called Pinsker inequality.

Lemma 1.1. *Let ψ be a C^2 convex function defined on $\mathbb{R}^+$. Assume that ψ is uniformly convex on $[0,A]$ for each $A>0$, that $\psi(1)=0$ and that $\lim_{u\to+\infty}(\psi(u)/u)=+\infty$. Then there exists some $c_\psi>0$ such that for all pair $(\mathbb{P},\mathbb{Q})$ of probability measures,*

$$\parallel \mathbb{P}-\mathbb{Q} \parallel_{TV} \leq c_\psi \sqrt{I_\psi(\mathbb{Q}|\mathbb{P})} \quad \text{where} \quad I_\psi(\mathbb{Q}|\mathbb{P}) = \int \psi\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) d\mathbb{P}$$

if $\mathbb{Q}$ is absolutely continuous w.r.t. $\mathbb{P}$, and $I_\psi(\mathbb{Q}|\mathbb{P})=+\infty$ otherwise.

Proof. For $0\leq u\leq A$ it holds

$$\psi(u)-\psi(1)-\psi'(1)(u-1) \geq \frac{1}{2}\left(\inf_{0\leq v\leq A}\psi''(v)\right)(u-1)^2 .$$

Thanks to convexity the left hand side in the previous inequality is everywhere positive. Since $\lim_{u\to+\infty}(\psi(u)/u)=+\infty$, it easily follows that there exists some constant c such that for all $0\leq u$,

$$(u-1)^2 \leq c\,(1+u)\,\big(\psi(u)-\psi(1)-\psi'(1)(u-1)\big) .$$

Take the square root of this inequality, apply it with $u=h(x)=(d\mathbb{Q}/d\mathbb{P})(x)$, integrate w.r.t. $\mathbb{P}$ and use Cauchy-Schwarz inequality. It yields

$$\left(\int |h-1|d\mathbb{P}\right)^2 \leq c\left(\int (1+h)d\mathbb{P}\right)\left(\int \big(\psi(h)-\psi(1)-\psi'(1)(h-1)\big)d\mathbb{P}\right) .$$

Since h is a density of probability the result follows with $c_\psi=\sqrt{2c}$. □

Remark 1.2. Note that we may replace the assumption $\psi(u)/u\to\infty$ by $\liminf_{u\to+\infty}(\psi(u)/u)-\psi'(1)=d>0$. For instance we may choose $\psi(u)=u-\frac{3}{2}+\frac{1}{u+1}$. ♢

The main idea now is to study the behavior of

$$t \mapsto I_\psi(t,h) = I_\psi(P_t^* h d\mu | d\mu) = \int \psi(P_t^* h)\, d\mu \tag{1.3}$$

as $t \to \infty$. Notice that with our assumptions $I_\psi(h) = I_\psi(0,h) \geq 0$ thanks to Jensen inequality. To this end we shall make the following additional assumptions. The main additional hypothesis we shall make is the existence of a "carré du champ", that is we assume that there is an algebra of uniformly continuous and bounded functions (containing constant functions) which is a core for the generator and such that for f and g in this algebra

$$L(fg) = fLg + gLf + \Gamma(f,g). \tag{1.4}$$

We also replace $\Gamma(f,f)$ by $\Gamma(f)$. Notice that with our choice there is a factor 2 which differs from many references, indeed if our generator is $\frac{1}{2}\Delta$, $\Gamma(f) = |\nabla f|^2$ which corresponds to $L = \Delta$ in many references. The correspondence is of course immediate changing our t into $2t$.
We shall also assume that Γ comes from a derivation, i.e. for f, h and g as before

$$\Gamma(fg,h) \;=\; f\Gamma(g,h) + g\Gamma(f,h)\,. \tag{1.5}$$

The meaning of these assumptions in terms of the underlying stochastic process is explained in the introduction of [10], to which the reader is referred for more details (also see [2] for the corresponding analytic considerations). Note that we may replace L by L^* without changing Γ.
Applying Ito's formula, we then get that for all smooth Ψ, and f as before,

$$\begin{aligned} L\Psi(f) \;&=\; \frac{\partial \Psi}{\partial x}(f)\, Lf \;+\; 1/2\, \frac{\partial^2 \Psi}{\partial x^2}(f)\, \Gamma(f)\,, \\ \Gamma(\Psi(f)) \;&=\; (\Psi'(f))^2\, \Gamma(f)\,. \end{aligned} \tag{1.6}$$

Under these hypotheses, we immediately obtain

$$\frac{d}{dt}\, I_\psi(t,h) \;=\; -\int 1/2\, \psi''(P_t^* h)\, \Gamma(P_t^* h)\, d\mu\,. \tag{1.7}$$

It follows

Proposition 1.8. *There is an equivalence between*

- *for all density of probability h such that $\int \psi(h) d\mu < +\infty$,*
$$I_\psi(t,h) \leq e^{-t/2C_\psi}\, I_\psi(h)\,,$$
- *for all nice density of probability h,*

$$\int \psi(h)\, d\mu \;\leq\; C_\psi \int \psi''(h)\, \Gamma(h)\, d\mu\,. \tag{1.9}$$

In this case the total variation distance

$$\parallel P_t^*(h\mu) - \mu \parallel_{TV} \leq M_\psi\, e^{-t/4C_\psi}\, I_\psi(h)\,,$$

goes to 0 with an exponential rate.

When there exists C_ψ such that for all nice h, (1.9) holds for μ, we will say that μ verifies an $\mathcal{I}_\psi$-inequality. Note that the proof of this last proposition is standard, the direct part is obtained by looking at $I_\psi(t,h) - I_\psi(h)$ when t goes to 0, while the converse part is a direct consequence of Gronwall lemma.

Slower decay can be obtained by weakening (1.9). Indeed replace (1.9) by

$$\int \psi(h)\,d\mu \,\leq\, \beta_\psi(s) \int \psi''(h)\,\Gamma(h)\,d\mu \,+ s\,G(h)\,, \tag{1.10}$$

supposed to be satisfied for all $s > 0$ for some non-increasing β_ψ, and some real valued functional G such that $G(P_t^* h) \leq G(h)$. An application of Gronwall's lemma implies that

$$I_\psi(t,h) \leq \xi(t)\ (I_\psi(h) + G(h))$$

with $\xi(t) = \inf\{s > 0\,, 2\beta_\psi(s)\,\log(1/s) \leq t\}$. Following Röckner-Wang [29], such an inequality may be called a weak $\mathcal{I}_\psi$-inequality. They consider the variance case, namely $\psi(u) = (u-1)^2$, when the entropy case, namely $\psi(u) = u\log u$ is treated in [12]. The only known converse statement is in the variance case.

In this work we shall push forward this approach in order to give some rate of convergence for all $h \in \mathbb{L}^1(\mu)$. The key is the following trick (see [12] section 5.2): if $h \in \mathbb{L}^1(\mu)$ and non-negative, for $K > 0$

$$\begin{aligned}\int |P_t^* h - 1| d\mu \ &\leq\ \int |P_t^*(h\wedge K) - P_t^* h| d\mu + \int |P_t^*(h\wedge K) - \int (h\wedge K) d\mu| d\mu + |\int (h\wedge K) d\mu - 1| \\ &\leq\ \int |P_t^*(h\wedge K) - \int (h\wedge K) d\mu| d\mu + 2\int (h-K) \mathbb{1}_{h\geq K} d\mu \qquad (1.11)\end{aligned}$$

where we have used the fact that P_t^* is a contraction in L^1. The second term in the right hand sum is going to 0 when K goes to $+\infty$, while the first term can be controlled by $\sqrt{I_\psi(t, h\wedge K)}$ according to Lemma 1.1. More precisely, according to De La Vallée-Poussin theorem,

$$\int h\,\varphi(h) d\mu < +\infty$$

for some nonnegative function φ growing to infinity. So

$$\int h \mathbb{1}_{h\geq K} d\mu \leq \frac{1}{\varphi(K)} \left(\int h\,\varphi(h) d\mu\right)\,,$$

and we get, provided (1.10) is satisfied

$$\int |P_t^* h - 1| d\mu \,\leq\, c_\psi \sqrt{\xi(t)\ (I_\psi(h\wedge K) + G(h\wedge K))} + 2\frac{1}{\varphi(K)} \left(\int h\,\varphi(h) d\mu\right)\,. \tag{1.12}$$

1.2. **About this paper.** Functional inequalities like (1.9) have a long story. When $\psi(u)$ behaves like u^2 (resp. $u\,\log(u)$) at infinity, they are equivalent to the Poincaré inequality (resp. the Gross logarithmic Sobolev inequality). We refer to [1] for an introduction to this topic. Many progresses in the understanding of such inequalities have been made recently. We refer to [9, 29, 5, 12] for their weak versions and to [31, 34, 6, 7, 28] for the so called F-Sobolev inequalities. All these inequalities will be recalled and discussed later. Links with long time behavior have been partly discussed in [13, 12, 3]. Note that in the recent [28], the authors study the decay of $P_t f$ for f belonging to smaller spaces than $\mathbb{L}^2$.

Our aim here is to give the most complete description of the decay to 0 in total variation distance using these inequalities, i.e. we want to give a general answer to the following question : if a density of probability h satisfies $\int \psi(h)d\mu < +\infty$ for some ψ convex at infinity, what can be expected for the decay to equilibrium in terms of a functional inequality satisfied by μ ?

To better see what we mean, let us describe the contents of the paper.

In Section 2 we recall old and recent results connected with Poincaré's like inequalities and logarithmic Sobolev like inequalities. Recall that log-Sobolev is always stronger than Poincaré. For short Poincaré (resp. log-Sobolev) inequality ensures an exponential decay for densities such that $\int h^2 d\mu < +\infty$ (resp. $\int h \log h d\mu < +\infty$). Actually we shall see in the examples of Section 2 that these integrability conditions can be replaced by $\int h^p d\mu < +\infty$ for some $p > 1$ (resp. $\int h \log_+^\beta h d\mu < +\infty$ for some $\beta > 0$) with still an exponential decay. For less integrable densities, weak forms of Poincaré and log-Sobolev inequalities furnish an explicit (but less than exponential) decay.

The questions are then :

- if $p > 2$ and $\int |h|^p d\mu$ if finite, can we obtain some exponential decay with a weaker functional inequality;
- if $u \log(u) \ll \psi(u) \ll u^2$, is it possible to characterize $\mathcal{I}_\psi$-inequality, thus ensuring and exponential decay of $I(t, \psi)$;
- if $\psi(u) \ll u \log(u)$ what can be said ?

The first question has a negative answer, at least in the reversible case, according to an argument in [29] (see Remark 3.22).

The answer to the second question is the aim of Section 3. It is shown that for each such ψ one can find a (minimal) F such that exponential decay is ensured by the corresponding F-Sobolev inequality (see (3.6) for the definition), and conversely (see Theorem 3.2, Theorem 3.13 and Remark 3.16). These inequalities have been studied in [31, 34, 6, 7, 28]. A key tool here is the use of capacity-measure inequalities introduced in [8] and developed in [6, 5, 7, 12]. Hence for exponential decay we know how to interpolate between Poincaré and log-Sobolev inequalities.

The third question is briefly discussed in Section 4. This section contains essentially negative results which will not be detailed. The reader is referred to the more complete version of this paper on Math ArXiv for details and proofs (see [15]). A particular case is the ultracontractive situation, i.e. when $P_t h \in \mathbb{L}^2(\mu)$ for all $h \in \mathbb{L}^1(\mu)$ and all $t > 0$. Indeed if a weak Poincaré inequality is satisfied, the true Poincaré inequality is also satisfied in this case, yielding a uniform exponential decay in total variation distance. In Section 4 we shortly explain why a direct study of the total variation distance, or of the almost equivalent Hellinger distance, furnishes bad results i.e. uniform (not necessarily exponential) decays are obtained under conditions implying ultracontractivity.

The next Section 5 contains a discussion inspired by the final section of [17], namely, what happens if instead of looking at the density $P_t^* h$ with respect to μ, one looks at the density $1/P_t^* h$ with respect to $P_t^* h \, d\mu$, that is we look at $d\mu/d\nu_t$ where ν_t is the law at time t. A direct study leads to new functional inequalities (one of them however is a weak version of the Moser-Trudinger inequality) which imply a strong form of ultracontractivity (namely the capacity of all non-empty sets is bounded from below by a positive constant). However, we also show that one can replace the integrability condition on h by a geometric condition ($h\mu$ satisfies some weak Poincaré inequality) provided the Bakry-Emery condition is satisfied (see Theorem 5.17). This yields apparently better results than the one obtained in Section 2 under the log-Sobolev inequality (which is satisfied with the Bakry-Emery condition).

Finally, let us mention that in [35], Wang gives some conditions for the convergence of P_t^* to the identity in operator norm ($\mathbb{L}^1$ in $\mathbb{L}^1$). Here we do not study this kind of uniform convergence since we consider initial densities in various integration spaces.

Note that in the whole paper, h will denote a density of probability, and φ is an increasing real valued function such that $u \mapsto u\varphi(u)$ is convex (as in the discussion yielding (1.12)).

