# RATE OF CONVERGENCE FOR ERGODIC CONTINUOUS MARKOV PROCESSES : LYAPUNOV VERSUS POINCARÉ.

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Abstract. We study the relationship between two classical approaches for quantitative ergodic properties : the first one based on Lyapunov type controls and popularized by Meyn and Tweedie, the second one based on functional inequalities (of Poincaré type). We show that they can be linked through new inequalities (Lyapunov-Poincaré inequalities). Explicit examples for diffusion processes are studied, improving some results in the literature. The example of the kinetic Fokker-Planck equation recently studied by Hérau-Nier, Helffer-Nier and Villani is in particular discussed in the final section.

*Key words :* Ergodic processes, Lyapunov functions, Poincaré inequalities, Hypocoercivity.

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## 1. Introduction, framework and first results.

Rate of convergence to equilibrium is one of the most studied problem in various areas of Mathematics and Physics. In the present paper we shall consider a dynamics given by a time continuous Markov process $(X_t, \mathbb{P}_x)$ admitting an (unique) ergodic invariant measure $\mu$, and we will try to describe the nature and the rate of convergence to $\mu$.
In the sequel we denote by $L$ the infinitesimal generator (and $D(L)$ the extended domain of the generator, see e.g. [6]), by $P_t(x,.)$ the $\mathbb{P}_x$ law of $X_t$ and by $P_t$ (resp. $P_t^*$) the associated semi-group (resp. the adjoint or dual semi-group), so that in particular for any regular enough density of probability $h$ w.r.t. $\mu$, $\int P_t(x,.)h(x)\mu(dx) = P_t^* h \, d\mu$.

Extending the famous Doeblin recurrence condition for Markov chains, Meyn and Tweedie developed stability concepts for time continuous processes and furnished tractable methods to verify stability [22, 23]. The most popular criterion certainly is the existence of a so called Lyapunov function for the generator [23, 11], yielding exponential (or geometric) convergence, via control of excursions of well chosen functionals of the process. Sub-geometric or polynomial convergence can also be studied (see [12, 28] among others for the diffusion case). A very general form of the method is explained in the recent work by Douc, Fort and Guillin [9], and we shall now explain part of their results in more details.

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**Definition 1.1.** *Let $\phi$ be a positive function defined on $[1,+\infty[$. We say that $V \in D(L)$ is a $\phi$-Lyapunov function if $V \geq 1$ and if there exist a constant $b$ and a closed petite set $C$ such that for all $x$*

$$LV(x) \leq -\phi(V(x)) + b\,\mathbb{1}_C(x)\,.$$

*Recall that $C$ is a petite set if there exists some probability measure $a(dt)$ on $\mathbb{R}^+$ such that for all $x \in C$ , $\int_0^{+\infty} P_t(x,.)a(dt) \geq \nu(.)$ where $\nu(.)$ is a non-trivial positive measure.*

When $\phi$ is linear ($\phi(u) = au$) we shall simply call $V$ a Lyapunov function. Existence of a Lyapunov function furnishes exponential (geometric) decay [11, 23], which is a particular case of

**Theorem 1.2.** [11] *Thm 5.2.c, and* [9] *Thm 3.10 and Thm 3.12.*
*Assume that there exists some increasing smooth and concave $\phi$-Lyapunov function $V$ such that $V$ is bounded on the petite set $C$. Assume in addition that the process is irreducible in some sense (see* [11, 9] *for precise statements). Then there exists a positive constant $c$ such that for all $x$,*

$$\| P_t(x,.) - \mu \|_{TV} \leq c\, V(x)\, \psi(t)\,,$$

*where $\psi(t) = 1/(\phi \circ H_\phi^{-1})(t)$ for $H_\phi(t) = \int_1^t (1/\phi(s))ds$, and $\| \ . \|_{TV}$ is the total variation distance.*

In particular if $\phi$ is linear, $\psi(t) = e^{-\rho t}$ for some positive explicit $\rho$.
Actually the result stated in Theorem 1.2 can be reinforced by choosing suitable stronger distances (stronger than the total variation distance actually weighted total variation distances) but to the price of slower rates of convergence (see [11, 9] for details). In the same spirit some result for some Wasserstein distance is obtained in [16]. An important drawback of this approach is that there is no explicit control (in general) of $c$. One of the interest of our approach will be to give explicit constants starting from the same drift condition.

The pointwise Theorem 1.2 of course extends to any initial measure $m$ such that $\int V dm$ is finite. In particular, choosing $m = h\mu$ for some nice $h$, convergence reduces to the study of $P_t^* h$ for large $t$. Long time behavior of Markov semi-groups is known to be linked to functional inequalities. The most familiar framework certainly is the $\mathbb{L}^2$ framework and the corresponding Poincaré (or weak Poincaré) inequalities, namely

**Theorem 1.3.** *The following two statements are equivalent for some positive constant $C_P$*

***Exponential decay.*** *For all $f \in \mathbb{L}^2(\mu)$,*

$$\| P_t f - \int f d\mu \|_2^2 \leq e^{-t/C_P} \ \| f - \int f d\mu \|_2^2 \ .$$

***Poincaré inequality.*** *For all $f \in D_2(L)$ (the domain of the Fredholm extension of $L$) ,*

$$Var_\mu(f) := \| f - \int f d\mu \|_2^2 \leq C_P \int -2f\, Lf\, d\mu\,.$$

*In the sequel we shall define $\Gamma(f) = -2f\,Lf$.*

Thanks to Cauchy-Schwarz inequality and since $P_t$ and $P_t^*$ have the same $\mathbb{L}^2$ norm, a Poincaré inequality implies an exponential rate of convergence in total variation distance, at least for initial laws with a $\mathbb{L}^2$ density w.r.t. $\mu$.

As for the Meyn-Tweedie approach, one can get sufficient conditions for slower rates of convergence, namely weak Poincaré inequalities introduced by Roeckner and Wang :

**Theorem 1.4.** [25] *Thm 2.1*
*Let $N$ be such that $N(\lambda f) = \lambda^2 N(f)$ , $N(P_t f) \leq N(f)$ for all $t$ and $N(f) \geq \| f \|_2^2$.*
*Assume that there exists a non increasing function $\beta$ such that for all $s > 0$ and all nice $f$ the following inequality holds*

$$\textbf{(WPI)} \quad \| f - \int f d\mu \|_2^2 \leq \beta(s) \int \Gamma(f) d\mu + s\, N\left(f - \int f d\mu\right) .$$

*Then*

$$\| P_t f - \int f d\mu \|_2^2 \leq \psi(t)\, N\left(f - \int f d\mu\right) ,$$

*where $\psi(t) = 2 \inf\{s > 0, \beta(s) \log(1/s) \ \leq \ t\}$.*

In the symmetric case one can state a partial converse to Theorem 1.4 (see [25] Thm 2.3). Note that this time one has to assume that $N(h)$ is finite in order to get $\mathbb{L}^2$ convergence for $P_t^* h$. In general (WPI) are written with $N = \| \, . \, \|_\infty$ (or the oscillation), criteria and explicit form of $\beta$ are discussed in [25, 3]. A particularly interesting fact is that any $\mu$ on $\mathbb{R}^d$ which is absolutely continuous w.r.t. Lebesgue measure, $d\mu = e^{-F} dx$ with a locally bounded $F$, satisfies some (WPI).

Actually, as for the Meyn-Tweedie approach, one can show slower rates of convergence for less integrable initial densities, as well as some results for an initial Dirac mass. We refer to [8] sections 4,5,6 for such a discussion in particular cases, we shall continue in this paper. Actually [8] is primarily concerned with (weak) logarithmic Sobolev inequalities, that is replacing the $\mathbb{L}^2$ norm by the Orlicz norm associated to $u \to x^2 \log(1 + x^2)$ i.e. replacing $\mathbb{L}^2$ initial densities by densities with finite relative entropy (Kullback-Leibler information) w.r.t $\mu$. According to Pinsker-Csiszar inequality, relative entropy dominates (up to a factor 2) the square of the variation distance, hence Gross logarithmic Sobolev inequality (or its weak version introduced in [8]) allows to study the decay to equilibrium in total variation distance too.

Generalizations (interpolating between Poincaré and Gross) have been studied by several authors. We refer to [32, 33, 4, 5, 24] for related results on super-Poincaré and general $F$-Sobolev inequalities, as well as their consequences for the decay of the semi-group in appropriate Orlicz norms. We also refer to [1] for an elementary introduction to the standard Poincaré and Gross inequalities.

If the existence of a $\phi$-Lyapunov function is a tractable sufficient condition for the Meyn-Tweedie strategy (actually is a necessary and sufficient condition for the exponential case), general tractable sufficient conditions for Poincaré or others functional inequalities are less known (some of them will be recalled later), and in general no criterion is known (with the notable exception of the one dimensional euclidean space). This is one additional reason to understand the relationship between the Meyn-Tweedie approach and the functional inequality approach, i.e. to link Lyapunov and Poincaré. This is the aim of this paper.

Before to describe the contents of the paper, let us indicate another very attractive related problem.

If $\mu$ is symmetric and ergodic, it is known that $\int \Gamma(f) d\mu = 0$ if and only if $f$ is a constant. In the non symmetric case this result is no more true, and we shall call fully degenerate (corresponding to the p.d.e. situation) these cases.
Still in the symmetric case (or if $L$ is normal, i.e $LL^* = L^*L$), it is known that an exponential decay

$$\| P_t f - \int f d\mu \|_2^2 \leq e^{-\rho t} N \left( f - \int f d\mu \right) ,$$

for some $N$ as in Theorem 1.4, actually implies a (true) Poincaré inequality (see [25] Thm 2.3).
A similar situation is no more true in the fully degenerate case. Indeed in recent works, Hérau and Nier [18] and then Villani [30] have shown that for the kinetic Fokker-Planck equation (which is fully degenerate) the previous decay holds with $N(g) = \| \nabla g \|_2^2$ ($\mu$ being here a log concave measure, $N(g)$ is greater than the $\mathbb{L}^2$ squared norm of $g$ up to a constant), and thanks to the hypoelliptic regularization property, it also holds with $N(g) = C \operatorname{Var}_\mu(g)$ for some constant $C > 1$ (recall that if $C \leq 1$, the Poincaré inequality holds). Of course the Bakry-Emery curvature of this model is equal to $-\infty$, otherwise an exponential decay with $N(g) = C \operatorname{Var}_\mu(g)$ would imply a Poincaré inequality, even for $C > 1$, so that this situation is particularly interesting.
It turns out that this model enters the framework of Meyn-Tweedie approach as shown in [35] (also see [9]). Hence relating Lyapunov to some Poincaré in such a case (called hypocoercive by Villani) should help to understand the picture.
We shall also study this problem.

Let us briefly describe now our framework.
Recall that in all the paper $\mu$ is an invariant measure for the process with generator $L$.
The main additional hypothesis we shall make is the existence of a "carré du champ", that is we assume that there is an algebra which is a core for the generator and such that for $f$ and $g$ in this algebra

$$L(fg) = fLg + gLf + \Gamma(f, g) \tag{1.5}$$

where $\Gamma(f, g)$ is the polarization of $\Gamma(f)$. We shall also assume that $\Gamma$ comes from a derivation, i.e. for $f$, $h$ and $g$ as before

$$\Gamma(fg, h) \;=\; f\Gamma(g, h) + g\Gamma(f, h) \,. \tag{1.6}$$

The meaning of these assumptions in terms of the underlying stochastic process is explained in the introduction of [6], to which the reader is referred for more details (also see [2] for the corresponding analytic considerations).
Applying Ito's formula, we then get that for all smooth $\Psi$, and $f$ as before,

$$L\Psi(f) \;=\; \frac{\partial \Psi}{\partial x}(f)\, Lf \;+\; 1/2\, \frac{\partial^2 \Psi}{\partial x^2}(f)\, \Gamma(f) \,. \tag{1.7}$$

Our plan will be the following: in the second Section we show how to get controls in variance or in entropy starting from the result of Theorem 1.2 which will be seen to be quite sharp. The Section 3 will be devoted to the introduction of (weak) Lyapunov Poincaré inequalities, leading to tractable criteria enabling us to give explicit control of convergence via

($\phi$-)Lyapunov condition, illustrated by the examples of Section 4. The next section presents similar results for the entropy, before presenting in the final Section an application in the particular fully degenerate case.

## 2. From Lyapunov to Poincaré and vice versa.

We first show that, in the symmetric case, the Meyn-Tweedie method immediately furnishes some Poincaré inequalities.

