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DIFFUSION OR FRACTIONAL DIFFUSION LIMIT FOR KINETIC FOKKER-PLANCK EQUATION WITH HEAVY TAILS EQUILIBRIA.

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Abstract. This paper is devoted to the diffusion and anomalous diffusion limit of the Fokker-Planck equation of plasma physics, in which the equilibrium function decays towards zero at infinity like a negative power function. We use probabilistic methods to recover and extend the results obtained in [20], showing that the small mean free path limit can give rise to a diffusion equation in particular in the critical case where the diffusion coefficient is no more well defined.

Key words : diffusion limit, Fokker-Planck, heavy tails, general Cauchy distributions, central limit theorem.

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1. **Introduction and main results.**

We consider a collisional kinetic equation given by

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f) & \text{in } [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \\ f(0,x,v) = f_0(x,v) & \text{in } \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{1.1}$$

Such a problem naturally arises when modeling the behavior of a cloud of particles. Provided $f_0 \geq 0$, the unknown $f(t,x,v) \geq 0$ can be interpreted as the density of particles occupying at time $t \geq 0$, the position $x \in \mathbb{R}^d$ with a physical state described by the variable $v \in \mathbb{R}^d$. This variable v represents the velocity of the particles.

As in [20], we focus in this paper on the Fokker-Planck equation when the collisional operator Q has a diffusive form:

$$Q(f) := \nabla_v \,.\, \left(\frac{1}{\omega}\, \nabla_v\, (f\, \omega)\right) \tag{1.2}$$

and where the equilibria are characterized by the choice of ω. In the whole paper, except the first section, we choose $\omega = \omega_\beta$ for some $\beta > d$ with

$$\omega_\beta(v) \,=\, (1+|v|^2)^{\beta/2}\,, \tag{1.3}$$

[1]This work is dedicated to the memory of Naoufel Ben Abdallah.

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so that there exists C_β such that $\displaystyle\int \frac{C_\beta}{\omega_\beta}\,dv = 1$. We shall denote by μ_β (or simply μ) the measure $\frac{C_\beta}{\omega_\beta}\,dv$. This corresponds to the so called *Barenblatt profile* or *general Cauchy distribution*.

In this equation, the scattering in velocity is modeled by a diffusion phenomena corresponding to a deterministic formulation of a Brownian motion.

When the scattering phenomenon is much stronger than the advection phenomenon, one expects that the solution of (1.1) can be approximated by a density depending on the time and space variable times a velocity profile given by the thermodynamical equilibrium.

More precisely, we introduce a small parameter $\varepsilon \ll 1$ which describes the mean free path of the particles, then we consider the following rescaling

$$x' = \varepsilon\, x \text{ and } \; t' = \theta(\varepsilon)\, t, \text{ with } \theta(\varepsilon) \to 0.$$

Typically, it means that we assume that the mean free path is very small and the time scale is very large. Then, we rescale the distribution function

$$f^\varepsilon(t', x', v) = f(t, x, v).$$

The function f^ε is now solution of (we skip the primes)

$$\begin{array}{rcl} \theta(\varepsilon)\;\partial_t f^\varepsilon + \varepsilon\; v \cdot \nabla_x f^\varepsilon & = & Q(f^\varepsilon), \\ f^\varepsilon(0, x, v) & = & f_0(x, v)\,. \end{array} \tag{1.4}$$

The goal is then to study the behavior of the solution as $\varepsilon \to 0$.

The usual diffusion limit corresponds to $\theta(\varepsilon) = \varepsilon^2$. It may be formally studied using the so-called *Hilbert expansion method*. In [20], two of us used the *moment method* which is popular for deriving limits of kinetic equations ([2, 3, 11, 18]), in the particular situation of (1.3). The main result in [20] roughly reads as follows

Theorem 1.1. *Assume that ω_β is given by (1.3) for some $\beta > d + 4$. For a nice enough nonnegative initial f_0 and the usual diffusion scaling $\theta(\varepsilon) = \varepsilon^2$, the solution f^ε of (1.4) weakly converges as $\varepsilon \to 0$ towards $\rho(t, x)\, C_\beta\, \omega_\beta^{-1}(v)$ where ρ is the unique solution of the heat equation*

$$\partial_t \rho - \nabla_x .(D\, \nabla_x)\rho = 0$$

with initial datum $\rho_0(x) = \int f_0(x, v) dv$ where D is the constant diffusion tensor given by

$$D = \int v \otimes Q^{-1}(v\, C_\beta\, \omega_\beta^{-1}(v)) dv.$$

Note that $Q^{-1}(v\, C_\beta\, \omega_\beta^{-1}(v))$ has an explicit formula and that the constraint on β corresponds to the range that ensure that the diffusion coefficient is finite.

The meaning of nice enough and of weak convergence is related to some weighted Lebesgue spaces. We refer to [20] for a precise statement and references to neighboring problems, in particular when the Fokker-Planck equation is replaced by the Boltzmann equation (see [18, 19, 2, 3]).