2. Examples using classical functional inequalities.

In this section we shall show how to apply the general method in some classical cases.

2.1. **Using Poincaré inequalities.** If we choose $\psi(u) = (u-1)^2$, (1.9) reduces to the renowned Poincaré inequality. In this case Lemma 1.1 reduces to Cauchy-Schwarz inequality. Recall the well known

Theorem 2.1. *The following two statements are equivalent for some positive constant C_P*

Exponential decay in $\mathbb{L}^2$. *For all $f \in \mathbb{L}^2(\mu)$,*

$$\| P_t f - \int f d\mu \|_2^2 \leq e^{-t/C_P} \; \| f - \int f d\mu \|_2^2 \; .$$

Poincaré inequality. *For all $f \in D_2(L)$ (the domain of the Friedrichs extension of L) ,*

$$Var_\mu(f) \; := \| f - \int f d\mu \|_2^2 \leq C_P \int \Gamma(f)\, d\mu \, .$$

Hence if a Poincaré inequality holds, for $\nu = h\mu$ with $h \in \mathbb{L}^2(\mu)$,

$$\| P_t^* \nu - \mu \|_{TV} = \| P_t^* h - 1 \|_{\mathbb{L}^1(\mu)} \leq e^{-t/2C_P} \; \| h - 1 \|_{\mathbb{L}^2(\mu)} \; .$$

If h is less integrable we obtain

Corollary 2.2. *Let $\alpha(u) = \sqrt{u}\,\varphi(u)$ and α^{-1} its inverse (recall that φ is increasing, hence α too) . If a Poincaré inequality holds, then*

$$\| P_t^* h - 1 \|_{\mathbb{L}^1(\mu)} \leq \left(\alpha^{-1}\left(2\left(\int h\,\varphi(h)d\mu\right) e^{t/2C_P}\right)\right)^{\frac{1}{2}} e^{-t/2C_P} \, . \tag{2.3}$$

If in addition $u \mapsto \varphi(u)/u$ is non increasing, then defining $\gamma(u) = u\,\varphi(u)$, the following better inequality holds

$$\| P_t^* h - 1 \|_{\mathbb{L}^1(\mu)} \leq \gamma^{-1}\left(4\left(\int h\varphi(h)d\mu\right) e^{t/C_P}\right) \, e^{-t/C_P} . \tag{2.4}$$

Proof. First remark that

$$\mathrm{Var}_\mu(h \wedge K) \leq \int (h \wedge K)^2\, d\mu \leq K \int (h \wedge K) d\mu \leq K \, .$$

We may now use Cauchy-Schwarz inequality to control $\int |P_t^*(h \wedge K) - \int (h \wedge K) d\mu| d\mu$ by the square root of the Variance in (1.11). The result then follows by an easy optimization in K. More precisely we may choose K in such a way that both terms in the right hand side of (1.11) are equal, yielding (2.3).
(2.4) is obtained following a suggestion of the referee. Indeed if $h \geq K$, then

$$(h \wedge K)^2 \leq hK \leq hK \, \frac{\varphi(h)}{\varphi(K)} \, .$$

If $h \leq K$ we have

$$(h \wedge K)^2 \leq h^2 \leq h^2 \, \frac{\varphi(h)}{\varphi(K)} \, \frac{K}{h} \leq hK \, \frac{\varphi(h)}{\varphi(K)}$$

since $u \mapsto \varphi(u)/u$ is non increasing. Hence under the former assumption

$$\mathrm{Var}_\mu(h \wedge K) \leq \int (h \wedge K)^2 \, d\mu \leq \frac{K}{\varphi(K)} \int h \, \varphi(h) \, d\mu \, .$$

Thus, according to (1.11), defining $I = \int h\varphi(h) d\mu$,

$$\begin{aligned} \int |P_t^* h - 1| d\mu \quad &\leq \quad \sqrt{\mathrm{Var}_\mu(P_t^*(h \wedge K))} + 2/\varphi(K) \int h\varphi(h) d\mu \\ &\leq \quad e^{-t/2C_P} \sqrt{\frac{K\, I}{\varphi(K)}} + 2I/\varphi(K) \, . \end{aligned}$$

Again we choose K in such a way that both terms are equal, i.e. $K = \gamma^{-1}(4I \, e^{t/C_P})$, yielding the result. $\square$

Example 2.5. Assume that $h \in \mathbb{L}^q(\mu)$ for some $1 < q < 2$. If a Poincaré inequality holds, (2.3) yields after some elementary calculation,

$$\| \, P_t^* h - 1 \, \|_{\mathbb{L}^1(\mu)} \leq 4^{\frac{q}{2q-1}} \left(\int h^q d\mu \right)^{\frac{1}{2q-1}} \; e^{-\frac{(q-1)t}{(2q-1)\, C_P}} \, . \tag{2.6}$$

Note that for $q = 2$ we do not recover the good rate $e^{-t/2C_P}$ but $e^{-t/3C_P}$. But if instead we use (2.4) (as we shall since $\varphi(u) = u^{q-1}$), we get

$$\| \, P_t^* h - 1 \, \|_{\mathbb{L}^1(\mu)} \leq \left(4 \int h^q d\mu \right)^{\frac{1}{q}} \; e^{-\frac{(q-1)t}{q\, C_P}} \, , \tag{2.7}$$

and this time recover the good exponent for $q = 2$. Note that, up to the constants, a similar result already appeared in [33]. Indeed if a Poincaré inequality holds then for $1 \leq p < 2$,

$$\int f^2 d\mu - \left(\int f^p d\mu \right)^{2/p} \; \leq \; C_P \int \Gamma(f) d\mu$$

which is a Beckner type inequality, called (I_p) in [33]. According to Proposition 4.1 in [33] (recall the extra factor 2 therein),

$$\int (P_t^* h)^{2/p} d\mu - 1 \; \leq \; e^{-\frac{(2-p)t}{C_P}} \left(\int h^{2/p} d\mu - 1 \right) \, ,$$

so that, taking $p = 2/q$ and applying Lemma 1.1 with $\psi(u) = u^q - 1$ in the left hand side of the previous inequality we recover the same exponential rate of decay as in (2.7). $\diamondsuit$

Remark that in the derivation of the Corollary we only used Poincaré's inequality for bounded functions. Hence we may replace it by its weak form introduced in [9, 29], that is, we take for G the square of the Oscillation of h in (1.10). It yields

Theorem 2.8. *(*[29] ***Theorem 2.1)*** *Assume that there exists some non-increasing function β_{WP} defined on $(0,+\infty)$ such that for all $s>0$ and all bounded $f\in D_2(L)$ the following inequality holds*

$$\textbf{\textit{Weak Poincaré inequality.}} \qquad Var_\mu(f) \,\leq\, \beta_{WP}(s) \int \Gamma(f)\, d\mu \,+\, s\, Osc^2(f)\,.$$

Then we get the following ***Decay***

$$Var_\mu(P_t^* f) \leq 2\xi_{WP}(t)\, Osc^2(f) \qquad \text{where} \qquad \xi_{WP}(t) = \inf\{s>0\,,\, \beta_{WP}(s)\,\log(1/s)\,\leq\, t\}\,.$$

Hence

Corollary 2.9. *If a weak Poincaré inequality holds,*

$$\parallel P_t^* h - 1 \parallel_{\mathbb{L}^1(\mu)} \leq 2\sqrt{2}\ \theta^{-1}\left(\sqrt{2}\left(\int h\,\varphi(h) d\mu\right)/\sqrt{\xi_{WP}(t)}\right)\, \sqrt{\xi_{WP}(t)}\,,$$

where $\theta(u) = u\varphi(u)$.

The proof of the corollary is similar to the proof of (2.3).

Remark 2.10. Since we are interested in functions such that $\int h\,\varphi(h)\,d\mu < +\infty$, instead of using the truncation argument we may directly try to obtain a weak inequality with $G(h) = \parallel h - m_h \parallel_\zeta$ where $\parallel . \parallel_\zeta$ denotes the Orlicz norm associated to $\zeta(u) = u\varphi(u)$, and m_h is a median of h. Actually as shown in [36] Theorem 29, provided $\varphi(h) \geq h$, such an inequality is equivalent to the weak Poincaré inequality replacing $\beta_{WP}(s)$ by

$$\beta_\zeta(s) = 6\,\beta_{WP}\left(\frac{1}{4}\,\bar\zeta(s/2)\right) \ \text{ where } \bar\zeta(u) = \frac{1}{\gamma^*(1/u)} \text{ with } \gamma(u) = \zeta(\sqrt{u})\,, \tag{2.11}$$

and γ^* is the Legendre conjugate of γ (assumed to be a Young function here). Since for a density of probability $m_h \leq 2$ and since there exists a constant c such that

$$\parallel g \parallel_\zeta \leq c\left(1 + \int g\,\varphi(g)\,d\mu\right)\,,$$

at least if ζ is moderate, we immediately get a decay result

$$\parallel P_t^* h - 1 \parallel_{\mathbb{L}^1(\mu)} \leq C\sqrt{\xi_\zeta(t)}\int h\,\varphi(h)\,d\mu\,, \tag{2.12}$$

with

$$\xi_\zeta(t) \,=\, \inf\{s\,;\, \beta_\zeta(s)\,\log(1/s)\,\leq\, t\}\,.$$

If $\varphi(u) = u^{p-1}$ for some $p>1$, Corollary 2.9 yields a rate of decay $(\xi_{WP}(t))^{\frac{p-1}{2p}}$.
Similarly, but if $p>2$, up to the constants, $\gamma(u) = u^{p/2}$, $\gamma^*(u) = u^{p/(p-2)}$ hence $\bar\zeta(u) = u^{p/(p-2)}$ so that we get $\xi_\zeta(t) = (\xi_{WP}(pt/(p-2)))^{(p-2)/p}$ hence a worse rate of decay. ♢

Of course our approach based on truncation extends to many other situations, in particular if we assume that $\int h\log h d\mu < +\infty$, a Poincaré inequality yields a polynomial behavior

$$\| P_t^* h - 1 \|_{\mathbb{L}^1(\mu)} \leq C\left(\int h\log h d\mu\right)\,(C_P/t)\,.$$

It was shown in [14] that a Poincaré inequality is equivalent to a restricted logarithmic Sobolev inequality (restricted to bounded functions). The truncation approach together with this restricted inequality do not furnish a better result. However with some extra conditions, which are natural for diffusion processes on $\mathbb{R}^n$, one can prove sub-exponential decay. We refer to [12] sections 4 and 5 for a detailed discussion.

2.2. **Using a logarithmic Sobolev inequality.** In the previous subsection we have seen (Example 2.5) that a Poincaré inequality implies an exponential decay for the total variation distance, as soon as $\nu = h\mu$ for $h \in \mathbb{L}^q(\mu)$ for some $q > 1$. In this section we shall see that a similar result holds if $\int h\log_+^\beta h d\mu < +\infty$ for some $\beta > 0$, as soon as a logarithmic Sobolev inequality holds. First of all we recall the following (corresponding to $\psi(u) = u\log u$ in the introduction)

Theorem 2.13. *The following two statements are equivalent for some positive constant C_{LS}*

Exponential decay for the entropy. *For all density of probability h*

$$\int P_t^* h\log(P_t^* h) d\mu \;\leq\; e^{-2t/C_{LS}} \int h\log h d\mu\,.$$

Logarithmic Sobolev inequality. *For all $f \in D_2(L)$,*

$$Ent_\mu(f^2) \;:=\; \int f^2\,\log\left(\frac{f^2}{\int f^2 d\mu}\right) d\mu \;\leq\; C_{LS}\int \Gamma(f)\,d\mu\,.$$

Hence if a logarithmic Sobolev inequality holds, for $\nu = h\mu$ with $Ent_\mu(h) < +\infty$,

$$\| P_t^*\nu - \mu \|_{TV} = \| P_t^* h - 1 \|_{\mathbb{L}^1(\mu)} \leq e^{-t/C_{LS}}\,\sqrt{2Ent_\mu(h)}\,.$$

Corollary 2.14. *Define $\bar\varphi(u) = \varphi(u)\sqrt{\log u}$ for $u \geq 1$. Then if a logarithmic Sobolev inequality holds,*

$$\| P_t^* h - 1 \|_{\mathbb{L}^1(\mu)} \leq 4\left(\log\circ\bar\varphi^{-1}\left(\left(\int h\,\varphi(h) d\mu\right)\,e^{t/C_{LS}}\right)\right)^{\frac{1}{2}}\,e^{-t/C_{LS}}\,. \tag{2.15}$$

If in addition $u \mapsto \varphi(u)/\log(u)$ is non increasing, we get the better

$$\| P_t^* h - 1 \|_{\mathbb{L}^1(\mu)} \leq 4\left(\log\circ\theta^{-1}\left(\left(\int h\,\varphi(h) d\mu\right)\,e^{2t/C_{LS}}\right)\right)\,e^{-2t/C_{LS}}\,, \tag{2.16}$$

where $\theta(u) = \varphi(u)\,\log(u)$.