Indeed let us assume that the hypothesis of Theorem 1.2 are fulfilled, and let $f$ be a bounded function such that $\int f d\mu = 0$. Then, if $f$ does not vanish identically, we may define $h = f_+ / \int f_+ d\mu$ which is a bounded density of probability. Thus, if $V \in \mathbb{L}^1(\mu)$, $\int hV d\mu < +\infty$. It follows that $\| P_t^* h - 1 \|_{\mathbb{L}^1(\mu)}$, which is the total variation distance between $\mu$ and the law at time $t$ (starting from $h\mu$), goes to 0 as $t \to +\infty$, with rate $c\psi(t)$ defined in Theorem 1.2. Hence for $0 < \beta < 1$,

$$\begin{aligned}
\int |P_t^* f_+ - \int f_+ d\mu|^2 \, d\mu &= \left(\int f_+ d\mu\right)^2 \int (P_t^* h - 1)^2 \, d\mu \\
&\leq \left(\int f_+ d\mu\right)^2 \int (P_t^* h - 1)^\beta \, (P_t^* h - 1)^{2-\beta} \, d\mu \\
&\leq \left(\int f_+ d\mu\right)^2 \left(\int |P_t^* h - 1| d\mu\right)^\beta \left(\int |P_t^* h - 1|^{\frac{2-\beta}{1-\beta}} d\mu\right)^{1-\beta} \\
&\leq c_\beta \, \psi^\beta(t) \left(\int f_+ V d\mu\right)^\beta \left(\int |f_+ - \int f_+ d\mu|^{\frac{2-\beta}{1-\beta}} d\mu\right)^{1-\beta} \\
&\leq c_\beta \, \psi^\beta(t) \left(\int f_+ V d\mu\right)^\beta \left(\int (2f_+)^{\frac{2-\beta}{1-\beta}} d\mu\right)^{1-\beta}
\end{aligned}$$

where we have used that $P_t^*$ is an operator with norm equal to 1 in all the $\mathbb{L}^p$'s ($p \geq 1$), the elementary $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for $p \geq 1$ and Hölder inequality.

Thus

$$\begin{aligned}
\int (P_t^* f)^2 d\mu &\leq 2 \int |P_t^* f_+ - \int f_+ d\mu|^2 \, d\mu + 2 \int |P_t^* f_- - \int f_- d\mu|^2 \, d\mu \\
&\leq 2^{4-\beta} \, c_\beta \, \psi^\beta(t) \left(\int |f| V d\mu\right)^\beta \left(\int |f|^{\frac{2-\beta}{1-\beta}} d\mu\right)^{1-\beta} .
\end{aligned}$$

In the symmetric case (or more generally the normal case) we may thus apply Theorem 2.3 in [25] so that we have shown

**Theorem 2.1.** *Under the hypotheses of Theorem 1.2, for any $f$ such that $\int f d\mu = 0$ and any $0 < \beta < 1$ it holds*

$$\int (P_t^* f)^2 d\mu \leq C_\beta \, \psi^\beta(t) \left(\int |f| V d\mu\right)^\beta \left(\int |f|^{\frac{2-\beta}{1-\beta}} d\mu\right)^{1-\beta} .$$

*The result extends to $\beta = 1$ provided $f$ is bounded, and in this case*

$$\int |P_t^* f|^2 d\mu \, \leq \, C \, (\int V d\mu) \, \parallel f \parallel_\infty^2 \, \psi(t) \, .$$

*If in addition $\mu$ is a symmetric measure for the process, then $\mu$ satisfies a Weak Poincaré Inequality with $N(f) = C(V) \parallel f \parallel_\infty^2$ and*

$$\beta(s) = s \, \inf_{u>0} \, \frac{1}{u} \, \psi^{-1}(u \, e^{(1-u/s)}) \;\; \textit{with} \;\; \psi^{-1}(a) := \inf\{b > 0 \, , \, \psi(b) \leq a\} \, .$$

*In particular if $\psi(t) = e^{-\rho t}$, $\mu$ satisfies a Poincaré inequality.*

The fact that a Lyapunov condition furnishes some Poincaré inequality in the symmetric case is already known see Wu [34, 36], but the techniques used by Wu are different and rely mainly on spectral ideas. Note also that a Lyapunov function is always in $\mathbb{L}^1(\mu)$ by integrating the Lyapunov condition, otherwise only $\phi \circ V$ is integrable w.r.t. $\mu$ (as a direct consequence of the Lyapunov inequality). In fact, the simple use, of this theorem enables us to derive very easily the correct rate of convergence to equilibrium and to extend known sharp weak Poincaré inequality in dimension one to higher dimension. The major drawback is that the constants are quite unknown in the general case, however we refer to [10] to results providing explicit constants.

The same idea furnishes without any effort a similar result for the decay of relative entropy. Indeed, if $h$ is a density of probability (w.r.t. $\mu$), using the concavity of the logarithm, we get for any $0 < \beta < 1$

$$\begin{aligned}
\int P_t^* h \, \log P_t^* h \, d\mu \;\; &= \;\; \int (P_t^* h \, - \, 1) \, \log P_t^* h \, d\mu \, + \, \int \log P_t^* h \, d\mu \\
&\leq \;\; \int (P_t^* h \, - \, 1) \, \log P_t^* h \, d\mu \, + \, \log \left( \int P_t^* h \, d\mu \right) \\
&\leq \;\; \int |P_t^* h \, - \, 1| \, |\log P_t^* h| \, d\mu \\
&\leq \;\; \left( \int |P_t^* h \, - \, 1| d\mu \right)^\beta \, \left( \int |P_t^* h \, - \, 1| \, |\log P_t^* h|^{\frac{1}{1-\beta}} \, d\mu \right)^{1-\beta} \, .
\end{aligned}$$

It is easily seen that the function $u \mapsto |u-1| |\log u|^p$ is convex on $]0, +\infty[$ for $p \geq 1$, so that

$$t \, \mapsto \, \int |P_t^* h \, - \, 1| \, |\log P_t^* h|^{\frac{1}{1-\beta}} \, d\mu$$

is decaying on $\mathbb{R}^+$. We thus have obtained

**Theorem 2.2.** *Under the hypotheses of Theorem 1.2, for any non-negative $h$ such that $\int h d\mu = 1$ and any $0 < \beta < 1$ it holds*

$$\int P_t^* h \, \log P_t^* h \, d\mu \, \leq \, C_\beta \, \psi^\beta(t) \, \left( \int h V d\mu \right)^\beta \, \left( \int |h - 1| |\log h|^{\frac{1}{1-\beta}} d\mu \right)^{1-\beta} \, .$$

*The result extends to $\beta = 1$ provided $h$ is bounded, and in this case*

$$\int P_t^* h \, \log P_t^* h \, d\mu \, \leq \, C \, (\int V d\mu) \, \parallel h \parallel_\infty \, \log(\parallel h \parallel_\infty) \, \psi(t) \, .$$

Note that (in the symmetric case) there is no analogue converse result for relative entropy as for the variance. Indeed recall that if $h$ is a density of probability, $\int h \log h d\mu \leq \text{Var}_\mu(h)$, hence relative entropy is decaying exponentially fast, controlled by the initial variance of $h$ as soon as a Poincaré inequality holds. But it is known that a Poincaré inequality may hold without log-Sobolev inequality. However, starting from Theorem 2.2 one can prove some (loose) weak log-Sobolev inequality, see [8] sections 4 and 5.

Of course Theorem 2.1 and Theorem 2.2 furnish (in the non-symmetric case as well) controls depending on the integrability of $V$. For instance if $V$ has all polynomial moments, we may control $\int |f| V d\mu$ by some $\int |f|^p d\mu$ in Theorem 2.1 and if $\int e^{qV} d\mu < +\infty$ for some $q > 0$ we may control $\int hV d\mu$ by the $u \log_+ u$ Orlicz norm of $h$ in Theorem 2.2. Recall that a Lyapunov function is in $\mathbb{L}^1(\mu)$.

We have seen that, in the symmetric case, the existence of a Lyapunov function implies a Poincaré inequality. Let us briefly discuss some possible converse.
If $P_t$ is $\mu$ symmetric for some $\mu$ satisfying a Poincaré inequality, then we know that $P_t$ has a spectral gap, say $\theta$. Let $f$ be an eigenfunction associated with the eigenvalue $-\theta$, i.e. $Lf + \theta f = 0$. If the semi-group is regularizing (in the ultracontractive case for instance), $f$ has to be bounded. Assume that $f$ is actually bounded and say continuous. Since $\int f d\mu = 0$, changing $f$ into $-f$ if necessary, we may assume that $\sup f \geq -\inf f = -M$. Then define $g = f + 1 + M$. $Lg = -\theta g + \theta(1 + M)$, so that for all $0 < \kappa < 1$,

$$Lg \, \leq \, -\kappa \, \theta \, g \, + \, \theta \, \kappa \, (1 + M) \, \mathbb{1}_C$$

with $C = \{f \leq (1 + M)\kappa/(1 - \kappa)\}$ a non empty (and non full) closed set.
Of course the previous discussion only covers very few cases, but it indicates that some converse has to be studied.

Another possible way to prove a converse result is the following. Assume that $d\mu(x) = e^{-2V(x)} dx$ where $V$ is $C^3$ and such that

$$|\nabla V|^2(x) \, - \, \Delta V(x) \, \geq \, -C_{min} \, > -\infty \tag{2.3}$$

for a nonnegative $C_{min}$ so that the process defined by (recalling that $B_t$ is an usual Brownian motion in $\mathbb{R}^d$)

$$dX_t \, = \, dB_t \, - \, (\nabla V)(X_t)dt \quad , \quad Law(X_0) = \nu \tag{2.4}$$

has a unique non explosive strong solution. Assume also that $\mu$ satisfies a Poincaré inequality. The difficulty here is that by using Poincaré inequality we inherit a control for all smooth $f$ with finite variance as

$$\text{Var}_\mu(P_t f) \leq e^{-\lambda t} \text{Var}_\mu(f).$$

But a drift inequality concerns the generator and its behavior towards some chosen function for all $x$. However it is known, see Down-Meyn-Tweedie [11] (Th. 5.2, Th. 5.3 and the remarks after Th. 5.3), that the existence of a drift condition is ensured by

$$\|P_t \delta_x - \pi\|_{TV} \leq M(x) \rho^t$$

for some larger than 1 function $M$ and $\rho < 1$. But it is once again a control local in $x$. In this direction, one can show (see [26, Theorem 3.2.7]) that $\text{Ent}_\mu P_t \delta_x$ is finite for all $t > 0$. But control in entropy is not useful as our assumption is a Poincaré inequality and thus a control in $L^2$ is needed. Actually the proof of Royer can be used in order to get the following result. Replacing the convex $\gamma$ therein by $\gamma(y) = y^2$ we obtain

$$\int (P_t\delta_x)^2 d\mu \leq Z\, e^{2V(x)}\, \mathbb{E}\left[e^{-2v(B_t)}\, e^{-\frac{1}{2}\int_0^t [|\nabla V|^2 - \Delta V](B_s)ds}\right] \leq Z\, e^{2V(x)}\, e^{\frac{1}{2}C_m t}$$

where $e^{-2v(y)} = (2\pi t)^{-d/2}\, e^{-|y-x|^2/2t}$. By the Poincaré inequality, we then get that for some $\lambda$ and $t_0$

$$\mathrm{Var}_\mu(P_t\delta_x) \leq e^{-\lambda(t-t_0)}\mathrm{Var}_\mu(P_{t_0}\delta_x) \leq Z\, e^{2V(x)}\, e^{\frac{1}{2}C_m t_0}\, e^{-\lambda(t-t_0)}$$

which ends the work as a control in $L^2$ enables us to control the $L^1$ distance, and we thus get the existence of a Lyapunov function. However, the Lyapunov function $V$ is not available in close form (see [21, 11] for a precise formula).

Finally, let us mention that it is not possible to get a converse result as previously starting from a weak Poincaré inequality as 1) we do not know how to control $\|P_t\delta_x\|_\infty$ (even if it should be controlled in many case) and 2) there is no converse part in the Meyn-Tweedie framework (even in the discrete time case) for sub geometric convergence in total variation towards $\phi$-Lyapunov condition.

## 3. From Lyapunov to Poincaré. Continuation.

Since we have seen in the previous section that the existence of Lyapunov functions furnishes functional inequalities in the symmetric case, in this section we shall study relationship between some modified Poincaré inequality (still yielding exponential decay) and the existence of a Lyapunov function (with $\phi(u) = \alpha u$), without assuming symmetry.