This problem has a natural probabilistic interpretation. Indeed, remark that Q is nothing else but the adjoint operator (in $\mathbb{L}^2(dv)$) of

$$L = \Delta_v - \frac{\nabla_v \omega}{\omega}.\nabla_v \,, \tag{1.5}$$

so that, provided f_0 is a density of probability, the solution $f(t,x,v)$ of (1.1) is the density of probability (with respect to Lebesgue's measure) of the law of the (stochastic) diffusion process given by the following stochastic differential equation (S.D.E.)

$$\begin{aligned} dv_t &= \sqrt{2}\, dB_t - \frac{\nabla_v \omega}{\omega}(v_t)\, dt \\ dx_t &= v_t\, dt \,, \end{aligned} \tag{1.6}$$

starting with initial distribution $f_0(x,v)dxdv$. We shall give rigorous statements later. The scaling we did corresponds to the following: $f^\varepsilon(t,x,v)$ is the density of probability of the joint law of

$$\left(x_0 + \varepsilon \int_0^{t/\theta(\varepsilon)} v_s\, ds \,,\, v_{t/\theta(\varepsilon)} \right) \tag{1.7}$$

when (x_0, v_0) are distributed according to $f_0(x,v)dxdv$. Notice that we rescaled the initial data so that we do not have to rescale f^ε by $1/\varepsilon^d$. We are thus reduced to study the joint law of $(v_s, \zeta(s) \int_0^s v_u du)$ when $s \to +\infty$, i.e. the joint law of the process and some particular *additive functional* of the process.

This approach was already used by the first named author together with D. Chafai and S. Motsch [5] in the study of the so called *persistent turning walker model* introduced in [12]. In [4], a rather general study of long time behavior of additive functionals of ergodic Markov processes is done. The fact that one can then derive the joint behavior of $(v_s, \int_0^s v_u du)$ is explained in section 3 of [5] (subsection: coupling with propagation of chaos and asymptotic independence) and is granted in general situations, in particular the ones we will look at. It is this *propagation of chaos (in time)* property which ensures the asymptotic splitting of f^ε as a product of a function of x times a function of v.

As remarked by J. Dolbeault during a short but nice discussion, one should try to attack the problem via hypo-coercive estimates in the spirit of C. Villani's work. The probabilistic counterpart of this approach is described in [1]. However, the invariant measure of the pair (x,v) is Lebesgue's measure times $\omega^{-1}\, dv$ which is not a probability measure. That is why some different normalizations are needed for both coordinates. If we introduce some repealing force (depending on x) in Q in order to apply hypo coercive estimates, the normalizations will be the same.

In this paper we shall first show how to recover Theorem 1.1 by using the arguments in [4]. The difference is that the notion of convergence is different, since here, in a probabilistic spirit, we are looking at convergence of the measures $f^\varepsilon\, dx\, dv$. These arguments apply to a large class of weights ω as recalled in the next section.

Then, when $\beta - d = 4$ we still get a diffusion limit as in Theorem 1.1. The covariance tensor D is actually given by a constant $\kappa/3$ times Identity. Such a phenomenon of *anomalous rate of convergence* to a diffusion limit was already observed on another examples (see [19]).

But a very interesting additional feature here is that some *variance breaking* occurs. Indeed, if we calculate

$$a_t^\varepsilon = \int x_i^2\, f^\varepsilon(t,x,v)\, dxdv$$

which does not depend on i, it is shown that $a_t^\varepsilon \to 2\,\kappa\, t$ as $\varepsilon \to 0$, while the similar second moment for the limiting density ρ is $2\,\kappa t/3$. This shows that there is a convergence of the measures but no convergence for the second moment to the expected one.
In probabilistic terms, the main result is thus the following

Theorem 1.2. *Assume that $\beta = d+4$. Then there exists $\kappa > 0$ such that, for each i,*

(1) *$Var_\mu(x_t^i)/t \ln t \;\to\; \kappa \;>\; 0$ as $t \to +\infty$,*
(2) *the normalized additive functional $x_t/\sqrt{Var_\mu(x_t^i)}$ converges in distribution to a centered gaussian vector with covariance matrix $(1/3)\, Id$.*

Thus, with $\theta(\varepsilon) = \varepsilon^2\, \ln(1/\varepsilon)$, for all initial density of probability f_0, the solution f_t^ε of (1.4) weakly converges as $\varepsilon \to 0$ towards $(v,x) \mapsto C_\beta\, \omega_\beta^{-1}(v)\,(h_0 * \rho_t)(x)$ where ρ_t is the density of a centered gaussian random vector with covariance matrix $(2\kappa/3)\, t\, Id$ and $h_0(x) = \int f_0(x,v)\, dv$.

The last statement is easily deduced from the previous ones. Indeed, we may apply (2) with $t' = t/\theta(\varepsilon)$ and a normalization

$$\sqrt{t'\,\ln t'} = \frac{1}{\varepsilon}\,\sqrt{t\,\ln(t/\theta(\varepsilon))/\ln(1/\varepsilon)} \sim \frac{\sqrt{2t}}{\varepsilon} \quad \text{as } \varepsilon \to 0\,.$$

The initial density of x_0 is then given by h_0, yielding the convergence of the distribution of the random vector defined by (1.7) to $C_\beta\, \omega_\beta^{-1}(v)\,(h_0 * \rho_t)(x)\, dv\, dx$.