Proof. The proof of (2.15) is similar to the one of Corollary (2.3), replacing the Variance by the Entropy, Cauchy-Schwarz inequality by Pinsker inequality an using the elementary

$$\mathrm{Ent}_\mu(h\wedge K) \leq \int (h\wedge K)\log(h\wedge K) d\mu + \frac{1}{e} \leq \log K + \frac{1}{e}\,.$$

We may then assume that $K > e^{1/e}$ and make an optimization in K.

For (2.16) we can show as for the proof of (2.4) that

$$(h \wedge K) \log(h \wedge K) \leq h\, \varphi(h)\, \frac{\log(K)}{\varphi(K)}$$

provided $u \mapsto \varphi(u)/\log(u)$ is non increasing, yielding the result.

□

Example 2.17. Assume that $\int h \log_+^\beta h\, d\mu < +\infty$ for some $0 < \beta \leq 1$. (2.15) yields, provided a logarithmic Sobolev inequality holds,

$$\| P_t^* h - 1 \|_{\mathbb{L}^1(\mu)} \leq 4 \left(\int h \log_+^\beta h\, d\mu \right)^{\frac{1}{2\beta+1}} e^{-2\beta t/(2\beta+1)C_{LS}} . \tag{2.18}$$

(2.16) yields (since $u \mapsto \varphi(u)/\log(u)$ is non increasing in our example)

$$\| P_t^* h - 1 \|_{\mathbb{L}^1(\mu)} \leq 4 \left(\int h \log_+^\beta h\, d\mu \right)^{\frac{1}{\beta+1}} e^{-2\beta t/(\beta+1)C_{LS}} . \tag{2.19}$$

Hence here again we get an exponential decay provided some "β-entropy" is finite, and (2.19) is optimal (up to the constants) for $\beta = 1$.

Actually, as in Example 2.5, a similar result can be obtained, provided a log-Sobolev inequality holds, using the more adapted

$$I_\psi(t,h) \;\leq\; e^{-C\beta t}\, I_\psi(h)$$

with $\psi(u) = u\,(\log^\beta(2+u) - \log^\beta(3)) = uF(u)$. It will be the purpose of the next section. In fact, in this example, we will even show that the assumption of a logarithmic Sobolev inequality to hold is not necessary, a well adapted F-Sobolev inequality will be sufficient. ◊

Remark 2.20. It is well known that a logarithmic Sobolev inequality implies a Poincaré inequality. Hence we may ask whether some stronger inequality than the log-Sobolev inequality, furnishes some exponential decay under weaker integrability conditions. But here we have to face a new problem : indeed classical stronger inequalities usually imply that P_t is ultracontractive (i.e. maps continuously $\mathbb{L}^1(\mu)$ into $\mathbb{L}^\infty(\mu)$). Hence in this case we get an exponential decay for the $\mathbb{L}^1(\mu)$ norm, combining ultracontractivity and Poincaré inequality for instance. We shall give some new insights on this in one of the next sections.

Examples of ultracontractive semi-groups can be found in [16, 22]. ◊

Remark 2.21. Since a logarithmic Sobolev inequality is stronger than a Poincaré inequality, it is interesting to interpolate between both inequalities. Several possible interpolations have been proposed in the literature, starting with [23]. In [6] a systematic study of this kind of F-Sobolev inequalities is done. Note that a homogeneous F-Sobolev inequality is written as

$$\int f^2\, F\left(\frac{f^2}{\int f^2 d\mu}\right) d\mu \;\leq\; \int \Gamma(f)\, d\mu$$

hence does not correspond to (1.9). That is why such inequalities are well suited for studying the convergence of $P_t f$ (see [28]), while we are interested here in the convergence of $P_t(f^2)$. Moreover their convergence are stated in Orlicz norm (clearly adapted to F-Sobolev), whereas ours are in more usual integral form.

The case of $F = \log$ corresponding to the log-Sobolev (or Gross) inequality appears as a very peculiar one since it is the only one for which the F-Sobolev inequality corresponds exactly to (1.9). It is thus natural to expect that the weak logarithmic Sobolev inequalities are well suited to furnish a good interpolation scale between Poincaré and Gross inequalities. This point of view is developed in [12]. We shall recall and extend some of these results below. $\diamondsuit$

Here again we may replace the logarithmic Sobolev inequality by a weak logarithmic Sobolev inequality

Theorem 2.22. *(*[12] ***Proposition 4.1)*** *Assume that there exists some non-increasing function β_{WLS} defined on $(0,+\infty)$ such that for all $s>0$ and all bounded $f \in D_2(L)$ the following inequality holds*

$$\textbf{\textit{Weak log-Sobolev inequality.}} \qquad Ent_\mu(f^2) \, \leq \, \beta_{WLS}(s) \int \Gamma(f)\, d\mu \, + \, s\, Osc^2(f)\,.$$

Then for all $\varepsilon > 0$, $Ent_\mu(P_t^* h) \leq (\frac{1}{e} + \varepsilon)\, \xi_{WLS}(\varepsilon, t)\, Osc^2(\sqrt{h})$ where $\xi_{WLS}(\varepsilon,t) = \inf\{s > 0\,,\, \beta_{WLS}(s)\, \log(\varepsilon/s) \, \leq \, 2t\}\,.$

Hence if a weak log-Sobolev inequality holds,

$$\| \, P_t^* h - 1 \, \|_{\mathbb{L}^1(\mu)} \leq \frac{4 \int h\, \varphi(h) d\mu}{(\varphi \circ \tilde{\varphi}^{-1})(\sqrt{2}\, \left(\int h\, \varphi(h) d\mu\right) / (\frac{1}{e} + \varepsilon)\, \sqrt{\xi_{WLS}(\varepsilon,t)})}\,,$$

where $\tilde{\varphi}(u) = \sqrt{u}\varphi(u)$. The proof is analogue to the variance case.
But it is shown in [12] that:

- if the Poincaré inequality does not hold, but a weak Poincaré inequality holds, a weak log-Sobolev inequality also holds (see [12] Proposition 3.1 for the exact relationship between β_{WP} and β_{WLS}) but yields a worse result for the decay in total variation distance, i.e. in this situation Theorem 2.22 is not as good as Theorem 2.8,
- if a Poincaré inequality holds, one can reinforce the weak log-Sobolev inequality into a restricted log-Sobolev inequality.

We shall thus describe this reinforcement.

Theorem 2.23. *Assume that μ satisfies a Poincaré inequality with constant C_P and a weak logarithmic Sobolev inequality with function β_{WLS}, and define $\gamma_{WLS}(u) = \beta_{WLS}(u)/u$. Then for all $t > 0$ and all bounded density of probability h, it holds*

$$Ent_\mu(P_t^* h) \, \leq \, e^{-\, t/2\gamma^{-1}_{WLS}(\sqrt{3C_P}\, \|h\|_\infty)}\, Ent_\mu(h)\,.$$

Hence if $\int h\varphi(h) d\mu < +\infty$,

- *if $\varphi(u) \geq c\, \log(u)$ at infinity for some $c > 0$, there exists a constant $c(\varphi)$ such that*
$$\| \, P_t^* h - 1 \, \|_{\mathbb{L}^1(\mu)} \leq \frac{c(\varphi)\, \int h\, \varphi(h) d\mu}{\varphi \circ \zeta^{-1}_{WLS}(t)}\,,$$
where $\zeta_{WLS}(u) = 2\, \log(\varphi(u))\, \gamma^{-1}_{WLS}(\sqrt{3C_P} u)$,
- *if $\varphi(u) \leq c \log(u)$ at infinity for all $c > 0$, there exists a constant $c(\varphi)$ such that*
$$\| \, P_t^* h - 1 \, \|_{\mathbb{L}^1(\mu)} \leq \frac{c(\varphi)\, \left(1 + \int h\, \varphi(h) d\mu\right)}{\varphi \circ \theta^{-1}_{WLS}(t)}\,,$$
where $\theta_{WLS}(u) = 2\, \log(\varphi(u) \log(u))\, \gamma^{-1}_{WLS}(\sqrt{3C_P} u)$.

Proof. The first result is mainly [12] Proposition 4.2. We just here give the explicit expression of γ_{WLS}. Using (1.11) and this result give the result if we add two remarks : in the first case we may find C_φ such that $\mathrm{Ent}_\mu(h \wedge K) \leq C_\varphi \left(\int h\varphi(h)d\mu\right)$ for all $K > e$, so that the result follows with $c_\varphi = 2 + C_\varphi$; in the second case we use $\mathrm{Ent}_\mu(h \wedge K) \leq \log(K)$. □

2.3. **The example of diffusion processes.** In the previous subsections, we introduce a bench of inequalities, Poincaré inequality or its weak version and logarithmic Sobolev inequality and also its weak version, for which necessary and sufficient conditions exist in dimension 1, and for which sufficient conditions are known in the multidimensional case. Results in dimension 1 relies mainly on explicit translation of capacity measure criterion established in [8, 5, 6, 12], and we refer to their works for further discussion. However, capacity measure conditions are (up to the knowledge of the authors) of no use in the multidimensional setting. Let us consider the following (simplified) case: assume that $d\mu = e^{-2V}dx$ for some regular V. A sufficient well known condition for a Poincaré inequality to hold (see [1] for example) is that there exists c such that

$$|\nabla V|^2 - \Delta V \geq c > 0$$

for large x's. The associated generator is $L = \frac{1}{2}\Delta - \nabla V.\nabla$. For general reversible diffusion the following (nearly sufficient for exponential decay) drift condition (see [3] Th.2.1 or for explicit expressions of constant Th. 3.6): $\exists u \geq 1,\ \alpha, b > 0$ and a set C such that

$$Lu(x) \leq -\alpha u(x) + b 1_C(x)$$

which are easy to deal with conditions which moreover extend to the weak Poincaré setting [3, Th.3.10 and Cor. 3.12]: $\exists u \geq 1,\ \alpha, b > 0$, a positive function ψ and a set C such that

$$Lu(x) \leq -\psi(u(x)) + b 1_C(x).$$

As a more precise example, consider the diffusion process

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

where the diffusion matrix σ has bounded smooth entries and is uniformly elliptic and assume

$$\exists 0 < p < 1, M, r > 0 \text{ such that } \forall |x| > M,\ x.b(x) \leq -r|x|^{1-p}.$$

Then the invariant measure satisfies a weak Poincaré inequality with $\beta_W(s) = d_p \log(2/s)^{2p/(1+p)}$ and Th. 2.8 implies for $0 < q < 1$

$$\| P_t^* h - 1 \|_{\mathbb{L}^1(\mu)} \leq C_{p,q} \left(\int h^{1+q} d\mu\right)^{1/(1+q)} e^{-D_{p,q} t^{\frac{1-p}{1+p}}}.$$

There are also well known conditions for logarithmic Sobolev inequalities. Among them the most popular is the Bakry-Emery condition: assume that $V(x) = v(x) + w(x)$ where w is bounded and v satisfies $Hess(v) \geq \rho Id$ for some positive ρ then a logarithmic Sobolev inequality holds with constant $e^{osc(w)}/\rho$. One may also cite Wang [32] and Cattiaux [11] for conditions in the lower bounded (possibly negative) curvature case plus integrability assumptions or on drift like conditions. Both are however non quantitative and are thus not interesting for our study. Concerning weak logarithmic Sobolev inequalities, in the regime between Poincaré and logarithmic Sobolev inequalities, the only multidimensional conditions known can be obtained through a F-Sobolev inequality we shall describe further in the next section.

3. Some general results on $\mathcal{I}_\psi$-inequalities.

In this section we shall give some general results on $\mathcal{I}_\psi$-inequalities, i.e there exists $C_\psi > 0$ such that for all nice functions h

$$\int \psi(h) d\mu \leq C_\psi \int \psi''(h)\Gamma(h) d\mu.$$

First, we use the usual way to derive Poincaré inequality from a logarithmic Sobolev inequality i.e. we write $h = 1 + \varepsilon g$ for some bounded g such that $\int g d\mu = 0$. For ε going to 0 (note that h is non-negative for ε small enough), we see that if ψ satisfies $\psi''(1) > 0$, an $\mathcal{I}_\psi$-inequality implies a Poincaré inequality

$$\mathrm{Var}_\mu(g) \;\leq\; 2\, C_\psi \int \Gamma(g)\, d\mu\,,$$

i.e. with a Poincaré constant $C_P = 2C_\psi$.

Next, for our purpose, what is important is to control some moment of h. Hence what really matters is the asymptotic behavior of ψ. In particular if η is a function which is convex at infinity (i.e. $\eta''(u) > 0$ for $u \geq b$) and such that $\eta(u)/u$ goes to infinity at infinity, we may build some ad-hoc ψ as follows.

For $a > 2 \wedge b$, we define

$$\begin{aligned}
\psi''(u) &= \frac{\eta''(u)}{\eta''(a)} \quad \text{if } u \geq a \quad , \quad \psi''(u) = 1 \quad \text{otherwise}, \\
\psi'(u) &= \int_{\frac{1}{2}}^{u} \psi''(v)\, dv \quad \text{and} \quad \psi(u) = \int_1^u \psi'(v)\, dv\,.
\end{aligned} \tag{3.1}$$

It is easily shown that $\psi(u) = \frac{1}{2}\,(u^2 - u)$ for $u \leq a$, while one can find some constants β and γ such that $\psi(u) = (\eta(u)/\eta''(a)) + \beta\, u + \gamma$ for $u \geq a$, so that there is a constant c such that $\psi(u) \leq c\,\eta(u)$ for $u \geq a$ (recall that $\eta(u)/u$ goes to infinity at infinity) and $\psi(u) \geq \frac{1}{2\,\eta''(a)}\,\eta(u)$ for u large enough.