### 3.1. Lyapunov-Poincaré inequalities.

The key tool is the following elementary lemma

**Lemma 3.1.** *For $\Psi$ smooth enough, $W \in D(L)$ and $f \in L^\infty$, define $I_W^\Psi(t) = \int \Psi(P_t f)\, W\, d\mu$. Then for all $t > 0$,*

$$\frac{d}{dt}\, I_W^\Psi(t) \;=\; -\int 1/2\, \Psi''(P_t f)\, \Gamma(P_t f)\, W\, d\mu \;+\; \int L^* W\, \Psi(P_t f)\, d\mu\,.$$

*In particular for $\Psi(u) = u^2$ we get (denoting simply by $I_W$ the corresponding $I_W^\Psi$)*

$$I'_W(t) \;=\; -\int \Gamma(P_t f)\, W\, d\mu \;+\; \int L^* W\, P_t^2 f\, d\mu\,.$$

*Proof.* Recall that $\int L(\Psi(g)W)d\mu = 0$. Using (1.5) and (1.7) with $g = P_t f$ we thus get

$$\begin{aligned}
\frac{d}{dt}\, I_W^\Psi(t) \;&=\; \int \Psi'(P_t f)\, L P_t f\, W\, d\mu \\
&=\; \int \big(L(\Psi(P_t f)) - 1/2\Psi''(P_t f)\Gamma(P_t f)\big)\; W\, d\mu
\end{aligned}$$

hence the result. ∎

This Lemma naturally leads to the following Definition and Proposition

**Definition 3.2.** *We shall say that $\mu$ satisfies a (W)-Lyapunov-Poincaré inequality, if there exists $W \in D(L)$ with $W \geq 1$ and a constant $C_{LP}$ such that for all nice $f$ with $\int f d\mu = 0$,*

$$\int f^2 \, W \, d\mu \, \leq \, C_{LP} \int \left( W \, \Gamma(f) \, - \, f^2 \, LW \right) \, d\mu \, .$$

Here and in all the paper, "nice" means that $f$ belongs to the domain of the generator and the set of nice functions is everywhere dense in the domain of the Dirichlet form (for instance smooth compactly supported functions in the usual euclidean cases).

**Proposition 3.3.** *The following statements are equivalent*

- *$\mu$ satisfies a (W)- Lyapunov-Poincaré inequality,*
- *$\int (P^*_t f)^2 \, W \, d\mu \, \leq \, e^{-(t/C_{LP})} \int f^2 W d\mu$ for all $f$ with $\int f d\mu = 0$.*

*In particular for all $f$ such that $\int f^2 W d\mu < +\infty$, $P_t f$ and $P^*_t f$ go to $\int f d\mu$ in $\mathbb{L}^2(\mu)$ with an exponential rate.*

*Proof.* We consider $I^*_W(t)$ replacing $P_t$ by $P^*_t$. Taking the derivative at time $t = 0$ furnishes as usual the converse part. For the direct one, we only have to use Gronwall's lemma. Indeed the Lyapunov-Poincaré inequality yields $(I^*_W)'(t) \, \leq \, - \, (1/C_{LP}) \, I^*_W(t)$. Since $I^*_W(t)$ is non-negative, this shows that $I^*_W(t)$ is non increasing, hence converges to some limit as $t$ tends to infinity, and this limit has to be 0 (otherwise $I^*_W$ would become negative). Since $I^*_W(+\infty) = 0$, the result follows by integrating the differential inequality above. $\square$

Note that a Lyapunov-Poincaré inequality is not a weighted Poincaré inequality (we still assume that $\int f d\mu = 0$) and depends on the generator $L$ (not only on the carré du champ). But as we already mentioned, Theorem 2.3 in [25] tells us that, in the symmetric case, if

$$\int P^2_t f \, d\mu \, \leq \, c \, e^{-\delta t} \, \parallel f \parallel^2_\infty$$

for all $f$ such that $\int f d\mu = 0$, then $\mu$ satisfies the usual Poincaré inequality, with $C_P = 1/\delta$. Hence

**Corollary 3.4.** *If $L$ is $\mu$ symmetric and $\mu$ satisfies a (W)-Lyapunov-Poincaré inequality for some $W \in \mathbb{L}^1$, then $\mu$ satisfies the ordinary Poincaré inequality, with $C_P = C_{LP}$.*

Now we turn to sufficient conditions for a Lyapunov-Poincaré inequality to hold.
Recall that we called $V$ a Lyapunov function if $LV \, \leq -\alpha V + b \mathbb{1}_C$. Note that integrating this relation w.r.t. $\mu$ yields $\alpha \int V d\mu \leq b\mu(C)$, so that, first we have to assume that $\int V d\mu < +\infty$, second since $V \geq 1$, $b$ and $\mu(C)$ have to be positive.
Before stating the first result of this section we shall introduce some definition

**Definition 3.5.** *Let $U$ be a subset of the state space $E$. We shall say that $\mu$ satisfies a local Poincaré inequality on $U$ if there exists some constant $\kappa_U$ such that for all nice $f$ with $\int_E f d\mu = 0$,*

$$\int_U f^2 \, d\mu \, \leq \, \kappa_U \int_E \Gamma(f) d\mu \, + \, (1/\mu(U)) \left( \int_U f \, d\mu \right)^2 \, .$$

Notice that the energy integral in the right hand side is taken over the whole space $E$. We may now state

**Theorem 3.6.** *Assume that there exists a Lyapunov function $V$ i.e. $LV \leq -2\alpha V + b\mathbb{1}_C$ for some set $C$ (non necessarily petite).*
*Assume that one can find a (large) set $U$ such that $\mu$ satisfies a local Poincaré inequality on $U$.*
*Assume in addition that*

1. *either $U$ contains $C' = C \cap \{V \leq b/\alpha\}$ and $\alpha\,\mu(U) > b\mu(U^c)$ ,*
2. *or $U$ contains $\{V \leq b/\alpha\}$ and $\mu(U) > \mu(U^c)$.*

*Then one can find some $\lambda > 0$ such that if $W = V + \lambda$, $\mu$ satisfies a (W)-Lyapunov-Poincaré inequality.*
*More precisely, corresponding to the two previous cases one can choose*

1. *$\lambda = (b\kappa_U - 1)_+$ and $1/C_{LP} = \alpha\left(1 - \frac{b\,\mu(U^c)}{\alpha\,\mu(U)}\right)/(1+\lambda)$,*
2. *or $\lambda = (b\kappa_U - 1)_+$ and $1/C_{LP} = \alpha\left(1 - \frac{\mu(U^c)}{\mu(U)}\right)/(1+\lambda)$.*

*Proof.* First remark the following elementary fact : define $C' = C \cap \{V \leq b/\alpha\}$. Then $LV \leq -\alpha V + b\mathbb{1}_{C'}$, that is we can always assume that $C$ is included into some level set of $V$. In the sequel $\theta = b/\alpha$. First we assume that $U$ contains $\{V \leq b/\alpha\}$, so that it contains $C'$. Let $\int f d\mu = 0$. Then for all $\lambda > 0$ it holds

$$\begin{aligned}\int f^2\,(V+\lambda)\,d\mu &\leq (1+\lambda)\int f^2\,V\,d\mu \\ &\leq (1+\lambda)/\alpha\int f^2\,\left(-L(V+\lambda) + b\mathbb{1}_{C'}\right)\,d\mu\,.\end{aligned}$$

But since $\int_U f d\mu = -\int_{U^c} f d\mu$ it holds

$$\begin{aligned}\int_{C'} f^2\,d\mu &\leq \int_U f^2\,d\mu \leq \kappa_U\int\Gamma(f)d\mu + (1/\mu(U))\left(\int_U f\,d\mu\right)^2 \\ &\leq \kappa_U\int\Gamma(f)d\mu + (1/\mu(U))\left(\int_{U^c} f\,d\mu\right)^2 \\ &\leq \kappa_U\int\Gamma(f)d\mu + (\mu(U^c)/\mu(U))\left(\int_{U^c} f^2\,d\mu\right) \\ &\leq \kappa_U\int\Gamma(f)d\mu + (\mu(U^c)/\theta\,\mu(U))\left(\int_{U^c} f^2\,V\,d\mu\right),\end{aligned}$$

where we used $V/\theta \geq 1$ on $U^c$. So, if we choose $\lambda = (b\kappa_U - 1)_+$ we get

$$b\int_{C'} f^2\,d\mu \leq \int\Gamma(f)\,(V+\lambda)\,d\mu + (b\mu(U^c)/\theta\,\mu(U))\left(\int f^2\,V\,d\mu\right).$$

It yields

$$\int\left(W\,\Gamma(f) - f^2\,LW\right)\,d\mu \geq \alpha\left(1 - \frac{b\,\mu(U^c)}{\theta\,\alpha\,\mu(U)}\right)\int f^2\,V\,d\mu\,,$$

hence the result with $1/C_{LP} = \alpha\left(1 - \frac{b\,\mu(U^c)}{\theta\,\alpha\,\mu(U)}\right)/(1+\lambda)$ since $\theta = b/\alpha$.

If $U$ does not contain the full level set $\{V \leq b/\alpha\}$ but only $C'$, the only difference is that we cannot divide by $\theta$, hence the result. □

**Remark 3.7.** The conditions on $U$ are not really difficult to check in practice. We have included the first situation because it covers cases where a bounded Lyapunov function exists, hence we cannot assume in general that $U$ contains some level set. The second case is the usual one on euclidean spaces when $V$ goes to infinity at infinity, so that we may always choose $U$ as a regular neighborhood of a level set of $V$.

One may think that the constant $C_{LP}$ we have just obtained is a disaster. In particular, contrary to the Meyn-Tweedie approach, the exponential rate given by $C_{LP}$ does not only depend on $\alpha$ but also on $b, C, V$. But recall that in Meyn-Tweedie approach the non explicit constant in front of the geometric rate depends on all these quantities (while we here have an explicit $\int f^2 W d\mu$). In addition to the stronger type convergence ($\mathbb{L}^2$ type), one advantage of Theorem 3.6 is perhaps to furnish explicit (though disastrous) constants.

3.2. **A general sufficient condition for a Poincaré inequality.** As we previously said, there are some situations for which a tractable criterion for Poincaré's inequality is known.

The most studied case is of course the euclidean space equipped with an absolutely continuous measure $\mu(dx) = e^{-2F}dx$ and the usual $\Gamma(f) = |\nabla f|^2$. Dimension one is the only one for which exists a general necessary and sufficient condition (Muckenhoupt criterion, see [1] Thm 6.2.2). A more tractable sufficient condition can be deduced (see [1] Thm 6.4.3) and can be extended to all dimensions using some isometric correspondence between Fokker-Planck and Schrödinger equations, namely $|\nabla F|^2(x) - \Delta F(x) \geq b > 0$ for all $|x|$ large enough (for a detailed discussion of the spectral theory of these operators see [17] in particular Proposition 3.1). Actually this condition can be extended to $\mu = e^{-2F}\nu$ if $\nu$ satisfies some log-Sobolev inequality, see [15] (as explained in [7] Prop 4.4).

We shall see now that these conditions actually are of Lyapunov type, hence can be extended to a very general setting.

**Lemma 3.8.** *Let $F$ be a nice enough function. Then if $V = e^{aF}$,*

$$LV - \Gamma(F,V) = aV\left(LF + (\frac{a}{2} - 1)\,\Gamma(F)\right).$$

The proof is immediate using (1.5), (1.6) and (1.7). We may thus deduce

**Theorem 3.9.** *Let $\nu$ be a ($\sigma$-finite positive measure) and $L$ be $\nu$ symmetric. Let $F \in D(L)$ be non-negative and such that $\mu = (1/Z_F)\,e^{-2F}\nu$ is a probability measure for some normalizing constant $Z_F$. For $0 < a < 2$ define*

$$H_a = LF + (\frac{a}{2} - 1)\,\Gamma(F)$$

*and for $\alpha > 0$, $C(a,\alpha) = \{H_a \geq -(\alpha/a)\}$.*

*Assume that for some $a$ and some $\alpha$, $H_a$ is bounded above on $C(a,\alpha)$.*

*Assume in addition that for $\varepsilon > 0$ small enough one can find a large subset $U \supseteq C(a,\alpha)$ with $\mu(U) \geq 1-\varepsilon$ such that $F$ is bounded on $U$, and $\mu$ satisfies the local Poincaré inequality on $U$.*

*Then $\mu$ satisfies the Poincaré inequality.*

*Proof.* Recall that the operator $L_F f = Lf - \Gamma(F,f)$ is $\mu$ symmetric. According to Lemma 3.8, if $V = e^{aF}$, $L_F V \leq -\alpha V$ outside $C(a,\alpha)$. But $H_a$ and $V$ being bounded on $C(a,\alpha)$, one can find some $b$ such that $V$ is a Lyapunov function. We may thus apply Theorem 3.6 which tells us that $\mu$ satisfies a Lyapunov-Poincaré inequality. Since we are in the symmetric case, we may conclude thanks to Corollary 3.4. $\square$

We defer to Section 4 further results, applications and comments of this Theorem.