The choice of $\frac{1}{2}$ in the definition of the derivative, ensures that ψ is non-positive for $u \leq 1$.

The function ψ fulfills the assumptions in Lemma 1.1. Of particular interest will be the associated inequality (1.9) which, as we already remarked implies a Poincaré inequality with $C_P = 2C_\psi$. We may look at sufficient conditions for an $\mathcal{I}_\psi$-inequality to be satisfied.

3.1. A capacity measure condition for an $\mathcal{I}_\psi$-inequality.

Let us first reduce the study of an $\mathcal{I}_\psi$-inequality to the large value case via the use of Poincaré inequality. Indeed, as previously pointed out, it is a natural assumption to suppose that μ satisfies some Poincaré inequality with constant C_P. To prove that μ satisfies (1.9) it is enough to find a constant C such that

$$\begin{aligned}
\int_{h\leq a} (h^2 - h)\, d\mu &\leq C \int_{h\leq a} \Gamma(h)\, d\mu \quad \text{and} \\
\int_{h>a} \eta(h)\, d\mu &\leq C \left(\int_{h\leq a} \Gamma(h)\, d\mu + \int_{h>a} \eta''(h)\, \Gamma(h)\, d\mu \right)\,.
\end{aligned}$$

Indeed, up to the constants, the sum of the left hand sides is greater than $\int \psi(h)\, d\mu$, while the sum of the right hand sides is smaller than $\int \psi''(h)\, \Gamma(h)\, d\mu$.

For the first inequality, let h be a nice density of probability (h belongs to the domain $\mathbb{D}(\Gamma)$ of the Dirichlet form $\mathcal{E}(h) = \int \Gamma(h) d\mu$). Remember that $\int (h \wedge a) d\mu \leq 1$. Hence

$$\begin{aligned} \int_{h \leq a} (h^2 - h)\, d\mu \ &\leq \ \int ((h \wedge a)^2 - (h \wedge a))\, d\mu \\ &\leq \ \int (h \wedge a)^2\, d\mu \ - \ \left(\int \, h \wedge a\, d\mu\right)^2 \\ &\leq \ C_P \int \Gamma(h \wedge a)\, d\mu \ = \ C_P \int_{h \leq a} \Gamma(h)\, d\mu \end{aligned}$$

applying the Poincaré inequality with $h \wedge a$ which belongs to $\mathbb{D}(\Gamma)$. For the latter equality we use the second part of (1.6) for a sequence Ψ_n approximating $u \mapsto u \wedge a$ and use Lebesgue bounded convergence theorem.

To manage the remaining term, we introduce some capacity-measure condition, whose origin can be traced back to Mazja [25]. Following [8, 6], for $A \subset \Omega$ with $\mu(\Omega) \leq 1/2$, we define

$$Cap_\mu(A, \Omega) \ := \ \inf \{ \int \Gamma(f)\, d\mu \ ; \ \mathbb{1}_A \ \leq f \ \leq \ \mathbb{1}_\Omega \} \, ,$$

where the infimum is taken over all functions in the domain of the Dirichlet form. By convention this infimum is $+\infty$ if the set of corresponding functions is empty.
If $\mu(A) < 1/2$ we define

$$Cap_\mu(A) \ := \ \inf \{ Cap_\mu(A, \Omega) \, ; \, A \subset \Omega \, , \, \mu(\Omega) \leq 1/2 \} \, .$$

A capacity measure condition is usually stated as the existence of some function γ such that $\gamma(\mu(A)) \leq C Cap_\mu(A)$. Such an inequality, and depending on the form of γ, is (qualitatively) equivalent to nearly all usual functional inequalities: (weak) Poincaré inequality, (weak) logarithmic Sobolev inequality, F-Sobolev inequality or generalized Beckner inequality. It is then a precious tool to compare those inequalities, translating then properties of one to the other or using known conditions for one to the other. It has moreover the good taste to be explicit in dimension 1. It is then natural to look at some capacity-measure condition for an $\mathcal{I}_\psi$-inequality. Our first result (similar to Theorem 20 in [6]) is the following

Theorem 3.2. *Assume that μ satisfies a Poincaré inequality with constant C_P. Suppose*

(H_η) *: let η be a C^2 non-negative function defined on $\mathbb{R}^+$ such that*
- $\lim_{u \to +\infty} \eta(u)/u = +\infty$,
- *there exists $b > 0$ such that $\eta''(u) > 0$ for $u > b$,*
- *η is non-decreasing on $[b, +\infty)$ and η'' is non-increasing on $[b, +\infty)$.*

(H_F) *: there exist $\rho > 1$ and a non-decreasing function F such that*
- *for all A with $0 < \mu(A) < 1/2$, $\mu(A)\, F(1/\mu(A)) \ \leq \ Cap_\mu(A)$,*
- *there exists a constant C_{cap} such that for all $u > a$,*

$$\frac{\eta(\rho\, u)}{u^2\, \eta''(u)\, F(u)} \ \leq \ C_{cap} \, .$$

Then μ satisfies an $\mathcal{I}_\psi$-inequality for ψ defined in (3.1), hence $I_\psi(t,h) \leq e^{-t/2C_\psi} I_\psi(h)$. In particular there exist constants M_η and C_η such that

$$\| P_t^*(h)\mu - \mu \|_{TV} \leq M_\eta \, e^{-t/4\,C_\eta} \left(1 + \int \eta(h) d\mu\right) .$$

Proof. According to the previous discussion, it remains to control $\int_{h>a} \eta(h)\, d\mu$. Define $\Omega = \{h > a\}$. By the Markov inequality $\mu(\Omega) \leq 1/a \leq 1/2$ since $a > 2$.
For $k \geq 0$, define $\Omega_k = \{h > a\rho^k\}$ for $\rho > 1$ previously defined. Again $\mu(\Omega_k) \leq 1/(a\,\rho^k)$ and

$$\int_{h>a} \eta(h)\, d\mu \,\leq\, \sum_{k\geq 0} \int_{\Omega_k \setminus \Omega_{k+1}} \eta(h)\, d\mu \,\leq\, \sum_{k\geq 0} \eta(a\,\rho^{k+1})\, \mu(\Omega_k)\,,$$

since η is non-decreasing on $[a, +\infty)$. But thanks to our hypothesis,

$$\mu(\Omega_k) \leq \frac{Cap_\mu(\Omega_k)}{F(1/\mu(\Omega_k))} \leq \frac{Cap_\mu(\Omega_k)}{F(a\,\rho^k)}\,,$$

since F is non-decreasing, provided $\mu(\Omega_k) \neq 0$. Since $\Omega_k \supseteq \Omega_{k+1}$ the previous sum has thus to be taken for $k < k_0$ where k_0 is the first integer such that $\mu(\Omega_{k_0}) = 0$ if such an integer exists. So from now on we assume that $\mu(\Omega_k) \neq 0$.
Consider now, for $k \geq 1$ the function

$$f_k := \min\left(1\,,\, \left(\frac{h \,-\, a\,\rho^{k-1}}{a\,\rho^k \,-\, a\,\rho^{k-1}}\right)_+\right) .$$

Since $\mu(\Omega_{k-1}) < 1/2$ and f_k vanishes on Ω_{k-1}^c, f_k vanishes with probability at least $1/2$. Hence

$$Cap_\mu(\Omega_k) \,\leq\, \int \Gamma(f_k)\, d\mu \,\leq\, \frac{\int_{\Omega_{k-1}\setminus\Omega_k} \Gamma(h)\, d\mu}{a^2\, \rho^{2(k-1)}\, (\rho-1)^2} \,\leq\, \frac{\int_{\Omega_{k-1}\setminus\Omega_k} \eta''(h)\, \Gamma(h)\, d\mu}{a^2\, \rho^{2(k-1)}\, (\rho-1)^2\, \eta''(a\,\rho^k)}\,,$$

since η'' is non-increasing.
Summing up all these estimates (for $k \geq 1$ remember) we obtain

$$\begin{aligned} \int_{h>\rho\, a} \eta(h)\, d\mu \quad &\leq \quad \sum_{k\geq 1} \left(\frac{\eta(a\,\rho^{k+1})}{a^2\, \rho^{2(k-1)}\, (\rho-1)^2 \eta''(a\,\rho^k) F(a\,\rho^k)}\right) \int_{\Omega_{k-1}\setminus\Omega_k} \eta''(h)\, \Gamma(h)\, d\mu \\ &\leq \quad \frac{\rho^2\, C_{cap}}{(\rho-1)^2} \int_{h>a} \eta''(h)\, \Gamma(h)\, d\mu\,, \end{aligned} \tag{3.3}$$

according to our hypothesis.
It remains to control $\int_{a<h\leq\rho\, a} \eta(h)\, d\mu$. But on $\{a < h \leq \rho\, a\}$, $\eta(h) \leq c\,(h^2 - h)$ for some $c > 0$, and as before

$$\int_{h<a\rho} (h^2 - h) d\mu \,\leq\, \int_{h<a\rho} \Gamma(h)\, d\mu \leq C' \left(\int_{h\leq a} \Gamma(h)\, d\mu + \int_{h>a} \eta''(h)\, \Gamma(h)\, d\mu\right)$$

for some C' since η'' is bounded from below on $[a, a\,\rho]$. The proof is completed. □

Remark 3.4. Remarks and examples.

(1) If $\eta(u) = u^2$ we may choose $F(u) = c$ for all u and conversely. The capacity-measure inequality $\mu(A) \leq (1/c)\, Cap_\mu(A)$ is known to be equivalent (up to the constants) to the Poincaré inequality. We thus recover (see below for more precise results) the usual $\mathbb{L}^2$ theory. Note that as we suppose F to be non decreasing, so that (H_F) already implies a Poincaré inequality, but with no precision on the constant.

Similarly if $\eta(u) = u\log(u)$ we may choose $F(u) = C\log(cu)$ for some well chosen c, C and conversely. Again the capacity-measure inequality $\mu(A)\,\log(c/\mu(A)) \leq (1/C)\, Cap_\mu(A)$ is known to be equivalent (up to the constants) to the logarithmic Sobolev inequality, and we recover the usual entropic theory.

(2) Since we know now what hypotheses on η are required we may follow more accurately the constants. Indeed since η'' is non-increasing, $\psi'' \leq 1$ for $u > a$. It is thus not difficult to check that $\psi(u) \leq (1 + (\rho-1)^2)(u^2 - u)$ on $[a, \rho\, a]$ (using $a > 2$). So it easily follows that

$$C_\eta \;=\; C_\psi \;\leq\; \max\left(\frac{\eta''(a)\,(1+(\rho-1)^2)\,C_P}{\eta''(\rho\, a)}\,,\,\frac{\rho^2\, C_{cap}}{(\rho-1)^2}\right)\,.$$

(3) Now choose $\eta(u) = u^p$ for some $2 \geq p > 1$ (recall that η'' is non-increasing). Again the best choice of F is a constant. More precisely choose $F = 3C_P$. It is known (see the lower bound of Theorem 14 in [6]) that $\mu(A) \leq F\, Cap_\mu(A)$. Hence we have $C_{cap} = (a^2\,\rho^{2p}/p(p-1))$. Then a rough estimate is

$$C_\eta \;\leq\; C_P\, \max\left(\rho^{2-p}\,(1+(\rho-1)^2)\,,\,\frac{\rho^{2+p}}{p\,(p-1)\,(\rho-1)^2}\right)\,.$$

Hence we obtain

$$\parallel P_t^*(h)\mu - \mu \parallel_{TV} \leq\; M_\eta\, e^{-\,c_p\, t/C_P}\,\left(1 + \int \eta(h) d\mu\right)\,,$$

for some constant c_p.

If p is close to one, it is easily seen that $c_p \geq (p-1)c$ for some universal constant c. So, Theorem 3.2 explains why the results in Example 2.5 are not so surprising.

A similar study is possible for $\eta(u) = u\log_+^\beta(u)$ for $\beta > 0$. In this case indeed, it is easily seen that one may choose

$$F(u) \;=\; \log(u) \qquad \text{and} \qquad C_{cap} = C(a,\rho)\,\frac{1 \wedge 2^{\beta-1}}{\beta}\,,$$

at least for u small enough. Such a capacity-measure is known to be equivalent to a logarithmic Sobolev inequality, and as before for $0 < \beta \leq 1$ we recover the results in Example 2.17 (with the linear dependence in β for β close to 0).

Interesting here is also the case $\beta > 1$. Indeed one could expect that the exponential decay of such a β-entropy would require a weaker inequality than the log-Sobolev inequality. It seems that this is not the case, even if, as we said, we cannot claim that the F obtained in Theorem 3.2 furnishes the best capacity-measure inequality.