### 3.3. **Weak Lyapunov-Poincaré inequalities and weak Poincaré inequalities.**

We shall conclude this section by extending the two previous subsections to the more general weak framework. We start with the following extension of Theorem 3.6

**Theorem 3.10.** *Assume that there exists a $2\phi$-Lyapunov function $V$ i.e. $LV \leq -2\phi(V)+b\mathbb{1}_C$ for some set $C$ (non necessarily petite). Recall that $\phi(u) \geq R > 0$.*
*Assume that one can find a (large) set $U$ such that $\mu$ satisfies a local Poincaré inequality on $U$.*
*Assume in addition that*

(1) *either $U$ contains $C' = C \cap \{\phi(V) \leq b\}$ and $R\,\mu(U) > b\mu(U^c)$ ,*
(2) *or $U$ contains $\{\phi(V) \leq b\}$ , $\phi$ is increasing and $\phi(b)\,\mu(U) > b\,\mu(U^c)$.*

*Then for $\lambda = (b\kappa_U - 1)_+$ and $W = V + \lambda$, $\mu$ satisfies a (W)-weak-Lyapunov-Poincaré inequality, i.e. for all $f$ with $\int f d\mu = 0$ and all $s > 0$,*

$$\int f^2 W d\mu \,\leq\, C_w\,\beta_W(s)\,\left(\int \, \big(W\,\Gamma(f)\,-\,f^2\,LW\big)\,d\mu\right)\,+\,s\,\parallel f\parallel_\infty^2$$

*with $\beta_W(s) = \inf\{u\,;\,\int_{V>u\phi(V)} V d\mu \leq s\}$, and where $C_w$ is given in the two corresponding cases by*

(1) $1/C_w = \left(1 - \frac{b\,\mu(U^c)}{R\,\mu(U)}\right)/(1+\lambda)$,
(2) *or* $1/C_w = \left(1 - \frac{b\mu(U^c)}{\phi(b)\mu(U)}\right)/(1+\lambda)$.

*Proof.* Looking at the proof of Theorem 3.6 we immediately see that, if $V$ is a $2\phi$-Lyapunov function (recall definition 1.1), then we may replace $C$ by $C' = C \cap \{\phi(V) \leq b\}$. In the first situation we obtain as in the proof of Theorem 3.6

$$\int_{C'} f^2\,d\mu \leq \kappa_U \int \Gamma(f)\,d\mu \,+\, (\mu(U^c)/R\,\mu(U))\,\left(\int f^2\,\phi(V)\,d\mu\right)\,,$$

so that

$$\int \,\big(W\,\Gamma(f)\,-\,f^2\,LW\big)\,d\mu \,\geq\, \left(1 - \frac{b\,\mu(U^c)}{R\,\mu(U)}\right)\int f^2\,\phi(V)\,d\mu\,.$$

In the second case we may replace $R$ by $\phi(b)$. It remains to note that

$$\int f^2 V d\mu \leq u \int_{V\leq u\phi(V)} f^2\phi(V)d\mu + \parallel f\parallel_\infty^2\,\left(\int_{V>u\phi(V)} V d\mu\right)$$

for all $u > 0$. $\square$

**Remark 3.11.** It is difficult to compare in full generality the previous weak Poincaré inequality with the one obtained in Theorem 2.1. More precisely, the previous result furnishes some decay for the variance (as Theorem 2.1) but the rate explicitly depends on $V$ (while

$V$ only appears through the constants in Theorem 2.1. We shall thus make a more accurate comparison on examples later on.
It is however worthwhile noticing that, in the first case, we do not need to impose any condition on $\phi$ except that $\phi$ is bounded below by some positive constant.
Also remark that Theorem 3.1 in [25] establishes a weak Poincaré inequality assuming that one can find an exhausting sequence of sets $U_n$ such that $\mu$ satisfies a local Poincaré inequality on each $U_n$. Here we only need one set $U$ (but large enough). Actually in the examples we have in mind the assumption in [25] is satisfied, but we shall see that we can improve upon the function $\beta_W$.

We shall now extend Theorem 3.9 to the weak context.

**Corollary 3.12.** *Let $\nu$ be a ($\sigma$-finite positive measure) and $L$ be $\nu$ symmetric. Let $F \in D(L)$ be non-negative and such that $\mu = (1/Z_F)\, e^{-2F} \nu$ is a probability measure for some normalizing constant $Z_F$. We assume in addition that there exists $p < 2$ such that $\int e^{-pF} d\nu = c_p < +\infty$.*
*Let $\eta$ be a non-increasing function such that $u\,\eta(\log(u))$ is bounded from below by a positive constant. For $0 < a < 2$ define $H_a = LF + (\frac{a}{2} - 1)\,\Gamma(F)$ and $C(a) = \{H_a \geq -\eta(F)\}$.*
*Assume that for some $0 < a < 2 - p$ , $H_a$ is bounded above on $C(a)$. Assume in addition that for $\varepsilon > 0$ small enough one can find a large subset $U \supseteq C(a)$ with $\mu(U) \geq 1 - \varepsilon$ and such that $F$ is bounded on $U$, and $\mu$ satisfies a local Poincaré inequality on $U$.*
*Then $\mu$ satisfies a weak Lyapunov-Poincaré, with $W = e^{aF} + \lambda$ (for some positive $\lambda$), inequality with*

$$\beta_W(s) \;=\; \frac{2}{\left(a\,\eta\left(\frac{\log(c_p/s)}{2-a-p}\right)\right)} \tag{3.13}$$

*hence for $\int f d\mu = 0$, $\int (P_t^* f)^2\, d\mu \;\leq\; \int (P_t^* f)^2\, W\, d\mu \;\leq\; \xi(t)\ \parallel f \parallel_\infty^2$ with*

$$\xi(t) = 2 \inf\{r > 0\,;\, -C_w\, \beta_W(r)\, \log(r) \;\leq\; t\}\,.$$

*Finally $\mu$ satisfies a weak Poincaré inequality with*

$$\beta(s) = s \inf_{u>0} \xi^{-1}(u\, e^{(1-u/s)}) \;\; \textit{with} \;\; \xi^{-1}(a) := \inf\{b > 0\,,\, \xi(b) \leq a\}\,.$$

We easily remark that $\beta$ and $\beta_W$ are of the same order and change only through constants.

*Proof.* With our hypotheses, for $0 < a < 2$, $e^{aF}$ is a $2\phi$-Lyapunov function for $\phi(u) = \frac{1}{2}\, au\, \eta(\log(u)/a)$. Recall that we do not need here $\phi$ to be increasing nor concave. We may thus apply Theorem 3.10 yielding some weak Lyapunov-Poincaré inequality for $\mu$.
We shall describe the function $\beta_W$. Recall that

$$\begin{aligned}
\beta_W(s) &= \inf\{u\,; \int_{V > u\phi(V)} V d\mu \leq s\} \\
&= \inf\{u\,; \int_{2 > au\eta(F)} e^{(a-2)F} d\nu \leq s\} \\
&= \inf\{u\,; \int_{F > \eta^{-1}(2/au)} e^{(a-2)F} d\nu \leq s\}\,.
\end{aligned}$$

But if $2-a=p+m$,

$$\int_{F>\eta^{-1}(2/au)} e^{(a-2)F} d\nu \quad \leq \quad \int e^{-pF}\, e^{-m\eta^{-1}(2/au)}\, d\nu$$

from which we deduce that $\beta_W(s) \leq 2/(a\eta(\frac{1}{m}\log(c_p/s)))$.

Using Lemma 3.1 we deduce as usual (see e.g. the proof of Theorem 2.1 in [25]) that $\int (P_t^* f)^2\, W\, d\mu \leq \xi(t) \parallel f \parallel_\infty^2$ with

$$\xi(t) = 2\inf\{r>0\,;\, -C_w\, \beta_W(r)\, \log(r) \leq 2t\} \tag{3.14}$$

for $\int f d\mu = 0$, $W$ and $C_w$ being as in the previous Theorem. □

**Remark 3.15.** In view of Theorem 2.1 it is interesting to replace the $\mathbb{L}^\infty$ norm above by $\mathbb{L}^p$ norms, with $p>2$. In the case of usual weak Poincaré inequalities it is known that we may replace the $\mathbb{L}^\infty$ norm by a $\mathbb{L}^p$ norm just changing the $\beta$ into $\beta_p(s) = c\,\beta(c' s^q)$ for some constants $c$ and $c'$, and $1/p+1/q=1$ (see e.g. [37] Theorem 29 for a more general result). But the proof in [37] (inspired by [8] Theorem 3.8) lies on a Capacity-Measure characterization of these inequalities introduced in [3].

The situation here is more complicated and a direct modification of the weak-Lyapunov-Poincaré inequality seems to be difficult. However, since we are interested in the rate of convergence to the equilibrium, we may mimic the truncation argument in [8]. Namely, let $f$ be such that $\int f d\mu = 0$, denote by $f_K = f \wedge K \vee -K$ and $m_K = \int f_K d\mu$, then if a weak-Lyapunov-Poincaré inequality holds we get for all $p>1$,

$$\begin{aligned}
\int (P_t^* f)^2 d\mu \;&\leq\; 2\,\left(\int (P_t^*(f_K - m_K))^2 W d\mu + \int (P_t^*(f - f_K + m_K))^2 d\mu\right)\\
&\leq\; 2\,\left(\int (P_t^*(f_K - m_K))^2 W d\mu + \int (f - f_K + m_K)^2 d\mu\right)\\
&\leq\; 2\,\xi(t)\, K^2 + 4\, \int_{|f|>K} (|f|-K)^2\, d\mu \,+\, 4 m_K^2\\
&\leq\; 2\,\xi(t)\, K^2 + 4\, \int_{|f|>K} (|f|-K)^2\, W d\mu \,+\, 4\, \left(\int_{|f|>K} ||f|-K| d\mu\right)^2\\
&\leq\; 2\,\xi(t)\, K^2 + 8\, \int_{|f|>K} (|f|-K)^2\, d\mu\\
&\leq\; 2\xi(t)\, K^2 + 8 \left(\int |f|^{2p} d\mu\right)\, K^{-2p/q}\,.
\end{aligned}$$

Now optimizing in $K$ furnishes

$$\int (P_t^* f)^2 d\mu \,\leq\, C\, \xi^{1/q}(t)\, \left(\int |f|^{2p} d\mu\right)^{1/p} \tag{3.16}$$

which is quite the result in Theorem 2.1, but with explicit constants.

**Remark 3.17.** It is perhaps more natural to try to obtain directly a weak Poincaré inequality starting from the existence of a $\phi$-Lyapunov function as follows.

For $\int f d\mu = 0$, we have

$$\int f^2 d\mu \leq \int \frac{-LV}{\phi(V)} f^2\, d\mu + \int f^2 \frac{b}{\phi(V)}\, \mathbb{1}_C\, d\mu\,.$$

We know how to manage the second term if a local Poincaré inequality holds, hence we focus on the first term in the right hand side of the previous inequality.

Assume that $L$ is $\mu$-symmetric. Integrating by parts we get

$$\int \frac{-LV}{\phi(V)} f^2 \, d\mu \;\; = \;\; \int \left( \frac{f\,\Gamma(f,V)}{\phi(V)} \; - \; \frac{f^2\phi'(V)\Gamma(V)}{2\phi^2(V)} \right) d\mu$$

but thanks to our hypotheses

$$\frac{f\,\Gamma(f,V)}{\phi(V)} \leq \frac{a}{2}\,\Gamma(f) + \frac{1}{2a}\,\frac{f^2\,\Gamma(V)}{\phi^2(V)}$$

for all $a > 0$ so that

$$\int \frac{-LV}{\phi(V)} f^2 \, d\mu \;\; \leq \;\; \int \frac{a}{2}\,\Gamma(f) d\mu \, + \, \int \left( \frac{f^2\,\Gamma(V)}{\phi^2(V)} \right) \left( \frac{1}{2a} - \phi'(V) \right) d\mu$$

Unless $\phi$ is linear, $\liminf \phi' = 0$ at infinity, so that we get an extra term that cannot be controlled. Of course if $\phi$ is linear, we may choose $a$ for this term to vanish identically, and so obtain another proof of the Poincaré inequality for $\mu$ (with more easily calculable constants).

## 4. **Examples.**

Due to the local Poincaré property, the most natural framework is the euclidean space $\mathbb{R}^d$. It will be our underlying space in all examples, but in many cases results extend to a Riemanian manifold as well.

4.1. **General weighted Poincaré inequalities.** Let $F$ be a smooth enough non-negative function such that $\mu = (1/Z_F)\, e^{-2F} dx$ is a probability measure. We may also assume that $F(x) \to +\infty$ as $x \to \infty$, so that the level sets of $F$ are compact. If

- either $\Delta\, F - \, |\nabla F|^2$ is bounded from above,
- or $\int |\nabla F|^2 d\mu \, < +\infty$ ,

it is known than one can build a conservative (i.e. non exploding) $\mu$ symmetric diffusion process with generator $L_F = \frac{1}{2}\,\Delta - \nabla F.\nabla$. We shall assume for simplicity that the first condition holds.

Assume in addition that

$$\liminf_{|x| \to +\infty} \left( |\nabla F|^2 \, - \, \Delta\, F \right) = \alpha > 0\,.$$

We may thus apply Theorem 3.9, with $L = \frac{1}{2}\,\Delta$, $\nu$ the Lebesgue measure (which is known to satisfy a Poincaré inequality on euclidean balls of radius $R$ with $C_P = CR^2$ , $C$ being universal, and for $\Gamma(f) = |\nabla f|^2$), $U$ a large enough ball, $a = 1$. Indeed, since $\nu$ satisfies a (true) Poincaré inequality on $U$, $\mu$ which is a log-bounded perturbation of $\nu$ on $U$ also satisfies a Poincaré inequality on $U$, hence a local one (since the energy on $U$ is smaller than the one on the full $E$). This yields

**Corollary 4.1.** *If $F$ is a $C^2$ non-negative function such that, $F(x) \to +\infty$ as $x \to \infty$, $\int e^{-2F} dx < +\infty$ and*

- *$\Delta\, F - \, |\nabla F|^2$ is bounded from above, and ,*
- *$\liminf_{|x| \to +\infty} \left( |\nabla F|^2 \, - \, \Delta\, F \right) = \alpha > 0\,.$*

*Then the following (weighted) Poincaré inequality holds for all $f$ smooth enough and some $C_P$,*

$$\int f^2\, e^{-2F}\, dx \,\leq\, C_P \int |\nabla f|^2\, e^{-2F}\, dx \,+\, \frac{\left(\int f\, e^{-2F}\, dx\right)^2}{\left(\int e^{-2F} dx\right)}\,.$$

This corollary immediately extends to uniformly elliptic operators in divergence form. The degenerate case is more intricate. Indeed, according to results by Jerison, Franchi, Lu ([19, 13, 20]) a Poincaré inequality holds on small metric balls for more general operators of locally subelliptic type. Let us describe the framework we are interested in.
Let $X_1, ..., X_m$ be $C^\infty$ vector fields defined on $\mathbb{R}^d$. We shall assume for simplicity that they are bounded with all bounded derivatives. We shall make the following Hörmander type assumption :

**Assumption 4.2.** *there exists $N \in \mathbb{N}^*$ and $c > 0$ such that for all $x$ and all $\xi \in R^d$, $\sum_Y \langle Y(x), \xi\rangle^2 \,\geq\, c|\xi|^2$, where the sum is taken over all Lie brackets $Y = [X_{i_1}, [...X_{i_k}]]$ of length less than or equal to $N$.*

This assumption is enough for ensuring that the natural associated subriemanian metric $\rho$ is locally equivalent to the usual one (see e.g. [13] Theorem 2.3). According e.g. to Theorem C in [20] (a similar result was first obtained by Jerison), the Lebesgue measure $dx = \nu$ satisfies a Poincaré inequality on small metric balls $B_\rho(y, s)$ for $s$ small enough and $\Gamma(f) = \sum_{i=1}^m |X_i f|^2$. But here we want some local Poincaré inequality on some large set.
If we replace the euclidean space by a connected unimodular Lie group with polynomial volume growth equipped with left invariant vector fields $X_1, ..., X_m$ generating the Lie algebra of $E$, then it is known that a Poincaré inequality holds for all metric balls (the result is due to Varopoulos and we refer to [27] p.275 for explanations). But in the euclidean case we can show that Lebesgue measure satisfies some local Poincaré inequality on euclidean balls centered at the origin.
Indeed let $|.|$ stands for the euclidean norm. Recall that there exist $R$ and $r$ such that

$$\{|x| \leq r\} \,\subset B_\rho(0, s) \,\subset \{|x| \leq R\}\,.$$

If $\int_{|x|\leq N} f dx = 0$, then for all $a$ it holds

$$\begin{aligned}
\int_{|x|\leq N} f^2(x)\, dx &= \int_{|x|\leq r} f^2(Nx/r)\, (\frac{N}{r})^d\, dx \\
&\leq \int_{|x|\leq r} (f(Nx/r) - a)^2\, (\frac{N}{r})^d\, dx \\
&\leq \int_{B_\rho(0,s)} (f(Nx/r) - a)^2\, (\frac{N}{r})^d\, dx\,,
\end{aligned}$$

so that if we choose $a = (\int_{B_\rho(0,s)} f(Nx/r)\, dx)/|B_\rho(0, s)|$ (where $|U|$ denotes the Lebesgue volume of $U$) we may use the Poincaré inequality in the metric ball, and obtain (denoting by

$g(x) = f(Nx/r))$

$$\begin{aligned}
\int_{|x|\leq N} f^2(x)\,dx \quad &\leq \quad C\,(\frac{N}{r})^d \int_{B_\rho(0,s)} \sum_{i=1}^m |X_i g|^2(x)\,dx \\
&\leq \quad C\left(\frac{N}{r}\right)^{d+2} \int_{B_\rho(0,s)} \sum_{i=1}^m |X_i f|^2(Nx/r)\,dx \\
&\leq \quad C\left(\frac{N}{r}\right)^{d+2} \int_{|x|\leq R} \sum_{i=1}^m |X_i f|^2(Nx/r)\,dx \\
&\leq \quad C\left(\frac{N}{r}\right)^{2} \int_{|x|\leq (RN/r)} \sum_{i=1}^m |X_i f|^2(x)\,dx\,,
\end{aligned}$$

Now, since $F$ is locally bounded, it is straightforward to show that $\mu$ satisfies a local Poincaré inequality on $\{|x| \leq N\}$ with $\kappa_N = C\left(\frac{N}{r}\right)^2 e^{4\sup_{|x|\leq(RN/r)} F(x)}$.

If we define a new vector field as $X_0 = \frac{1}{2}\sum_{i=1}^m divX_i.X_i$ , then $dx$ is symmetric for the generator $L = \frac{1}{2}\sum_{i=1}^m X_i^2 + X_0$ and $\mu$ is symmetric for the generator $L_F = \frac{1}{2}\sum_{i=1}^m X_i^2 + X_0 - \sum_{i=1}^m X_i F.X_i$ written in Hörmander form. Hence the following generalizes Corollary 4.1

**Corollary 4.3.** *Assume that Assumption 4.2 is fulfilled, and let* $L = \frac{1}{2}\sum_{i=1}^m X_i^2 + X_0$ *be as above. If* $F$ *is a* $C^2$ *non-negative function such that,* $F(x) \to +\infty$ *as* $x \to \infty$, $\int e^{-2F}dx < +\infty$ *and*

- $LF - 1/2\sum_{i=1}^m |X_i F|^2$ *is bounded from above, and ,*
- $\liminf_{|x|\to+\infty} \left(1/2\sum_{i=1}^m |X_i F|^2 - LF\right) = \alpha > 0\,.$

*Then the following (weighted) Poincaré inequality holds for all* $f$ *smooth enough and some* $C_P$,

$$\int f^2\, e^{-2F}\,dx \;\leq\; C_P \int \sum_{i=1}^m |X_i f|^2\, e^{-2F}\,dx \;+\; \frac{\left(\int f\, e^{-2F}\,dx\right)^2}{\left(\int e^{-2F}dx\right)}\,.$$

**Remark 4.4.** The choice $V = e^{aF}$ is not necessarily the best possible. Indeed one wants to get the smallest possible Lyapunov function. For example if $F(x) = |x|^2$ (i.e. the gaussian case) one can choose $V(x) = 1 + a|x|^2$ for $a > 0$. This is related to some sufficient condition for the Gross logarithmic Sobolev inequality to hold (see [7]). In the same way, if $F(x) = |x|^p$ (at least away from 0 for $F$ to be smooth) for some $2 > p \geq 1$, it is easy to see that $V(x) = \exp\left(a|x|^{2-p}\right)$ is a Lyapunov function (at least for a good choice of $a$), and of course $2 - p < p$ when $p > 1$, so that this choice is better than $e^{aF}$. These laws of exponent $1 < p < 2$ are the generic examples of laws satisfying interpolating inequalities (called $F$-Sobolev inequalities see [4], take care that this $F$ is not the potential). It clearly suggests that the best possible choice for the Lyapunov function is connected with the $F$-Sobolev inequality satisfied by $\mu$.

4.2. **General weighted weak Poincaré inequalities.** In this subsection we shall compare various weak Poincaré inequalities obtained in [25], [3], Theorem 2.1 and Corollary 3.12; as well as the various rates of convergence to equilibrium. The framework is the same as in the previous subsection.

4.2.1. **Sub-exponential laws.** We consider here for $0 < p < 1$ the measures $\mu_p(dx) = C_p\, e^{-2|x|^p} dx$ where $C_p$ is a normalizing constant and $|.|$ denotes the euclidean norm. It is shown in [3] that if $d = 1$, $\mu_p$ satisfies a weak Poincaré inequality with $\beta_p(s) = d_p\, \log^{(2/p)-2}(2/s)$ this function being sharp. Note that the previous result does not extend to higher dimensions via the tensorization result (Theorem 3.1 in [3]). In any dimension however, [25] furnishes $\beta_p(s) = d_p\, \log^{(4(1-p)/p}(2/s)$. Note that for $d = 1$ the result in [3] improves on the one in [25]. These bounds furnish a sub-exponential decay

$$\int (P_t^* f)^2\, d\mu \;\leq\; c_1\, e^{-c_2\, t^\delta} \;\parallel f \parallel_\infty^2$$

for any $\mu$ stationary semi-group, with $\delta = p/(2-p)$ if $d = 1$ and $\delta = p/(4-3p)$ for any $d$.

But sub-exponential laws enter the framework of subsection 3.3 with $V = e^{a|x|^p}$ , $\eta(u) = c\, u^{2(1-\frac{1}{p})}$ hence $\beta_W(s) = C\, \log^{(2/p)-2}(c/s)$ for some constants $c$ and $C$. Note that we recover the right exponent $(2/p)-2$ for $\beta_W$, hence the right sub-exponential decay in any dimension. Up to the constants, we also recover, thanks to Theorem 2.3 in [25], that $\beta$ behaves like $\beta_W$. Also note that in this case the rate given by Theorem 2.1 is again $\psi(t) = c_1\, e^{-c_2\, t^{p/(2-p)}}$.

These results extend to any $F$ going to infinity at infinity and satisfying

$$(1-a/2)|\nabla F|^2 - \Delta F \geq c\, F^{2(1-\frac{1}{p})}$$

at infinity, generalizing to the weak Poincaré framework similar results for super-Poincaré inequalities (see [4] and [5]).

4.2.2. **Heavy tails laws.** Let us deal now with measures $\mu_p(dx) = C_p(1+|x|)^{-(d+p)}$ where $p > 0$, $C_p$ is a normalizing constant, and $|.|$ denotes once again the usual euclidian norm. The sharp result in dimension 1 has been given in [3] with $\beta_p(s) = d_p s^{-2/p}$, but cannot be extended to higher dimensions. Röckner-Wang [25] furnishes in any dimension $\beta_p(s) = cs^{-\tau}$ where $\tau = \min\{(d+p+2)/p, (4p+4+2d)/(p^2-4-2d-2p)_+\}$. This result is not sharp in dimension one but enables to quantify the polynomial decay of the variance in any dimension. Once again, we may use the results of section 3.3 with $V(x) = (1+|x|)^{a(d+p)/2}$, so that $F(x) = \frac{d+p}{2}\log(1+|x|)$ and $\eta(u) = C(p,d)e^{-4u/(p+d)}$. Use now (3.13) to get that $\beta_W(s) = C(p',d)s^{\frac{2}{p'}}$ for any $p' < p$ (and $C(p',d) \to \infty$ as $p' \to p$). This result enables us to be nearly optimal in any dimension and thus improves on the result of [25]. Note that once again, results of [12, 9] would give, via Theorem 2.1 the same result, but without explicit constants.