(4) One may be surprised of the intervention of a new function F, in (H_F), rather than an usual capacity-measure condition. In fact, it enables us to relax the assumptions on η. In particular, if there exists a and $\rho > 1$ such that for $u > a$, $\eta(\rho u)/(u^2\rho''(u))$

is non decreasing then instead of (H_F) one may use the capacity-measure condition: there exists C_c such that

$$\frac{\eta''(1/\mu(A))}{\mu(A)\,\eta(\rho/\mu(A))} \leq C_c\,Cap_\mu(A).$$

$\diamondsuit$

Remark 3.5. Theorem 3.2 allows to cover the class of F-Sobolev inequalities. Indeed combining the results in section 5 of [6] and Lemma 17 in [7], if μ satisfies a Poincaré inequality and the F-Sobolev inequality

$$\int f^2\,F\left(\frac{f^2}{\int f^2 d\mu}\right)\,d\mu \,\leq\, C\int \Gamma(f)\,d\mu \tag{3.6}$$

for all nice f, then the capacity-measure inequality in Theorem 3.2 is satisfied, provided $u \mapsto F(u)/u$ is non-increasing and $F(\lambda u) \leq (\lambda/4)\,F(u)$ for some $\lambda > 4$ and all u large enough (Theorem 22 and Remark 23 in [6]).
Conversely, Theorem 20 in [6] tells us that the capacity-measure inequality in Theorem 3.2 implies the $\tilde{F}$-Sobolev inequality with $\tilde{F}(u) = (F(u/\rho) - F(2))_+$ for $\rho > 1$. With the previous hypotheses on F, and up to the constants, we may replace $\tilde{F}$ by F_+.
For instance if, for $1 \leq \alpha \leq 2$, we choose $F(u) = \log^{2(1-\frac{1}{\alpha})}(1+u) - \log^{2(1-\frac{1}{\alpha})}(2)$ the Boltzmann measure $\mu(dx) = (1/Z)e^{-2U(x)}dx$ with $U(x) = |x|^\alpha$ for large x, satisfies a F-Sobolev inequality (see [6] section 7). An elementary calculation shows that we can choose

$$\eta(u) = u\,\log^{2(1-\frac{1}{\alpha})}(u)\;e^{\log^{(2/\alpha)-1}(u)}\,,$$

for large u. We thus get an interpolation result between Poincaré and Gross inequalities. $\diamondsuit$

3.2. **Links between $\mathcal{I}_\psi$-inequalities and F-Sobolev inequalities.** In view of the previous remark it is natural to relate an $\mathcal{I}_\psi$-inequality and F-Sobolev inequalities. To this end define

$$H(u) = \int_0^u \sqrt{\psi''(s)}\,ds \tag{3.7}$$

which is a continuous increasing function, whose inverse function is denoted by H^{-1}. We assume that $H(u) \to +\infty$ as $u \to +\infty$ so that H^{-1} is everywhere defined on $\mathbb{R}^+$. Remark that the derivative of $\psi \circ H^{-1}$ is equal to $(\psi'/\sqrt{\psi''}) \circ H^{-1}$, so is non-decreasing if ψ'' is non-increasing, that is $\psi \circ H^{-1}$ is a convex function.
For $f \geq 0$, denote by

$$N(f) = \inf\{\lambda > 0\,;\,\int H^{-1}(f/\lambda)d\mu \leq 1\}\,. \tag{3.8}$$

Then an easy change of variables shows that an $\mathcal{I}_\psi$-inequality is equivalent to

$$N^2(f)\int \psi\left(H^{-1}\left(\frac{f}{N(f)}\right)\right)\,d\mu \,\leq\, C_\psi\int \Gamma(f)\,d\mu\,, \tag{3.9}$$

for all nice $f \geq 0$. (3.9) looks like a F-Sobolev inequality except that the normalization is not the $\mathbb{L}^2$ norm but N. As before, up to the constants, both coincide if $F = \log$ explaining why entropy is particularly well suited.

We see that (3.9) is exactly

$$\int f^2\, F(f^2/N^2(f))\, d\mu \leq C_\psi \int \Gamma(f)\, d\mu \quad \text{for} \quad F(u) = (\psi \circ H^{-1})(\sqrt{u})/u\,. \tag{3.10}$$

We can thus get immediate comparison results, assuming that F is non-decreasing (we will see in the proof of the next Theorem that one can always modify (3.10) for this property to hold). Indeed we have two interesting cases (at least for large u and up to constants):

$$\text{either} \quad H(u) \geq \sqrt{u} \quad \Leftrightarrow u^2 \geq H^{-1}(u) \Leftrightarrow \int f^2 d\mu \geq N^2(f) \tag{3.11}$$

$$\text{or} \quad H(u) \leq \sqrt{u} \quad \Leftrightarrow u^2 \leq H^{-1}(u) \Leftrightarrow \int f^2 d\mu \leq N^2(f) \tag{3.12}$$

since H and H^{-1} are non-decreasing. In the first case, (3.10) implies the F-Sobolev inequality (3.6) while in the second case the F-Sobolev inequality implies (3.10). Note that once again the limiting case $H(u) = \sqrt{u}$ corresponds to log-Sobolev.
The first case gives some converse to Theorem 3.2. Note that $\psi(v) = H^2(v)\, F(H^2(v)) \geq H^2(v)\, F(v)$ since F is non-decreasing, hence we get a F-Sobolev inequality for some F such that $F(v) \leq \psi(v)/H^2(v)$. With some additional (but reasonable) assumptions we can improve this result. Indeed

Theorem 3.13. *Let η and ψ be as in Theorem 3.2, and H defined in (3.7). We assume that $H(+\infty) = +\infty$. Assume in addition that for u large*

- *$u \mapsto \bar{F}(u) = (\psi/H^2)(u)$ is non-decreasing and satisfies $\bar{F}(\lambda u) \leq \lambda \bar{F}(u)/4$, for some $\lambda > 4$,*
- *$u \mapsto \bar{F}(u)/u$ is non-increasing.*

If μ satisfies a Poincaré inequality with constant C_P and an $\mathcal{I}_\psi$-inequality for some C_ψ, then for $\mu(A)$ small enough, the capacity-measure inequality

$$\mu(A)\, \bar{F}(1/\mu(A)) \;\leq\; D\, Cap_\mu(A)$$

is satisfied for some $D > 0$. Accordingly (see Remark 3.5) μ satisfies the $\bar{F}_+$-Sobolev inequality (with some constant D_F).
Conversely if μ satisfies a Poincaré inequality with constant C_P and the $\bar{F}$-Sobolev inequality, and if $H(u) \geq \sqrt{u}$ for large u, an $\mathcal{I}_\psi$-inequality is satisfied for some C_ψ.

Proof. Note that $\lim_{u \to +\infty} \bar{F}(u)/u$ exists by monotonicity. Denote it by m. We have $\bar{F}(u)/4u \geq \bar{F}(\lambda u)/(\lambda u)$ so that letting u go to infinity we get $m/4 \geq m$ hence $m = 0$. In particular the capacity-measure inequality when $\mu(A) = 0$ reduces to $Cap_\mu(A) \geq 0$ which is of course satisfied. We shall thus assume now that $\mu(A) > 0$.
First we write (3.10) in the form

$$\int f^2\, F(f/N(f))\, d\mu \leq C_\psi \int \Gamma(f)\, d\mu \quad \text{for} \quad F(u) = (\psi \circ H^{-1})(u)/u^2\,. \tag{3.14}$$

The first part of the proof is mimicking the proof of Lemma 17 in [7]. Note that the derivative of F (defined in (3.14)) is given by

$$u \mapsto \frac{u\, \psi'(H^{-1}(u)) \;-\; 2\, (\psi\, \sqrt{\psi''})(H^{-1}(u))}{u^3\, \sqrt{\psi''(H^{-1}(u))}}$$

which is non-negative for u large enough since $u \mapsto (\psi/H^2)(u)$ is non-decreasing.

Choose some $\rho > 1$ large enough, such that $F(2\rho) \geq 0$ and define $\tilde{F}(u) = F(u) - F(2\rho)$ which is thus non-negative for and non-decreasing on $[2\rho, +\infty)$ if ρ is large enough. We thus have

$$\int f^2 \, \tilde{F}_+ \left(\frac{f}{N(f)}\right) \, d\mu \, \leq \, C_\psi \, \int \, \Gamma(f) \, d\mu \, + \, M \, \int f^2 d\mu \, ,$$

with $M = \sup_{0 \leq u \leq 2\rho} |F(u)|$.

Now we can follow [7] with some slight modifications. We give the details for the sake of completeness. Let χ defined on $\mathbb{R}^+$ as follows : $\chi(u) = 0$ if $u \leq 2$, $\chi(u) = u$ if $u \geq 2\rho$ and $\chi(u) = 2\rho\,(u-2)/(2\rho) - 2)$ if $2 \leq u \leq 2\rho$. Since $\chi(f) \leq f$, $N(\chi(f)) \leq N(f)$ so that since $\tilde{F}_+$ is non-decreasing,

$$\begin{aligned} \int f^2 \tilde{F}_+(f/N(f)) d\mu \;&=\; \int \chi^2(f) \tilde{F}_+(\chi(f)/N(f)) d\mu \\ &\leq\; \int \chi^2(f) \tilde{F}_+ \left(\frac{\chi(f)}{N(\chi(f))}\right) d\mu \\ &\leq\; BC_\psi \int \Gamma(f) \, d\mu + M \int \chi^2(f) d\mu \\ &\leq\; BC_\psi \int \Gamma(f) \, d\mu + M \int_{f^2 \geq 2 \int f^2 d\mu} f^2 d\mu \end{aligned}$$

where $B = (\rho/\rho - 1)^2$. But as shown in [6], $\int_{f^2 \geq 2 \int f^2 d\mu} f^2 d\mu \leq 12 C_P \int \Gamma(f) \, d\mu$ so that we finally obtain the existence of D_ψ such that

$$\int f^2 \, \tilde{F}_+ \left(\frac{f}{N(f)}\right) \, d\mu \, \leq \, D_\psi \, \int \, \Gamma(f) \, d\mu \, . \tag{3.15}$$

The second part of the proof is mimicking the one of Theorem 22 in [6]. Let $\mu(A) < 1/2$ and $\mathbb{1}_A \leq f \leq \mathbb{1}_\Omega$ with $\mu(\Omega) \leq 1/2$. For $k \in \mathbb{N}$ we define $\Omega_k = \{f \geq 2^k \, N(f)\}$ and

$$f_k = \min\left((g - 2^k \, N(f))_+ \, ; \, 2^k \, N(f)\right) \, .$$

Note that f_k is equal to 0 on Ω_k^c and to $2^k \, N(f)$ on Ω_{k+1}.

In addition, since $H^{-1}(0) = 0$,

$$\begin{aligned} \int H^{-1} \left(\frac{f_k \, H(1/\mu(\Omega_k))}{2^k \, N(f)}\right) d\mu \;&=\; \int_{\Omega_k} H^{-1} \left(\frac{f_k \, H(1/\mu(\Omega_k))}{2^k \, N(f)}\right) d\mu \\ &\leq\; \int_{\Omega_k} H^{-1} \left(H(1/\mu(\Omega_k))\right) d\mu \;=\; 1 \end{aligned}$$

so that $N(f_k) \leq 2^k N(f)/H(1/\mu(\Omega_k))$. Therefore, applying (3.15) (we need here a non-negative F)

$$\begin{aligned} D_\psi \int \Gamma(f) \, d\mu \;&\geq\; D_\psi \int \Gamma(f_k) \, d\mu \geq \int_{\Omega_{k+1}} f_k^2 \, \tilde{F}_+ \left(\frac{f_k}{N(f_k)}\right) d\mu \\ &\geq\; \mu(\Omega_{k+1}) \, 2^{2k} \, N^2(f) \, \tilde{F}_+(H(1/\mu(\Omega_k)) \, . \end{aligned}$$

We are thus in the situation of the proof of Theorem 22 in [6] replacing $\mu(g^2)$ therein by $N^2(f)$ and F therein by $\tilde{F}_+ \circ H$. We may conclude since for u large, $(\tilde{F}_+ \circ H)(u) \geq c\,\psi(u)/H^2(u)$ and for $\mu(A)$ small enough according to Remark 23 in [6].

The direct part being proven let us briefly indicate how to prove the converse part. Again we may modify $\bar{F}$ into a non-negative G thanks to Poincaré inequality (this is exactly Lemma 17 in [7]). The properties of G ensure that we may apply Theorem 22 in [6], i.e. the G-Sobolev inequality implies a capacity-measure inequality (with the same G). Next just remark that the proof of Theorem 20 in [6] applies to any homogeneous inequality (i.e. we may replace $\int f^2 d\mu$ therein by $N^2(f)$ for example). We thus get that (3.10) holds with G in place of F. But as we remarked $F \leq \bar{F}$ for large u, and with our hypotheses $\bar{F} \leq c\,G$ at infinity. We may thus replace (changing the constants) G by F for large u, small values of u can be controlled again (if necessary) by using Poincaré inequality. □

Remark 3.16. At least if $H(u) \geq \sqrt{u}$ (up to a constant actually), we have two results saying that some F-Sobolev inequality implies an $\mathcal{I}_\psi$-inequality: the first one with $F(u) \geq C\,\eta(\rho u)/u^2\,\eta''(u)$ at infinity, the second one with $\bar{F}(u) = \eta(u)/H^2(u)$. It seems not easy to compare them in full generality. However one can use some asymptotic estimates.
First recall that ψ'' (hence $\sqrt{\psi''} := g$) is supposed to be non-increasing at infinity. Since we have assumed that $H(+\infty) = +\infty$ it implies that $g'(u)/g(u) = (1/2)(\psi'''(u)/\psi''(u)) \geq -1/u$ near infinity. Now write the elementary

$$\int_m^u (g(s) + sg'(s))\, ds \;=\; ug(u) \;-\; mg(m)\,.$$

It immediately follows that

$$\text{if } \frac{u\,\psi'''(u)}{\psi''(u)} \to 0 \text{ as } u \to +\infty\,, \text{ then } H(u) \sim_{u\to+\infty} u\,\sqrt{\psi''(u)}\,, \tag{3.17}$$

while

$$\text{if } \liminf_{u\to+\infty} \frac{u\,\psi'''(u)}{2\,\psi''(u)} \;=\; -\,d\,, \text{ for some } d<1, \text{ then } H(u) \leq_{u\to+\infty} \frac{1}{1-d}\,u\,\sqrt{\psi''(u)}\,. \tag{3.18}$$

Hence we always get that

$$\bar{F}(u) \;\geq c\,\frac{\psi(u)}{u^2\,\psi''(u)}\,, \tag{3.19}$$

that is in general the same condition in both Theorems. This is very satisfactory but of course we have made additional assumptions on $\bar{F}$ in Theorem 3.13. ◊

One of the very interesting feature of F-Sobolev inequality is that they are linked to contraction properties for the semi-group. We now recall these general results taken from [32].

According to Wang's beautiful results ([32] chapter 3.3), a F-Sobolev inequality is equivalent to a super-Poincaré inequality, i.e. for all nice f and all $s \geq 1$,

$$\int f^2 d\mu \leq \beta_{SP}(s) \int \Gamma(f) d\mu + s \left(\int |f| d\mu\right)^2\,. \tag{3.20}$$

If the F-Sobolev inequality holds, (3.20) holds with $\beta_{SP}(s) = c/F(s)$ for s large enough ([32] Theorem 3.3.1). For a somewhat intricate converse see [32] Theorem 3.3.3.
Assume that the F-Sobolev inequality holds. The associated super-Poincaré inequality implies some boundedness for the associated semi-group. Of particular interest here are Theorem 3.3.13 (2) and Theorem 3.3.14 in [32]. The first one tells us that P_t is super-bounded (i.e.

is bounded from $\mathbb{L}^2(\mu)$ in $\mathbb{L}^p(\mu)$ for all $p > 2$ and all $t > 0$) as soon as $F(u)/\log(u) \to +\infty$ as $u \to \infty$ (some converse statement is also true), while the second one tells us that P_t is ultracontractive (or ultrabounded in Wang's terminology) as soon as

$$\int^{+\infty} \frac{1}{u\,F(u)}\,du \,<\, +\infty\,.$$

Let us come back to the second situation in (3.11). Roughly speaking this case is the one of stronger inequalities than the log-Sobolev inequality, for which with the mild additional previous assumptions, we know that the semi-group is ultracontractive. However we can give another interesting example, and will continue the discussion in the next section.

Example 3.21. For $F(u) = \log(u)\,\log(\log(u))$ at infinity, Wang's results show that the semi-group P_t^* is super-bounded but not ultracontractive. An elementary calculation show that we can choose $\eta(u) = u\,\log(\log(u))$ in this case. $\diamondsuit$

The study of weak inequalities should be interesting. The two extreme cases, weak Poincaré and weak logarithmic Sobolev inequalities have already been studied. As remarked in [12] the main interest of weak log-Sobolev inequalities is to describe some interpolation between Poincaré and Gross (if a Poincaré inequality does not hold, the weak log-Sobolev inequality furnishes worse results than the corresponding weak Poincaré inequality). So the potential weak inequalities should give better results than the weak log-Sobolev inequality (recall Theorem 2.23). However, the technical intricacies are certainly too much for a potential reader since we do not have (yet) any convincing application.

Remark 3.22. Finally we may ask whether it is possible to get some exponential decay using a weaker inequality than Poincaré inequality but for η's larger than $u \mapsto u^2$ at infinity. Assume for instance that for all density of probability h bounded by $M \geq 2$ we have for some function ξ decaying to 0,

$$\int |P_t^* h - 1| d\mu \leq \xi(t)\,.$$

Let f be in $\mathbb{L}^2(\mu)$ such that $\int f d\mu = 0$ and $\parallel f \parallel_\infty \leq 1$. Then $h = (f+2)/2$ is a density of probability, bounded by $3/2$ hence

$$\mathrm{Var}_\mu(P_t^* f) \,\leq\, 2 \int |P_t^* h - 1| d\mu \,\leq\, 2\,\xi(t) \,\leq\, 2\,\xi(t)\,\mathrm{Osc}^2(f)$$

and the previous inequality extends to all f in $\mathbb{L}^2(\mu)$ by homogeneity.
In the symmetric case ($P_t = P_t^*$) this result implies a weak Poincaré inequality (see [29] Theorem 2.3). In particular if $\xi(t) = c\,e^{-\lambda t}$ for some $\lambda > 0$ the same Theorem shows that μ satisfies a Poincaré inequality. Hence in the symmetric case we cannot obtain any exponential decay for the total variation distance even for bounded densities without assuming that a Poincaré inequality is satisfied. If it is not we have to use the results of the previous section.
$\diamondsuit$

Remark 3.23. An aficionado of functional inequalities may have remarked that we have not discussed usual properties introduced when dealing with a new functional inequality like $\mathcal{I}_\psi$: tensorization and concentration of measure. In fact, concentration is not at all our purpose here and in fact it may be directly deduced from the capacity measure condition imposed in Theorem 3.2 or inherited by Theorem 3.13. Concerning tensorization, it is more relevant for applications concerning diffusion to deal directly in multidimensional space rather than the limiting setting of tensorization and perturbation argument. Note also that by the equivalence obtained via Theorem 3.13, of an $\mathcal{I}_\psi$ inequality and an F-Sobolev inequality, we get all the tensorization property (and concentration) via F-Sobolev inequalities, see [6, 7] for details.

3.3. **Further examples.** The major difference between Theorems 3.2 and 3.13 is that in the first one we do not explicitly suppose an F-Sobolev inequality. Therefore we may put less stringent assumptions on F, and still have an explicit condition in dimension 1: namely (H_F) can be translated in

(H'_F) : there exist $\rho > 1$ and a non-decreasing function F such that

- let m be a median of μ, and denoting μ_c the density of the absolutely continuous part of μ w.r.t. the Lebesgue measure, if

$$\sup_{x>m} \mu([x,\infty[)F(1/\mu([x,\infty[)) \int_m^x \mu_c^{-1}(t)dt \, < \, \infty$$

$$\sup_{x<m} \mu(]-\infty,x])F(1/\mu(]-\infty,x])) \int_x^m \mu_c^{-1}(t)dt \, < \, \infty$$

- there exists a constant C_{cap} such that for all $u > a$,

$$\frac{\eta(\rho\, u)}{u^2\, \eta''(u)\, F(u)} \, \leq \, C_{cap} \, .$$

However this measure capacity condition is no more tractable in the multidimensional case whereas we have known conditions in the multidimensional case for F-Sobolev inequalities. Indeed, by [7, Th. 21], assume that $d\mu = e^{-2V}dx$ with V a C^2 potential such that $Hess(V) \geq R$ for some real R and let F be C^1 on $]0,\infty[$ such that

- $F(x) \to \infty$ as $x \to \infty$, $F(x) \leq c \log_+ x$, $F(xy) \leq \hat{c} + F(x) + F(y)$ and $xF'(x) \leq \tilde{c}$ for some positive $c, \tilde{c}$ and real $\hat{c}$;
- the following drift like condition is verified: $F(e^{2V}) + C(LV - |\nabla V|^2) \leq K$ for some positive C and K

then μ verifies a (F-B)-Sobolev inequality for some positive B. Using then Theorem 3.2 via [7, Th. 18] for the implied capacity-measure condition (H_F) we get an $\mathcal{I}_\psi$-inequality, hence an exponential decay for the total variation distance using Lemma 1.1.

Consider for example, for $1 < \alpha < 2$, $V(x) = |x|^\alpha + \log(1 + |x|\sin^2(x))$, then μ satisfies a Poincaré inequality and the previous conditions with $F(u) = \log(1+u)^{2(1-1/\alpha)} - \log(2)^{2(1-1/\alpha)}$, so that we get for some $c_1, c_2 > 0$

$$\| \, P_t^* h - 1 \, \|_{\mathbb{L}^1(\mu)} \leq c_1 \, e^{-c_2 t} \left(\int h \log^{2(1-1/\alpha)}(h) e^{\log^{(2/\alpha)-1}(h)} d\mu) \right)^{1/2} .$$

4. **Is a direct study of the total variation distance possible ?**

A natural question is of course : is it possible to directly study the possible decay of the total variation distance, instead of looking at larger quantities like the variance or the relative entropy ? Due to the non smoothness of $u \mapsto |u-1|$ the answer is no, but one can try to replace the total variation distance by almost equivalent quantities.

Before to do this, let us mention the results by Mao ([24]) and Wang ([35]) who gave some example of a diffusion process for which a logarithmic Sobolev does not hold, while uniform (hence exponential) convergence holds in $\mathbb{L}^1$. So we cannot expect to find some functional inequality equivalent to the exponential convergence in total variation distance.

Let us briefly gives some more hints (we refer to [15] for the details).

4.1. **The linear case.** In the preceding section we only looked at functions ψ such that $\psi(u)/u \to \infty$ at infinity. However Remark 1.2 shows that it is possible to consider cases where ψ is almost linear at infinity.

Consider (at least for large u) $\eta(u) = u + \theta(u)$ where θ is a convex function such that $\theta(u)/u \to 0$ as $u \to \infty$. Necessarily $\theta'(u) \leq 0$ for large u and goes to 0 as $u \to \infty$. If η'' is non-increasing, so does θ'', and according to the previous property $\theta''(u) \to 0$ at infinity.

Define ψ as in (3.1). Then for $u > a$, $\psi(u) = \frac{-\theta'(a)}{\theta''(a)}\, u + \nu(u)$ where $\nu(u)/u \to 0$ at infinity. In order to apply Remark 1.2 it is thus enough to have $2\theta'(a)+\theta''(a) < 0$ (since $\psi'(1) = 1/2$). Assuming this condition, we may extend Theorem 3.2 to this η. This yields $F(u) \geq \eta(\rho\, u)/(u^2\, \theta''(u))$ and since what is important is the behavior of F near infinity and η is moderate, the key is the behavior of $u \mapsto 1/(u\, \theta''(u))$ when u goes to infinity. A capacity-measure inequality is interesting only if $F(u)/u \to 0$ as u goes to infinity (otherwise we already know that the semi-group is ultracontractive) so that the only interesting cases are those for which $u^2\, \theta''(u) \to \infty$ as $u \to \infty$.

The main question is: is it possible to build such θ's ? The simplest way to do so is to write

$$\theta(u) = -\int_a^u (1/\tau(s))ds \quad , \quad \theta'(u) = -\,(1/\tau(u)) \quad , \quad \theta''(u) = (\tau'(u)/\tau^2(u)) \tag{4.1}$$

where τ is a non-negative, non-decreasing function. Fix some F. In our situation what we have to do is to find some τ such that

$$\frac{\tau'(u)}{\tau^2(u)} = \frac{1}{u\, F(u)}\,.$$

Since $\theta' = -1/\tau$ goes to 0 at infinity, it implies Wang's integrability condition, hence ultracontractivity.

4.2. **Using the Hellinger distance.** Another possibility to control the total variation distance is to use Hellinger distance, defined for $\nu = h\mu$ by

$$d_H(\nu,\mu) = 2\int (1-\sqrt{h})\, d\mu\,. \tag{4.2}$$

It is elementary to check that

$$d_H(\nu,\mu) \;\leq\; 2\; \parallel \mu - \nu \parallel_{TV} \leq 4\,\sqrt{d_H(\nu,\mu)} \tag{4.3}$$

hence both distances are "almost" equivalent. Using the concavity of $u \mapsto \sqrt{u}$ it is also immediate that

$$d_H\left(\frac{\nu+\mu}{2},\mu\right) \leq \frac{1}{2}\, d_H(\nu,\mu) \leq \parallel \mu-\nu \parallel_{TV} = 2\, \parallel \mu - \frac{\nu+\mu}{2} \parallel_{TV} \leq 8\sqrt{d_H\left(\frac{\nu+\mu}{2},\mu\right)} \tag{4.4}$$

so that (with some changes in the constants) we may assume that $\nu = h\mu$ with $h \geq 1/2$.