4.3. **Drift conditions for diffusion processes.** Consider a $d$ dimensional diffusion process

$$dX_t \;=\; \sigma(X_t)dB_t + \beta(X_t)dt\,. \tag{4.5}$$

We assume that the (matrix) $\sigma$ has smooth and bounded entries, and is either uniformly elliptic or hypoelliptic in the sense of Assumption 4.2. We also assume the following drift condition

$$\text{there exist } M \text{ and } r > 0 \text{ such that for all } |x| \geq M\ ,\ \langle \beta(x), x\rangle \;\leq\; -r|x|\,. \tag{4.6}$$

We also assume that the diffusion has an unique invariant probability measure $d\mu = e^F dx$. This is automatically satisfied if $\sigma$ is uniformly elliptic and (4.6) holds (see [9] Proposition 4.1).

Consider a smooth function $V$ which coincides with $e^{a|x|}$ outside the ball of radius $M$, $|x|$ denoting the euclidean distance. Then on this set

$$LV(x) = a\langle \beta(x), \frac{x}{|x|}\rangle + a^2\eta(x)$$

where $\eta(x) \to 0$ as $|x| \to +\infty$. Hence, according to (4.6) for all $a$, $V$ is a Lyapunov function (but $C$ and $b$ depend on $a$).

We may thus apply Theorem 3.6 (thanks to the local Poincaré property discussed in the previous subsection) and get that for any density of probability $h$,

$$\int |P_t^* h - 1|^2 \, d\mu \, \leq \, e^{-\delta_a t} \int (h-1)^2 \, e^{a|x|} \, d\mu \, .$$

Indeed we know that $\mu$ satisfies a $(V+\lambda)$-Lyapunov-Poincaré inequality, hence apply Proposition 3.3 and then replace $W$ by $1+\lambda$ in the left hand side, and $(V+\lambda)$ by $(1+\lambda)V$ in the right hand side. Hence we get an exponential convergence for initial densities in $\mathbb{L}^2(e^{a|x|}\mu)$ for some $a > 0$.

Remark that if $\sigma = Id$ and $\beta = -\nabla F$, $d\mu = e^{-2F}dx$ and (4.6) which reads

$$\text{there exist } M \text{ and } r > 0 \text{ such that for all } |x| \geq M \ , \ \langle \nabla F(x), x\rangle \, \geq \, r|x| \, ,$$

thus implies the Poincaré inequality.

We may now complete the picture in the sub-exponential case (the polynomial case being handled similarly), namely we assume

(4.7)
$$\text{there exist } 0 < p < 1, \ M \text{ and } r > 0 \text{ such that for all } |x| \geq M, \ , \ \langle \beta(x), x\rangle \, \leq \, -r|x|^{1-p} \, .$$

One may then show as in [9] that for sufficiently small $a$, $V(x) = e^{a|x|^{1-p}}$ is a $\phi$-Lyapunov function with $\phi(v) = v \, \log(v)^{-2\frac{p}{1-p}}$ and get via the use of theorem 3.10 as in the preceding paragraph a weak Lyapunov-Poincaré inequality with $W(x) = V(x) + \lambda$ and $\beta_W(s) = d_p \log(2/s)^{2p/(1+p)}$. It then implies that for any density of probability $h$,

$$\int |P_t^* h - 1|^2 \, d\mu \, \leq \, C_{a,p} e^{-\delta_a t^{\frac{1-p}{1+p}}} \int (h-1)^2 \, e^{a|x|^{1-p}} \, d\mu \, .$$

Let us remark that for this diffusion case, the use of Lyapunov function was already present in Röckner-Wang [25, Th. 3.2 and 3.3] to obtain weak Poincaré inequality. They however always propose as Lyapunov function the distance to the origin, combined with local approximations, which is not optimal as seen in the previous subsections. Remark however that as in [9], Röckner-Wang [25] also considers the case of Markov processes with jumps. We leave this for further research.

## 5. **Entropy and weighted entropy.**

In all the previous sections we studied the behaviour of the Variance or some weighted Variance. The only exception is Theorem 2.2 where we obtained the rate of convergence for relative entropy. In many significant cases, for physical relevance, $\mathbb{L}^2$ bounds are too demanding, so that it is of some interest to look at less demanding bounds.

Using Lemma 3.1 the following Proposition is obtained exactly as Proposition 3.3, after stating the analogue of Definition 3.2

**Definition 5.1.** *Let $\Psi$ be a non-negative function such that $\Psi(1) = 0$. We shall say that $\mu$ satisfies a (W)-Lyapunov-$\Psi$-Sobolev inequality, if there exists $W \in D(L)$ with $W \geq 1$ and a constant $C_\Psi$ such that for all nice non-negative $h$ with $\int h d\mu = 1$,*

$$\int \Psi(h)\, W\, d\mu \,\leq\, C_\Psi \int \left(\frac{1}{2}\, W\, \Psi''(h)\, \Gamma(h) \,-\, \Psi(h)\, LW\right) d\mu\,.$$

**Proposition 5.2.** *Let $\Psi$ be a non-negative function such that $\Psi(1) = 0$. The following statements are equivalent*

- *$\mu$ satisfies a (W)-Lyapunov-$\Psi$-Sobolev inequality,*
- *$\int \Psi(P_t^* h)\, W\, d\mu \,\leq\, e^{-(t/C_\Psi)} \int \Psi(h) W d\mu$ for all non-negative $h$ with $\int h d\mu = 1$.*

Since the goal of this section is to deal with densities of probability $h$ with very few moments (in particular not in $\mathbb{L}^2$), we shall not discuss the analogous weak versions of these inequalities. The interested reader will easily derive the corresponding results.

Note that for Definition 5.1 to be interesting, we do certainly have to assume that $\Psi''(u) > 0$ for all $u$. This is a big difference with the (homogeneous) $F$-Sobolev inequalities studied in [4] where $F$ often vanishes on some neighborhood of 0.
Indeed if we want to mimic what we have done in Theorem 3.6, we have to introduce some local version of some new $\Psi$-Sobolev inequality, replacing the local Poincaré inequality. Instead of looking at such a complete theory, we shall focus on a typical example which will give the flavor of the results one can obtain. The first remark is, see for instance [14], that the Lebesgue measure satisfies a logarithmic Sobolev inequalities on the interval $I = [-R, R]$ with constant $8R^2/\pi^2$ which by tensorization holds also on the tensor product $I^d$ with the same constant so that we obtain the equivalent of the local Poincaré inequality.

Now if $d\mu = e^{-2F}dx$ is a probability measure, the normalized measure $\bar{\mu} = \mu/\mu(I^d)$ also satisfies a log-Sobolev inequality on $I^d$ as soon as $F$ is locally bounded.
But $u \mapsto u \log u$ is not everywhere non-negative so that we have to modify it.
First, since $\bar{\mu}$ also satisfies a Poincaré inequality on $I^d$, we may apply Lemma 17 in [5] and obtain the following $G$-Sobolev inequality with $G(u) = (\log u - \log 4)_+$ and some universal $C$ (all universal constants will be denoted by $C$ in the sequel)

$$\int_{I^d} f^2\, G\left(\frac{f^2}{\int_{I^d} f^2 d\bar{\mu}}\right) d\bar{\mu} \,\leq\, C\,(1 + R^2) \int_{I^d} |\nabla f|^2\, d\bar{\mu}\,. \tag{5.3}$$

Now consider $\Psi$ defined on $\mathbb{R}^+$ by

$$\Psi(u) = (u-1)^2\, \mathbb{1}_{u\leq 2} + (1 + (1 - 4\log 2)(u-2) + 4(u\log u - u - 2\log 2 + 2))\, \mathbb{1}_{u>2}\,, \tag{5.4}$$

so that

$$\Psi''(u) = 2\mathbb{1}_{u\leq 2} + \frac{4}{u}\,\mathbb{1}_{u>2}\,,$$

is everywhere positive. $\Psi$ is non-negative and $\Psi(u)=0$ if and only if $u=1$. It is easy to see that $u\mapsto \Psi(u)/u$ is non-decreasing on $[1,+\infty[$ and of course $\Psi$ behaves like $4G$ at infinity. Thus combining (5.3) and Lemma 21 in [4] we obtain that for any nice $g$ with $\int_{I^d} g^2 d\bar\mu = 1$ it holds

$$\int_{I^d} \Psi(g^2)\,\mathbb{1}_{g^2>1}\,d\bar\mu \;\leq\; C\,(1+R^2)\int_{I^d} |\nabla g|^2\,d\bar\mu\,. \tag{5.5}$$

We may thus state

**Theorem 5.6.** *Let $\mu = e^{-2F}dx$ be a probability measure on $\mathbb{R}^d$ (supposed to be $L$ invariant) satisfying a Poincaré inequality (on the whole $\mathbb{R}^d$) with constant $C_P$. Assume that there exists a Lyapunov function $V$ i.e. $LV \leq -2\alpha V + b\mathbb{1}_C$ for some set $C$ (not necessarily petite), such that either $C$ or the level sets of $V$ are compact.*

*Then $\mu$ satisfies a (W)-Lyapunov-$\Psi$-Sobolev inequality for $W = V+\lambda$ where $\lambda$ is a large enough constant and $\Psi$ is defined in (5.4).*

**Remark 5.7.** According to Corollary 3.4 and Theorem 3.6, if $L$ is $\mu$ symmetric, the Poincaré inequality automatically holds here.

*Proof.* Since we assumed that $C$ or the level sets of $V$ are compact, as for the proof of Theorem 3.6 what we have to do is to control $\int_{I^d}\Psi(h)d\mu$ for a large enough $I^d$ and a non-negative $h$ such that $\int_{\mathbb{R}^d} h d\mu = 1$. In the sequel we write $h=f^2$ (we may first assume that $f\geq\varepsilon>0$ and then go to the limit if necessary).

First, applying Poincaré inequality we get

$$\begin{aligned}
\int_{I^d}\Psi(h)\,\mathbb{1}_{h\leq 2}\,d\mu \;&=\; \int_{I^d}(h-1)^2\,\mathbb{1}_{h\leq 2}\,d\mu \;=\; \int_{I^d}(h\wedge 2-1)^2\,\mathbb{1}_{h\leq 2}\,d\mu\\
&\leq\; \int_{\mathbb{R}^d}(h\wedge 2-1)^2\,d\mu\\
&\leq\; C_P\int_{\mathbb{R}^d}|\nabla h|^2\,\mathbb{1}_{h\leq 2}\,d\mu + \left(\int_{\mathbb{R}^d}(h\wedge 2-1)\,d\mu\right)^2\\
&\leq\; C_P\int_{\mathbb{R}^d}|\nabla h|^2\,\mathbb{1}_{h\leq 2}\,d\mu + \left(\int_{\mathbb{R}^d}\left((h-1)\mathbb{1}_{h<2}+\mathbb{1}_{h\geq 2}\right)\,d\mu\right)^2\\
&\leq\; C_P\int_{\mathbb{R}^d}|\nabla h|^2\,\mathbb{1}_{h\leq 2}\,d\mu + \left(\int_{\mathbb{R}^d}\left((1-h)\mathbb{1}_{h\geq 2}+\mathbb{1}_{h\geq 2}\right)\,d\mu\right)^2\\
&\leq\; C_P\int_{\mathbb{R}^d}|\nabla h|^2\,\mathbb{1}_{h\leq 2}\,d\mu + \left(\int_{\mathbb{R}^d}(2-h)\,\mathbb{1}_{h\geq 2}\,d\mu\right)^2\\
&\leq\; C_P\int_{\mathbb{R}^d}|\nabla h|^2\,\mathbb{1}_{h\leq 2}\,d\mu + \left(\int_{\mathbb{R}^d}(h-2)\,\mathbb{1}_{h\geq 2}\,d\mu\right)^2\\
&\leq\; C_P\int_{\mathbb{R}^d}|\nabla h|^2\,\mathbb{1}_{h\leq 2}\,d\mu + \left(\int_{\mathbb{R}^d}h\,\mathbb{1}_{h\geq 2}\,d\mu\right)^2\\
&\leq\; C_P\int_{\mathbb{R}^d}|\nabla h|^2\,\mathbb{1}_{h\leq 2}\,d\mu + \int_{\mathbb{R}^d}h\,\mathbb{1}_{h\geq 2}\,d\mu\,,
\end{aligned}$$

since $\int_{\mathbb{R}^d} h\,\mathbb{1}_{h\geq 2}\,d\mu \leq 1$. Since $\mu$ satisfies a Poincaré inequality, Remark 22 in [4] shows that

$$\int f^2\,\mathbb{1}_{f^2\geq 2\int f^2 d\mu}\,d\mu \;\leq\; C\int |\nabla f|^2 d\mu\,,$$

so that (recall that $\Psi''(u) = 2\mathbb{1}_{u\leq 2} + \frac{4}{u}\,\mathbb{1}_{u>2}$) we finally obtain for some constant $C$,

$$\int_{I^d} \Psi(h)\,\mathbb{1}_{h\leq 2}\,d\mu \;\leq\; C\int_{\mathbb{R}^d} \Psi''(h)\,|\nabla h|^2\,d\mu\,.$$