Introduce as usual $I(t) = d_H(P_t^* h\mu, \mu)$ for some density of probability h, and differentiating w.r.t. t, we get

$$\frac{d}{dt}\, I(t) = -\frac{1}{4} \int \frac{|\nabla P_t^* h|^2}{(P_t^* h)^{3/2}}\, d\mu\,. \tag{4.5}$$

As in the preceding subsections we may state

Proposition 4.6. *Assume that there exists some non-increasing function β_H defined on $(0, +\infty)$ such that for all $s > 0$ and all f belonging to $D_2(L)$ the following inequality holds*

$$\left(\int f^4\, d\mu\right)^{1/2} - \int f^2 d\mu \leq \beta_H(s) \int \Gamma(f) d\mu + s\, Osc(f^2)\,, \tag{4.7}$$

then for all $\nu = h\mu$, $d_H(P_t^* h\mu, \mu) \leq 3\xi_H(t)\, \parallel h \parallel_\infty^{1/2}$ with

$$\xi_H(t) = \inf\{s > 0\,,\, \beta_H(s)\, \log(1/s)\, \leq\, 4t\}\,.$$

Hence, if $\tilde\eta(u) = u^{1/4}\varphi(u)$,

$$\parallel P_t^*\nu - \mu \parallel_{TV} \leq \frac{4 \int h\varphi(h) d\mu}{(\varphi \circ \tilde\eta^{-1})\left(2 \int h\varphi(h) d\mu / \sqrt{3\xi_H(t)}\right)}\,.$$

Proof. Apply (4.7) with $f = (P_t^* h)^{1/4}$. It yields

$$\frac{d}{dt}\, I(t) \leq -\frac{4}{\beta_H(s)} I(t) + \frac{4s}{\beta_H(s)} \parallel P_t^* h \parallel_\infty$$

hence the result (because $\mathrm{Osc}(h^{1/2}) \leq \parallel h \parallel_\infty^{1/2}$ and $d_H(\nu, \mu) \leq 2$). □

Note that (4.7) implies the following

$$\mathrm{Var}_\mu(f^2)\, \leq\, 2\left(\beta_H(s) \int \Gamma(f) d\mu + s\, \mathrm{Osc}(f^2)\right) \left(\int f^4 d\mu\right)^{1/2}, \tag{4.8}$$

just multiplying both hand sides in (4.7) by $\left(\int f^4 d\mu\right)^{1/2} + \int f^2 d\mu$ and applying Cauchy-Schwarz inequality. Conversely (4.8) implies (4.7) up to a factor 2 (majorizing $(\int f^4 d\mu)^{1/2}$ in the right hand side by $\left(\int f^4 d\mu\right)^{1/2} + \int f^2 d\mu$ and then dividing both hand sides by this quantity).

We refer to [15], section 4, for a detailed study of these inequalities. For our purpose, (4.7) is unfortunately not convincing. Indeed it is shown in Corollary 4.10 of [15] that if $\beta_H(s) = c/s$, (4.7) is equivalent to a Poincaré inequality, and if $\beta_H(s) = c/s \log(1/s)$ it is equivalent to a logarithmic Sobolev inequality. In the first case a direct application of Proposition 4.6 for h such that $\int h^2 d\mu < +\infty$, i.e. with $\varphi(u) = u$, yields a polynomial decay $c/t^{2/5}$ which is disastrous, since Poincaré inequality yields an exponential decay.

Now, (4.7) with β_H constant is equivalent to the exponential decay $I(t) \leq e^{-\alpha t}\, I(0)$ for some $\alpha > 0$, which implies according to (4.3), $\parallel P_t^* h - 1 \parallel_{\mathbb{L}^1(\mu)} \leq 2\sqrt{2}\, e^{-\alpha t/2}$. But, according again to Corollary 4.10 of [15], β_H constant implies a super Poincaré inequality with $\beta_{SP} = c\, s^{-1/2}$, hence again P_t is ultracontractive according to Wang's result.

Hence, the direct study of the Hellinger distance furnishes no convincing results. However, in [18], where inequalities in the spirit of (4.8) were introduced under the name of L^q-Poincaré inequalities, applications of those type of inequalities concern large time behavior of nonlinear diffusions, namely porous media equation $\partial_t u = L(u^m)$ for $m \geq 1$. Formal calculations indicate that (4.8) could have the same role for other nonlinear diffusions. We leave this for further research.

5. **Other related inequalities, reversing the roles.**

One of the main feature of the use of functional inequalities for studying the total variation distance, is that symmetry is broken. Indeed $I_\psi(\mathbb{Q}|\mathbb{P})$ is in general not symmetric. If it seems natural to privilege the invariant measure μ by looking at $I_\psi(P_t^*\nu|\mu)$, one may ask what happens if we reverse the roles. This idea is not completely new since in [17] the authors have studied the evolution of the total variation distance between $P_t^*\nu$ and $P_t^*\nu'$ for any initial ν and ν', but under strong conditions on one of them.
In second place, since

$$\parallel P_t^* h - 1 \parallel_{\mathbb{L}^1(\mu)} = 2 \parallel P_t^* \left(\frac{h+1}{2}\right) - 1 \parallel_{\mathbb{L}^1(\mu)} \tag{5.1}$$

we may assume that $h \geq \frac{1}{2}$, i.e. $P_t^* h \geq \frac{1}{2}$. Thus if $\nu = h\mu$

$$\frac{d\mu}{dP_t^*\nu} = \frac{1}{P_t^* h} \leq 2\,.$$

We thus have, denoting $P_t^*\nu = \nu_t$,

$$\begin{aligned} \parallel P_t^*\nu - \mu \parallel_{TV} &\leq \sqrt{\mathrm{Var}_{\nu_t}(1/P_t^* h)} \\ \parallel P_t^*\nu - \mu \parallel_{TV} &\leq \sqrt{2\,\mathrm{Ent}_{\nu_t}(1/P_t^* h)} \end{aligned}$$

so that we shall study

$$V(t) = \mathrm{Var}_{\nu_t}(1/P_t^* h) = \int \frac{1}{P_t^* h} d\mu - 1\,, \tag{5.2}$$

and

$$E(t) = \mathrm{Ent}_{\nu_t}(1/P_t^* h) = \int \log(1/P_t^* h)\, d\mu\,. \tag{5.3}$$

Assuming first that h is also bounded from above (for the forthcoming calculation to be rigorous), we immediately get using the chain rule

$$\frac{d}{dt}\, V(t) = - \int \frac{1}{(P_t^* h)^3}\, \Gamma(P_t^* h)\, d\mu \quad \text{ and } \quad \frac{d}{dt}\, E(t) = -\frac{1}{2} \int \frac{1}{(P_t^* h)^2}\, \Gamma(P_t^* h)\, d\mu\,. \tag{5.4}$$

Remark now that the exponential decay

$$V(t) \,\leq\, e^{-\lambda t}\, V(0)$$

is equivalent to

$$\int (1/P_t^* h) d\mu - 1 \,\leq\, (1/\lambda) \int \frac{1}{(P_t^* h)^3}\, \Gamma(P_t^* h)\, d\mu \tag{5.5}$$

for all $t \geq 0$, and that the exponential decay

$$E(t) \,\leq\, e^{-\lambda t}\, E(0)$$

is equivalent to

$$\int \log(1/P_t^* h) d\mu \;\leq\; (1/2\lambda) \int \frac{1}{(P_t^* h)^2}\, \Gamma(P_t^* h)\, d\mu \tag{5.6}$$

for all $t \geq 0$.

There are now two approaches which can be seen as static or dynamic: the static one is to consider equations (5.5) and (5.6) as functional inequalities and as before look at capacity-measure conditions for these inequalities ; the dynamic one starts from the assumption that $hd\mu$ satisfies some inequalities (say Poincaré for example) and study the propagation along the semigroup of such an inequality which enables us to get a direct control of $V(t)$.

5.1. **The static approach.** Using (5.5) with $u = \sqrt{1/P_t^* h}$, such an exponential decay for all h ($\geq 1/2$) is equivalent to

$$\int u^2\, d\mu \,-\, 1 \;\leq\; C_{WE} \int \Gamma(u) d\mu \tag{5.7}$$

for all u belonging to $D_2(L)$ such that $0 \leq u \leq \sqrt{2}$ and $\int (1/u^2) d\mu = 1$. The weak version

$$\int u^2\, d\mu \,-\, 1 \;\leq\; \beta_{WE}(s) \int \Gamma(u) d\mu + s\,, \tag{5.8}$$

for some non-increasing function β_{WE} defined on $(0, +\infty)$ and all $s > 0$ implies that for all $\nu = h\mu$,

$$\| \, P_t^* \nu - \mu \, \|_{TV} \leq \sqrt{V(t)} \leq C \sqrt{\xi_{WE}(t)}\,, \tag{5.9}$$

for some universal constant C where $\xi_{WE}(t) = \inf \{ s > 0\,,\, \beta_{WE}(s)\, \log(1/s) \;\leq\; 4t \}$.

But, it is shown in details in [15] section 5, that (5.8) (thus (5.7)) implies the very strong consequence: there exists $\beta > 0$ such that for all A with $\mu(A) > 0$ we have $Cap_\mu(A) \geq \beta$, which implies of course ultracontractivity.

A similar analysis can be done with (5.6). Here again we may state :
assume that there exists some non-increasing function β_{MT} defined on $(0, +\infty)$ such that for all $s > 0$ and all v belonging to $D_2(L)$ such that $v \geq -\log 2$ and $\int v d\mu = 0$, the following inequality holds

$$\log \left(\int e^v \, d\mu \right) \;\leq\; \beta_{MT}(s) \int \Gamma(v) d\mu + s\, \mathrm{Osc}^2(v)\,, \tag{5.10}$$

then for all $\nu = h\mu$,

$$\| \, P_t^* \nu - \mu \, \|_{TV} \leq \sqrt{2\, \xi_{MT}(t)}\, \sqrt{\log 2 + \mathrm{Osc}^2(\log h)}\,,$$

for some universal constant C, where

$$\xi_{MT}(t) = \inf \{ s > 0\,,\, 2\beta_{MT}(s)\, \log(1/s) \;\leq\; t \}\,.$$

Hence for all $\nu = h\mu$, for $\eta(u) = \varphi(u)\, \log(u)$,

$$\| \, P_t^* \nu - \mu \, \|_{TV} \leq \frac{4 \int h\, \varphi(h)\, d\mu}{(\varphi \circ \eta^{-1}) \left(\left(\int h\, \varphi(h)\, d\mu \right) / \sqrt{\xi_{MT}(t)} \right)}\,.$$

Indeed, if (5.10) holds, we may choose

$$v_t = \log(P_t^* h) - \int \log(P_t^* h) d\mu \,,$$

and apply (5.4). We obtain

$$\frac{d}{dt}\, E(t) \leq \frac{-1}{2\beta_{MT}(s)}\, E(t) \,+\, \frac{s}{2\beta_{MT}(s)}\, \mathrm{Osc}^2(v_t)\,.$$

Gronwall's lemma immediately yields $E(t) \leq \xi_{MT}(t)(E(0)+\mathrm{Osc}^2(\log h))$ because $\mathrm{Osc}^2(\log P_t^* h) \leq \mathrm{Osc}^2(\log h)$. Since $E(0) \leq \log 2$ if $h \geq \frac{1}{2}$ the conclusion follows.

Inequality (5.10) is a weak version of the so called Moser-Trudinger inequality, i.e. for all nice v such that $\int v d\mu = 0$,

$$\log\left(\int e^v \, d\mu\right) \,\leq\, C_{MT} \int \Gamma(v) d\mu \,,$$

for some constant C_{MT}, which appears as some limit case of Sobolev inequalities.

But, as equation (5.8) one can show again that the capacity of all sets (with positive measure) is uniformly bounded from below (see [15] section 5.1.2.

5.2. **The dynamic approach.** As previously said, another direct approach of (5.5), close to [17], is the following.

Assume that $\nu_t = P_t^* h \, \mu$ satisfies a Poincaré inequality

$$\mathrm{Var}_{\nu_t}(g) \,\leq\, C_P(t) \int \Gamma(g)\, d\nu_t \,. \tag{5.11}$$

Applying (5.11) with $g = 1/P_t^* h$ yields precisely (5.5) with $\lambda = 1/C_P(t)$ and consequently

$$V(t) \,\leq\, e^{-\int_0^t (1/C_P(s))\, ds}\, V(0)\,. \tag{5.12}$$

Here we have to be much more accurate. Indeed in the derivation of (5.9) we may first establish (5.8) and (5.4) for nice h's bounded from below and from above, and then extend (5.9) to general h's. Here, since we are using Poincaré inequality for ν_t, we need (5.4) for the h we are interested in. It is not difficult to see that $h \in \mathbb{L}^2(\mu)$ and $h \geq 1/2$ are sufficient for all these derivations to be correct. Dealing with initial densities in $\mathbb{L}^2(\mu)$ is certainly disappointing, however we shall see how to use approximations by such functions, but to this end we have to weaken our assumptions.