For the other part we have to be accurate with normalization in order to use (5.5). Indeed the latter applies for normalized functions for the normalized measure on $I^d$.
Let $m = \int_{I^d}(h\vee a)\,d\bar{\mu}$ for some $2 > a > 0$.
If $m \leq 1$ then

$$\Psi(h)\,\mathbb{1}_{h>2} = \Psi(h\vee a)\,\mathbb{1}_{h>2} \leq \Psi(h\vee a/m)\,\mathbb{1}_{h>2} \leq \Psi(h\vee a/m)\,\mathbb{1}_{(h\vee a/m)>2}$$

so that we may apply (5.5) with $g = (h\vee a/m)^{\frac{1}{2}}$ (we can of course replace $\mathbb{1}_{g>1}$ by $\mathbb{1}_{g>2}$). Of course $|\nabla g|^2$ is up to some constant (the normalization by $m$ disappears) equal to $\mathbb{1}_{h>a}(|\nabla h|^2/h)$ hence up to the constants to $\mathbb{1}_{h>a}\,\Psi''(h)\,|\nabla h|^2$. Remark that we need $h > a$ for $1/h$ to be bounded (since $\Psi''(u) = 2$ when $u \leq 2$) at least for $h < 2$.
If $m \geq 1$ the situation is more delicate. But

$$m \leq \int_{I^d} h d\bar{\mu} + a \leq (1/\mu(I^d)) + a$$

so that if we choose $R$ (the length of the edge of $I^d$) large enough we may assume that $\mu(I^d) \geq 3/4$, choose $a = 1/3$ so that $m \leq 5/3 < 2$. In other words on $\{h > 2\}$, $h/m \geq 6/5$. It follows that $\Psi(h) = \Psi(m\,\frac{h}{m}) \leq c\,\Psi(\frac{h}{m})$ on $\{h > 2\}$, for some constant $c$ (recall the form of $\Psi$). Furthermore $\mathbb{1}_{h>2} \leq \mathbb{1}_{\frac{h}{m}>\frac{6}{5}}$ so that one more time we may apply (5.5), and conclude as in the case $m \leq 1$.
We have thus shown the existence of some $C$ such that

$$\int_{I^d} \Psi(h)\,\mathbb{1}_{h>2}\,d\mu \;\leq\; C\int_{\mathbb{R}^d} \Psi''(h)\,|\nabla h|^2\,d\mu\,.$$

With the previous result the proof is completed. □

**Remark 5.8.** Since $\Psi(u)$ behaves like $u\log u$ at infinity, the previous result has the following consequence : if $V$ has some exponential moment, then

$$\int P_t^* h\,\log P_t^* h d\mu \;\leq\; C\,e^{-\eta t}\left(1\vee\int \Psi(h)\,\log_+(\Psi(h))\,d\mu\right) \;\leq\; C'\,e^{-\eta t}\left(1\vee\int h\,\log_+^2(h)\,d\mu\right).$$

This result is (at a qualitative level) a little bit weaker than the one we obtain in this case in Theorem 2.2, since there we can replace the exponent 2 by any exponent greater than 1.
It should also be interesting to extend this kind of result to (strongly) hypoelliptic operators as in Corollary 4.3. The key would be to prove a local log-Sobolev inequality for the corresponding $\Gamma$. We strongly suspect that some inequality of this type is true, but we did not find any reference about it.

## 6. **Fully degenerate cases, towards hypocoercivity.**

Proposition 3.3 and Theorem 5.6 are hypocoercive results in Villani's terminology. The former shows a coercivity property in $\mathbb{L}^2(W\mu)$ norm, which is stronger than the $\mathbb{L}^2(\mu)$ norm, while the latter can be interpreted in terms of semi-distances. We refer to [29] for a nice presentation of hypocoercivity. In studying fully degenerate cases, Villani introduces higher order functional inequalities (reminding the celebrated $\Gamma_2$ criterion for logarithmic Sobolev inequality), see equation (11) in [29] and more generally [30]. These higher order inequalities enable him to introduce Lie brackets of the diffusion vector fields with the drift vector field, hence are clearly related to some hypoelliptic situation of Hörmander type. A deep study of the spectral theory of hypoelliptic operators is done in [17], and we refer to the references in both [17, 30] for more details and contributors. Also notice that the hypocoercivity phenomenon was first studied by Hérau and Nier (see [18]) by using pseudo-differential calculus (also see some recent work by Hérau on his Web page).

Since the existence of a Lyapunov function does not immediately rely on non degeneracy, it is natural to consider fully degenerate cases from this point of view. Note that Theorem 3.6 requires a local Poincaré inequality, hence is not adapted, while the method in section 2 furnishes some exponential decay for the variance but controlled by some $\mathbb{L}^p$ norm.

In this section we shall recall the results in [30] for the particular example of the kinetic Fokker-Planck equation. Then we shall see that this example enters the framework of Meyn-Tweedie approach, following [35] and [9] who indicated how to build some Lyapunov function.

First we recall what the kinetic Fokker-Planck equation is. Let $F$ be a smooth function on $\mathbb{R}^d$. We consider on $\mathbb{R}^{2d}$ the stochastic differential system ($x$ stands for position and $v$ for velocity)

$$\begin{aligned} dx_t &= v_t\, dt \\ dv_t &= dB_t - v_t\, dt - \nabla F(x_t)\, dt \end{aligned} \tag{6.1}$$

associated with

$$L = \frac{1}{2}\Delta_v + v\,\nabla_x - (v + \nabla F(x))\,\nabla_v\,.$$

Define

$$\mu(dx, dv) = e^{-\,(|v|^2+2F(x))}\, dx\, dv = e^{-H(x,v)}\, dx\, dv \tag{6.2}$$

which is assumed to be a bounded measure (in the sequel we shall denote again by $\mu$ the normalized (probability) measure $\mu/\mu(\mathbb{R}^{2d})$).

If $F$ is bounded from below, it is known that (6.1) has a pathwise unique, non explosive solution starting from any $(x, v)$. Actually the statement in [35] Lemma 1.1 is for a weak solution since Wu is using Girsanov theory. But introduce the stopping time $\tau_R = \inf\{s \geq 0; |v_t| \geq R\}$. Since $|x_{t\wedge\tau_R}| \leq Rt + |x|$ pathwise uniqueness holds up to each time $\tau_R$ and the explosion time is the limit of the $\tau_R$'s as $R$ goes to infinity. That this limit is almost surely $+\infty$ is proved by Wu at the top of p.210 in [35].

Furthermore $\mu$ is in this case the unique invariant measure.

Let us make three additional remarks

- $\mu$ is not symmetric,

- $L$ is fully degenerate, in particular since $\Gamma f = |\nabla_v f|^2$ any function $f(x,v) = g(x)$ with $\int f d\mu = 0$ is such that $\Gamma f = 0$ so that the Poincaré inequality (with $\Gamma$) is not true for $\mu$,
- the Bakry-Emery curvature of the semi-group (see [1] Definition 5.3.4) is equal to $-\infty$.

The main results in [30] about convergence to equilibrium for this equation are collected below

**Theorem 6.3.** ***Villani*** [30] ***Theorems 29,31,32.***

(1) *Define* $H^1(\mu) := \{f \in \mathbb{L}^2(\mu); \nabla f \in \mathbb{L}^2(\mu)\}$ *equipped with the semi-norm* $\| f \|_{H^1(\mu)} = \| \nabla f \|_{\mathbb{L}^2(\mu)}$.

*Assume that* $|\nabla^2 F| \leq c(1 + |\nabla F|)$ *and that the marginal law* $\mu_x(dx) = e^{-2F(x)}dx$ *satisfies the classical Poincaré inequality for all nice* $g$ *defined on* $\mathbb{R}^d$

$$Var_{\mu_x}(g) \, \leq \, C \int_{\mathbb{R}^d} |\nabla g|^2(x) \, \mu_x(dx) \, .$$

*Then there exist* $C$ *and* $\lambda$ *positive such that for all* $f \in H^1(\mu)$,

$$\| \, P_t^* f \, - \, \int f d\mu \, \|_{H^1(\mu)} \leq \, C \, e^{-\lambda t} \, \| \, f \, \|_{H^1(\mu)} \, .$$

(2) *With the same hypotheses, there exists* $C$ *such that for all* $1 \geq \varepsilon > 0$ *and all* $t > \varepsilon$,

$$Var_\mu(P_t^* f) \, \leq \, C \, \varepsilon^{-3/2} \, e^{-\lambda(t-\varepsilon)} \, Var_\mu(f) \, .$$

(3) *Assume that* $|\nabla^j F| \leq c_j$ *for all* $j \geq 2$ *and that* $\mu_x$ *satisfies a (classical) log-Sobolev inequality*

$$Ent_{\mu_x}(g^2) \, \leq \, C \int_{\mathbb{R}^d} |\nabla g|^2(x) \, \mu_x(dx) \, .$$

*Then for all* $h \geq 0$ *such that* $\int h d\mu = 1$ *and satisfying*

$$\forall k \in \mathbb{N} \quad , \quad \int (1 + |x| + |v|)^k \, h(x,v) \, d\mu \, < \, +\infty \, ,$$

*it holds for some* $\lambda > 0$,

$$\int P_t^* h \, \log(P_t^* h) \, d\mu \, \leq \, C(h) \, e^{-\lambda t}$$

*where* $C(h)$ *depends on the above moments.*

It is worthwhile noticing that since $\mu$ is a product measure of $\mu_x$ and a gaussian measure, $\mu$ inherits the classical Poincaré or log-Sobolev inequality as soon as $\mu_x$ satisfies one or the other. Part (2) in the previous result is simply an hypoelliptic regularization property, and some hypotheses can be slightly improved (see [30] Theorems 29,31 and 32 for the details). However, it has to be noticed that $C > 1$ (otherwise $\mu$ would satisfy a Poincaré inequality with $\Gamma$) and that the Bakry-Emery curvature has to be $-\infty$ for the same reason.

In [35], Wu gave some sufficient conditions for the existence of a Lyapunov function for this (and actually more general) model (see [35] Theorem 4.1). We recall and extend this result below. First define

$$\Lambda_{a,b}(x,v) = aH(x,v) + b(\langle v, \nabla G(x) \rangle + G(x)) \tag{6.4}$$

where $G$ is smooth, $a$ and $b$ being positive parameters.