¿From now on we assume that $1/2 \leq h \leq K$ is a (nice) bounded density of probability such that $\nu = h\mu$ satisfies a weak Poincaré inequality with function β_{WP}, we expect not depending on K. (5.4) is thus satisfied, and we want to study a possible weak Poincaré inequality for $\nu_t = P_t^* h\mu$. It holds

$$\begin{aligned}
\mathrm{Var}_{\nu_t}(f) &= \int f^2\, P_t^* h\, d\mu - \left(\int f\, P_t^* h\, d\mu\right)^2 = \int P_t(f^2)\, h\, d\mu - \left(\int P_t f\, h\, d\mu\right)^2 \\
&= \int P_t(f^2)\, h\, d\mu - \int (P_t f)^2\, h\, d\mu + \int (P_t f)^2\, h\, d\mu - \left(\int P_t f\, h\, d\mu\right)^2 \\
&\leq \left(\int P_t(f^2)\, h\, d\mu - \int (P_t f)^2\, h\, d\mu\right) + \beta_{WP}(s) \int \Gamma(P_t f)\, h\, d\mu \,+\, s\, \mathrm{Osc}^2(f)\,.
\end{aligned}$$

It remains to exchange Γ and P_t and to control the first term in the left hand side. Both controls are known to be equivalent to a curvature assumption introduced by Bakry and Emery. Recall some definitions (see e.g. [1] section 5.3 and Proposition 5.4.1)

Proposition 5.13. *Introduce $\Gamma_2(f,g) := (1/2)\left(L\Gamma(f,g) - \Gamma(Lf,g) - \Gamma(f,Lg)\right)$. We shall say that the curvature of L is bounded from below by $\rho \in \mathbb{R}$ if for all nice f, $\Gamma_2(f) := \Gamma_2(f,f) \geq \rho\,\Gamma(f)$.*
Then the following three assertions are equivalent

- *the curvature of L is bounded from below by $\rho \in \mathbb{R}$,*
- *for all nice f and all $t > 0$, $\Gamma(P_t f) \,\leq\, e^{-\rho t}\, P_t(\Gamma(f))$,*
- *for all nice f and all $t > 0$, $P_t(f^2) - (P_t f)^2 \,\leq\, \frac{1-e^{-\rho t}}{\rho}\, P_t(\Gamma(f))$ (for $\rho = 0$ replace the coefficient by t).*

We immediately deduce

Corollary 5.14. *If the curvature of L is bounded from below by $\rho \in \mathbb{R}$, and $\nu = h\mu$ satisfies a weak Poincaré inequality with function β_{WP}, then $\nu_t = P_t^* h\mu$ satisfies a weak Poincaré inequality with function*

$$\beta_{WP}(t,s) = \frac{1-e^{-\rho t}}{\rho} \,+\, e^{-\rho t}\,\beta_{WP}(s)\,.$$

Plugging this estimate (with $f = 1/P_t^* h$) in (5.4) yields (since $\mathrm{Osc}(f) \leq 2$)

$$\frac{d}{dt}\,V(t) \leq -\,\frac{V(t)}{\beta_{WP}(t,s)} + \frac{4s}{\beta_{WP}(t,s)}\,.$$

Defining

$$r(t,s) = \int_0^t \frac{1}{\beta_{WP}(u,s)}\,du \tag{5.15}$$

we obtain

$$V(t) \leq e^{-r(t,s)}\,V(0) + \int_0^t \frac{4s}{\beta_{WP}(u,s)}\,e^{r(u,s)-r(t,s)}\,du \,\leq\, e^{-r(t,s)}\,V(0) \,+\, 4s\,.$$

But an elementary calculation yields

$$r(t,s) = \log\left(\frac{e^{\rho t} + \rho\,\beta_{WP}(s) - 1}{\rho\,\beta_{WP}(s)}\right)$$

so that finally

$$V(t) \,\leq\, \frac{\rho\,\beta_{WP}(s)}{e^{\rho t} + \rho\,\beta_{WP}(s) - 1} + 4s\,.$$

Of course this inequality has some interest only if $r(t,s) \to +\infty$ as $t \to +\infty$, hence if $\rho \geq 0$. It also yields

$$\parallel P_t^* h\mu - \mu \parallel_{TV} \leq \left(\inf_{s>0}\left\{\frac{\rho\,\beta_{WP}(s)}{e^{\rho t} + \rho\,\beta_{WP}(s) - 1} + 4s\right\}\right)^{1/2}\,. \tag{5.16}$$

This inequality is more tractable since it extends to any h (up to a factor 2, recall (5.1)), if we can approximate $(h+1)/2$ by a sequence of h_n with $n \geq h_n \geq 1/2$ such that each $h_n\mu$ satisfies a weak Poincaré inequality with the same β_{WP}. As a consequence $((h+1)/2)\mu$ will satisfy the same inequality. We have thus shown

Theorem 5.17. *Assume that the curvature of L is bounded from below by $\rho \geq 0$. Assume that one can find a sequence h_n with $n \geq h_n \geq 1/2$ such that each $h_n\mu$ satisfies a weak Poincaré inequality with the same β_{WP}, such that $h_n \to (1+h)/2$ in $\mathbb{L}^1(\mu)$ as n goes to infinity. Then (5.16) holds.*

In particular if $s \mapsto \beta_{WP}(s)/s$ is non-increasing (at least for small s), define $\theta(u) = \inf\{s\,;\,(\beta_{WP}(s)/s) \leq 4u/\rho\}$. Then there exists a constant C such that

$$\| P_t^* h\mu - \mu \|_{TV} \leq C\, \theta^{1/2}(e^{\rho t})\,.$$

In particular if $\beta_{WP}(s) \leq cs^{-q}$ for some $q \geq 0$, $\| P_t^* h\mu - \mu \|_{TV} \leq C\, e^{-\rho t/2(1+q)}$.

Recall that $\rho > 0$ implies that μ satisfies a log-Sobolev inequality with $C_{LS} = 2/\rho$. Thus, since h_n is bounded below (by 1/2) and above (say by n), $h_n\mu$ satisfies a log-Sobolev inequality with a constant depending on n. Using this constant and estimating $\| h\mu - h_n\mu \|_{TV}$ is similar (actually a little bit worse) to the truncation method we used in subsection 2.2.

Nevertheless, since μ satisfies a log-Sobolev inequality, we have to compare the result obtained in (5.16) and the ones in Corollary 2.14, for densities h which are not in the space $\mathbb{L}\log\mathbb{L}$ (nor in any $\mathbb{L}\log_+^\beta \mathbb{L}$ for $\beta > 0$ according to (2.18) in Example 2.17). Since there is no general criterion for the weak Poincaré inequality, we shall make this comparison on examples only.

Example 5.18. Let us assume that $\mu(dx) = e^{-V(x)}dx$ is a probability measure on $\mathbb{R}$, $Lf = f'' - V'f'$ so that μ is a symmetric measure for L and $\Gamma(f) = 2(f')^2$. If $V''(x) \geq \rho > 0$, then the curvature of L is bounded from below by ρ.

Choose $h = e^V g$, with $g \geq 0$ and $\int g dx = 1$. For simplicity we assume that g is symmetric. Then it is known (see [5] Theorem 2.3) that $\nu = h\mu$ satisfies a weak Poincaré inequality with a function $\beta_{WP}(s) = C\,\beta(s)$, if β is a non increasing function, for $12B \geq C \geq (1/4)b$ where

$$b = \sup_{x>0} \frac{\nu([x,+\infty))}{\beta\left(\frac{\nu([x,+\infty))}{4}\right)} \int_0^x (1/g)(y)dy \quad \text{and} \quad B = \sup_{x>0} \frac{\nu([x,+\infty))}{\beta(\nu([x,+\infty)))} \int_0^x (1/g)(y)dy\,.$$

We immediately see that if $g \gg e^{-V}$, $((1+h)/2)\mu$ will satisfy a weak Poincaré inequality with the same β and a modified constant C, and that $((1+h\wedge n)/2)\mu$ will also satisfy a weak Poincaré inequality with the same β and a constant D which can be chosen independent of n.

As in Remark 3.16 we may evaluate

$$\int_0^x (1/g)(y)dy \sim \frac{-1}{g'(x)} \quad \text{and} \quad \int_x^{+\infty} g(y)dy \sim \frac{-g^2(x)}{g'(x)}\,,$$

provided $g' < 0$ near infinity and $\lim_{x\to+\infty}(g(x)\,g''(x)/(g')^2(x)) = 1$ (see e.g. [1] Proposition 6.4.1).

We shall give some explicit examples

- If $\nu([x,+\infty)) \sim x^{-p}$ for some $p > 0$, i.e. g behaves like $x^{-(1+p)}$ at infinity, ν satisfies a weak Poincaré inequality with $\beta_{WP}(s) \sim s^{-2/p}$, so that Theorem 5.17 furnishes an exponential decay $e^{-\rho pt/2(p+2)}$. For such a result using Corollary 2.14 we need that $h \in \mathbb{L}\log_+^\beta L$ for some $\beta > 0$, that is we need $\int V^\beta(x)g(x)dx < +\infty$. Hence if $V(x) \sim x^k$ near infinity, we need $\beta < p/k$ and obtain a decay slightly worse than $e^{-\rho pt/2(p+k)}$. In all cases (since $k \geq 2$) this is a worse decay than $e^{-\rho pt/2(p+2)}$.

If $V \sim e^x$ at infinity, the situation is still worse since Corollary 2.14 does not furnish the exponential decay which is still true according to Theorem 5.17.

- If $g(x) \sim (1/x \log^2(x))$ at infinity, we get $\beta_{WP}(u) \sim e^{2/\sqrt{u}}$. This yields a polynomial decay c/t for the total variation distance. Now it is easily seen that, if $V(x) = x^2$, $\int h \log_+^{1-\varepsilon}(\log_+(h))\, d\mu < +\infty$ for $\varepsilon > 0$ and infinite for $\varepsilon = 0$. Corollary 2.14 furnishes a decay $c/t^{(1-\varepsilon)/2}$, hence still a worse rate. Again for larger V's the result is unchanged with Theorem 5.17 and is getting worse with Corollary 2.14.

It seems in the one dimensional case that Theorem 5.17 gives better results than Corollary 2.14, in particular because it does not take into account the moments of h with respect to μ (this is not completely true since these moments have an influence on β_{WP} but we may change μ without changing nor ρ nor β_{WP}).
A better understanding of general weak Poincaré inequalities is however necessary to claim that it has to be a general fact. In addition, μ is supposed to satisfy a strong form of the log-Sobolev inequality (the Bakry-Emery condition). Finally it is quite possible that for very oscillating densities (not satisfying the conditions for the tail estimates to be true for instance) one can have finite entropy but a bad weak Poincaré function. ♢

Remark 5.19. We have previously studied the propagation of weak Poincaré inequalities along the semi-group. It is then natural to look at the propagation of Super-Poincaré inequalities. Indeed, assume that $\nu = h\mu$ satisfies a Super-Poincaré inequality (see [7] or [4] for explicit conditions on h and μ), i.e. there exists β_{SP} defined on $[1, \infty[$ such that

$$\mathrm{Var}_\nu(f) \leq \beta_{SP}(s) \int \Gamma(f) d\nu + s \left(\int |f| d\nu \right)^2$$

and if the curvature of L is bounded below by ρ, then, by the same proof as before $\nu_t = P_t^* h\mu$ satisfies a Super-Poincaré inequality with $\beta_{SP}(t,s) = \rho^{-1}(1 - e^{-\rho t}) + e^{-\rho t}\beta_{SP}(s)$. We can then as before (with the same precautions on h as before) plug these estimate into (5.4) to get

$$\| P_t^* h\mu - \mu \|_{TV} \leq \left(\inf_{s>1} \left\{ \frac{\rho\, \beta_{WP}(s)}{e^{\rho t} + \rho\, \beta_{WP}(s) - 1} + (s-1) \right\} \right)^{1/2}, \tag{5.20}$$

which does not give better result than a Poincaré inequality. ♢

Remark 5.21. Concerning the same approach for entropy, let us point out the following remarks. First recall Proposition 5.4.5 in [1] which applies here since we are in the framework called "the diffusion case" therein. Hence under the assumptions previously set we may replace the weak Poincaré inequality by a weak log-Sobolev inequality, and (up to some constants) replace β_{WP} by β_{WLS} in (5.16). This is of course not very clever since β_{WLS} is much bigger than β_{WP}. ♢

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Patrick CATTIAUX,, Ecole Polytechnique, CMAP, F- 91128 Palaiseau cedex,, and Université Paul Sabatier Institut de Mathématiques. Laboratoire de Statistique et Probabilités, UMR C 5583, 118 route de Narbonne, F-31062 Toulouse cedex 09.

E-mail address: `cattiaux@cmapx.polytechnique.fr`

Arnaud GUILLIN,, Ecole Centrale Marseille and LATP UMR CNRS 6632,, Centre de Mathematiques et Informatique Technople Chteau-Gombert,, 39, rue F. Joliot Curie, F-13453 Marseille Cedex 13.

E-mail address: `guillin@cmi.univ-mrs.fr`