**Theorem 6.5.** *Assume that $F$ is bounded from below and that there exists some $G$ satisfying*

(1) $\liminf_{|x|\to+\infty}\langle \nabla G(x), \nabla F(x)\rangle = 2c > 0$,
(2) $\parallel \nabla^2 G \parallel_\infty < c/16d$ ,
(3) *there exists $\kappa > 0$ such that for all $x$, $|\nabla G(x)|^2 \leq \kappa(1 + |\langle \nabla F(x), \nabla G(x)\rangle|)$,*
(4) *$\Lambda_{a,b}$ is bounded from below.*

*Then for all $0 < \varepsilon$ one can find a pair $(a, b)$ such that $\max(a, b) \leq \varepsilon$ for which $V_{a,b}(x, v) = e^{\Lambda_{a,b}(x,v)-\inf_{x,v}\Lambda_{a,b}(x,v)}$ is a Lyapunov function.*
*Hence if there exists $\eta > 0$ such that $\int e^{\Lambda_{\eta,\eta}(x,v)}d\mu < +\infty$, for each $p > 1$ one can find a Lyapunov function $V_p \in \mathbb{L}^p(\mu)$, so that there exists $\lambda > 0$ such that for each $q > 2$ there exists $C_q$ such that*

$$Var_\mu(P_t^* f) \;\leq\; C_q\, e^{-(\frac{q-2}{q-1})\,\lambda\, t}\; \parallel f - \int f d\mu \parallel_q^2 \;.$$

*Proof.* Elementary computation yields

$$LV_{a,b}/V_{a,b} = -2a|v|^2(1-a) + ad + 2ab\langle v, \nabla G\rangle + \frac{1}{2}b^2|\nabla G|^2 + b\langle \nabla^2 G\, v, v\rangle - b\langle \nabla F, \nabla G\rangle\,.$$

Our aim is to choose $G$ for the right hand side to be negative outside some compact set. A rough majorization gives

$$LV_{a,b}/V_{a,b} \leq (-2a(1-a) + b|\nabla^2 G(x)| + 4ab)|v|^2 - b\langle \nabla G, \nabla F\rangle + (\frac{b^2}{2} + 4ab)|\nabla G|^2 + ad\,.$$

We have thanks to (3)

$$-b\langle \nabla G, \nabla F\rangle + (\frac{b^2}{2} + 4ab)|\nabla G|^2 + ad \leq b(-1 + \kappa(\frac{b}{2} + 4a))\langle \nabla G, \nabla F\rangle + (ad + \kappa b(\frac{b}{2} + 4a))$$

so that if we choose $a$ and $b$ small enough for $\kappa(\frac{b}{2} + 4a) \leq \frac{1}{2}$ the first term is less than $-cb$ for $|x|$ large enough thanks to (1). Hence if we choose $ad + \kappa b(\frac{b}{2} + 4a) < cb/2$ we get $LV_{a,b}/V_{a,b} \leq -cb/2$ for $|x|$ large and all $v$ as soon as

$$-2a(1-a) + b|\nabla^2 G(x)| + 4ab \leq 0\,.$$

We may thus first choose $a$ and $b$ small enough for $\kappa(\frac{b}{2} + 4a) < c/4$, so that it remains to choose $a < cb/4d$.
Now if $|x| \leq L$, $(LV_{a,b}/V_{a,b})(x, v) \to -\infty$ as $|v| \to +\infty$ as soon as

$$-2a(1-a) + b|\nabla^2 G(x)| + 4ab < 0\,.$$

We may choose $b \leq 1/8$ and $a \leq 1/2$ so that we only have to check $-a/2 + b|\nabla^2 G(x)| < 0$, i.e. $a/2 > cb/16d$ thanks to (2). This is possible since our unique constraint is $a/2 < cb/8d$.
We have thus obtained the existence of a Lyapunov function for some pair $(a, b)$ with both $a$ and $b$ as small as we want. This Lyapunov function thus belongs to $\mathbb{L}^p$ if $a$ and $b$ are small enough, according to our integrability hypothesis. It remains to apply Theorem 2.1 to conclude (all the other hypotheses in Theorem 1.2 are satisfied here, see [35, 9] for the details). □

**Example 6.6.** Let us describe some examples.

(1) **(Wu** [35]**)** Assume the drift condition $\liminf_{|x|\to+\infty}\langle x, \nabla F(x)\rangle/|x| = 2c > 0$. Then we may choose $G(x) = |x|$ for $|x|$ large, and $|\nabla^2 G(x)| \leq \varepsilon$ for all $x$. This is the situation discussed in [35]. Notice that $\mu_x$ satisfies a classical Poincaré inequality (see e.g. section 4) so that the hypotheses of Theorem 6.3 are satisfied.

(2) A little more general situation is for $F$ going to infinity, satisfying

$$\liminf_{|x|\to+\infty} |\nabla F(x)|^2 = 2c > 0 \text{ and } |\nabla^2 F| \ll |\nabla F| \text{ at infinity.}$$

In this case also $\mu_x$ satisfies a classical Poincaré inequality as we saw in section 4 (if $d = 1$ the converse is true). If $|\nabla^2 F(x)| \to 0$ as $|x| \to +\infty$ we may choose a function $G$ such that $|\nabla^2 G(x)| \leq \varepsilon$ for all $x$ and $G(x) = F(x)$ for $x$ large. This function will satisfy all (1),(2),(3). For (4) and the integrability condition to be satisfied it is enough to assume in addition that

$$|\nabla F(x)|^2/F(x) \text{ goes to 0 at infinity.}$$

This is the case for $F(x) = |x|^p$ at infinity for $1 \leq p < 2$.

(3) If the latter condition is not satisfied we may take $G = F^\alpha$ for some $\alpha \leq 1$. But in this situation we can obtain a better Lyapunov function and study convergence in entropy.

**Remark 6.7.** The $\mathbb{L}^2$ convergence in Theorem 6.3 is optimal, hence we cannot expect to improve it and actually the controls we obtained in Theorem 6.5 are weaker. In addition, in the last version of his work (see [31]) Villani gives some explicit bounds for the constants involved. As we said, such estimates are not yet available in Theorem 1.2.

However, Villani's approach uses the classical Poincaré inequality in an essential way, and only gives exponential decay results. Examples for the existence of $\phi$-Lyapunov functions for this kinetic model are given in [9] section 4.3 Indeed consider $F(x) \sim |x|^p$ for large $|x|$ with $0 < p < 1$. Attentive calculations show that one can consider smooth $G$ with $\nabla G(x) = |x|^m$ for large $|x|$ with $1 - p < m \leq 1$,

$$e^{s\Lambda^\delta_{a,b}(x,v) - \inf_{x,v} s\Lambda^\delta_{a,b}(x,v)} \qquad (m+1)\delta \leq p,$$

as a $\phi$-Lyapunov function for well chosen $s, a, b$, with $\phi(t) = t/\ln^{1/\delta - 1} t$. Combined with Theorem 2.1 we thus get a subexponential decay in a situation where it is known that there is no exponential decay, thanks to an argument by Wu [35]. We refer to [9] for the polynomial decay case. We shall not go further in this direction here, but Theorems 1.2 and 2.1 thus allow to study a larger field of potentials.

As we said before we turn to the study of entropy decay.

This time we shall directly use $\Lambda_{a,b} + M = V_{a,b}$ as a Lyapunov function, for $M$ large enough. Indeed

$$LV_{a,b}(x,v) = ad - 2a|v|^2 - b\langle \nabla F(x), \nabla G(x)\rangle + b\langle \nabla^2 G(x)v, v\rangle\,.$$

Our aim is to find $G$ and $\eta > 0$ such that $LV_{a,b} \leq -\eta V_{a,b}$ outside some compact set. We shall choose $G(x) = F^{1-\alpha}(x)$ for large $x$, for some $0 \leq \alpha < 1$, assuming that $F$ is non-negative outside some compact set. Actually we shall assume that $F$ goes to infinity at infinity. With all these choices

$$\Lambda_{a,b}(x,v) \geq a|v|^2 + 2aF(x) - b|v|\,\frac{|\nabla F(x)|}{F^\alpha(x)}$$

is bounded from below as soon as $|\nabla F(x)|^2/F^{1+2\alpha}(x)$ goes to 0 at infinity or if this ratio is bounded and $b/a$ small enough.

Now if $\alpha > 0$,

$$\langle \nabla^2 G(x)v, v\rangle = (1-\alpha)F^{-\alpha}(x)\,\langle \nabla^2 F(x)v, v\rangle - \alpha(1-\alpha)\,F^{-(1+\alpha)}(x)\,\langle \nabla F(x), v\rangle^2\,,$$

so that for $x$ large,

$$LV_{a,b}(x,v) \leq ad - 2a|v|^2 - b(1-\alpha)F^{-\alpha}(x)\,|\nabla F(x)|^2 + b(1-\alpha)F^{-\alpha}(x)\langle \nabla^2 F(x)v, v\rangle\,. \tag{6.8}$$

To show that $V_{a,b}$ is a Lyapunov function, using the same majorization as in the proof of the latter Theorem, it is enough to show that we can find some $\eta > 0$ such that for $x$ large

$$\left((2-\eta)a - 2b\eta - b(1-\alpha)\frac{|\nabla^2 F(x)|}{F^{\alpha}(x)}\right)|v|^2 \,+\, b\frac{|\nabla F(x)|^2}{F^{\alpha}(x)}\left(1-\alpha-\frac{2\eta}{F^{\alpha}(x)}\right) - \\ - \left(M + ad + 2a\eta F(x) + b\eta F^{1-\alpha}(x)\right) \,\geq\, 0\,. \tag{6.9}$$

Note that the same result holds true for $\alpha = 0$.

The situation is now quite simple : first we shall assume that $|\nabla F(x)|^2 \geq \kappa F^{1+\alpha}(x)$ for large $x$, so that for any $b$ we may choose $\eta$ small enough for the sum of the last two terms to be positive; next we have to assume that $|\nabla^2 F(x)|/F^{\alpha}(x)$ is bounded, so that we may choose $b$ small enough for the coefficient of $|v|^2$ to be positive. Of course for $|x| \leq L$ (6.9) has to be replaced by the correct one involving $G$, but $G$ being smooth it is enough again to choose $b$ and $\eta$ small enough.

Choosing $a$ small enough we see that $\int e^{pV_{a,b}} d\mu < +\infty$, so that applying Theorem 2.2 and Hölder-Orlicz inequality to bound $\int V h d\mu$ we have obtained

**Theorem 6.10.** *Assume that $F(x) \to +\infty$ as $|x| \to +\infty$ (hence is bounded from below) and that there exists $0 \leq \alpha < 1$ such that the following holds*

(1) *there exist $c$ and $C$ such that for $|x|$ large,*

$$c\,F^{1+\alpha}(x) \leq |\nabla F(x)|^2 \,\leq C\,F^{1+2\alpha}(x)\,,$$

(2) *$|\nabla^2 F(x)|/F^{\alpha}(x)$ is bounded (for $|x|$ large).*

*Then for all $p > 1$ one can find a Lyapunov function $V_p$ such that $\int e^{pV} d\mu < +\infty$. Hence there exists $\lambda > 0$ such that for any $1 > \beta > 0$ there exists $C_\beta$ such that for all density of probability $h$,*

$$\int P_t^* h\, \log P_t^* h\, d\mu \leq\, C_\beta\, e^{-\beta\,\lambda\, t}\left(1 + \int h \log h d\mu\right)^{\beta}\left(\int |h-1||\log h|^{\frac{1}{1-\beta}} d\mu\right)^{1-\beta}.$$

**Example 6.11.** If $F(x) = |x|^p$ for some $p \geq 2$ and large $|x|$, then we may apply the previous Theorem with

$$\frac{p-2}{2p} \,\leq\, \alpha \,\leq\, \frac{p-2}{p}\,.$$

**Remark 6.12.** As it is shown in [7] the condition $|\nabla F(x)|^2 \geq \eta F(x) + \Delta F(x)$ for large $x$ implies a classical logarithmic Sobolev inequality for $\mu$. Hence if $|\nabla^2 F| \leq C(1 + \nabla|F|)$ our hypothesis (1) in Theorem 6.10 implies a classical logarithmic Sobolev inequality, as it is asked in Theorem 6.3 (3).

But case (3) in Theorem 6.3 is (very) roughly the case where $c|x|^2 \leq F(x) \leq C\,|x|^2$ for some positive $c$. Our result covers more "convex at infinity" cases.

Finally, even if we do not have explicit constants, our hypotheses on $h$ seem to be weaker than the moment conditions in Theorem 6.3. For instance if $F(x) = |x|^2/2$ we may choose with $a > 0$

$$h(x,v) = \frac{e^{|x|^2+|v|^2}}{\left(1+|x|^{d+1}+|v|^{d+1}\right)^{a+1}}$$

for any $\beta < 1 - 2/(a(d+1))$, while this $h$ does not fulfill the hypotheses of Theorem 6.3 (3) (requires all $\beta < 1!$).

**Remark 6.13.** Of course, since for any density of probability $h$ it holds $\int h \log h d\mu := Ent_\mu(h) \leq Var_\mu(h)$, the relative entropy is decaying at least with the same rate as the variance, hence Theorem 6.5 furnishes some decay. The study of relative entropy in [18] is based on this argument.

**Remark 6.14.** Remark that the generator $L$ can be written in Hörmander's form $L = \frac{1}{2}X_1^2 + X_0$ where the vector fields $X_i(x,v)$ are given by $X_1(x,v) = \partial_v$ and $X_0(x,v) = v\partial_x - (v + \nabla F(x))\partial_v$. Hence the Lie bracket $[X_1, X_0](x,v) = \partial_x - \partial_v$ is such that $X_1$ and $[X_1, X_0]$ generate the tangent space at any $(x,v)$. Furthermore $|X_1|^2 + |[X_1, X_0]|^2$ is uniformly bounded from below by a positive constant. Hence Malliavin calculus allows us to show that, for any $t > 0$, the law of $(x_t, v_t)$ starting from any point $(x,v)$ has a $C^\infty$ density $p_t$ w.r.t Lebesgue measure, hence a smooth density $h_t$ w.r.t. $\mu$. Furthermore $p_t$ satisfies some gaussian upper bound. However we do not know how to show that $h_t \in \mathbb{L}^2(\mu)$. The latter is shown in [18], but starting with some particular initial absolutely continuous laws. Due to the gaussian part of $\mu$, exponent 2 is optimal for such a result.

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