There are some differences between the cases of a Boltzmann type operator Q and the diffusive type operator Q we are looking here. In the first situation, it is shown in [19, 3] that the usual diffusion limit for $f_.^\varepsilon$ holds for $\beta - d > 2$, the anomalous rate of convergence for $\beta - d = 2$ and, up to an appropriate scaling $\theta(\varepsilon)$, convergence to an anomalous diffusion equation (driven by a fractional Laplacian) for $\beta - d < 2$.
In our situation, another phase transition occurs for $\beta - d = 4$, still with some Maxwellian (gaussian) limit but with an anomalous variance rate. We think that presumably, a second one occurs for $\beta - d = 2$, as in the Boltzmann case, with anomalous diffusion limits.

The proofs, as easily guessed, are quite technical and lie on the Lindeberg method in the central limit theorem for mixing sequences.

We shall make the following abuse of notation, denoting simply by v the function $v \mapsto v$.
$< U, V >$ will denote the scalar product in $\mathbb{R}^d$ when U, V are vectors in $\mathbb{R}^d$. $< M >$ will denote the martingale bracket, when M is a martingale.
C will denote a constant that may change from line to line.

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2. Classical rate of convergence.

In this section we recall the basic facts about long time behavior of stochastic diffusion processes we will need to provide different proofs for an alternative result to Theorem 1.1.

2.1. **Ergodic behavior, long time behavior.**

Come back to the S.D.E. (1.6). We will assume that
Hypotheses H.

(1) **H1** $\omega > 0$ is smooth (C^2 or C^∞) and such that there exists C with $\int C\,\omega^{-1}\,dv = 1$. We thus define the probability measure $\mu(dv) = C\omega^{-1}(v)\,dv$.
(2) **H2** there exists $c \in \mathbb{R}$ such that for all v, $\langle v, \nabla\omega\rangle \geq c\,\omega(v)$. The latter condition is sometimes called a *drift condition*.

If (H) is satisfied, it is known that (1.6) has an unique non explosive solution starting from any v_0 (H2 ensures that $v \mapsto |v|^2$ is a Lyapunov function for Hasminskii's explosion test). In addition μ is the unique invariant probability measure for the process, and is actually symmetric. This means that for all $g, h \in C_0^\infty$,

$$\int gLh\,d\mu = \int hLg\,d\mu\,.$$

We may thus define for $g \in C_b^\infty$,

$$P_t g(v) = e^{tL} g(v) = \mathbb{E}_v(g(v_t))$$

the associated semi-group, $\mathbb{E}_v$ being the expectation when the process starts from v. This semi-group extends to all $\mathbb{L}^p(\mu)$ spaces, and is self-ajoint in $\mathbb{L}^2(\mu)$. It is a Markov semi-group, i.e. $P_t 1 = 1$ (1 being here the constant function). Furthermore, the operator norm of P_t acting on $\mathbb{L}^p(\mu)$ is equal to 1.
Thanks to symmetry, if $h \geq 0$ satisfies $\int h d\mu = 1$ and if the law of v_0 is given by $h\mu$, the law of v_t is exactly $P_t h\,\mu$. In other words the solution of

$$\partial_t f = Q(f) \quad \text{ with } \; f_0(v) = (h\,\omega^{-1})\,(v) \tag{2.1}$$

is given by

$$P_t\,(f_0\,\omega)\,(v)\,\omega^{-1}(v)\,.$$

Hence if we look at (1.4) without the transport term (or if one wants with an initial datum only depending on v), the asymptotic behavior as $\varepsilon \to 0$ is given by the long time behavior of the semi-group P_t.
We shall now recall some known facts about this long time behavior.

Denote

$$\mathbb{L}_0^p(\mu) = \mathbb{L}^p(\mu) \cap \{g \in \mathbb{L}^1(\mu); \int g d\mu = 0\}$$

the hyperplane of $\mathbb{L}^p$ whose elements have zero mean. If $1 \leq p \leq r \leq +\infty$, and T is a bounded operator from $\mathbb{L}_0^r(\mu)$ into $\mathbb{L}_0^p(\mu)$ introduce

$$|T|_{rp} = \sup\left\{ g \neq 0 \in \mathbb{L}_0^r(\mu); \frac{\parallel Tg \parallel_p}{\parallel g \parallel_r} \right\} \tag{2.2}$$

the operator norm of T.
Of course, P_t is bounded from $\mathbb{L}_0^p(\mu)$ into $\mathbb{L}_0^p(\mu)$, and $|P_t|_{pp} \leq 1$. The first next result is due to Roeckner and Wang [21]. A stronger version is contained in [9].

Proposition 2.1. *Assume that hypotheses (H) are satisfied.*
Then $\alpha(t) := |P_t|_{\infty,2} \to 0$ *as* $t \to +\infty$.

Notice that thanks to the semi-group property and the stability of $\mathbb{L}^p_0(\mu)$, as soon as $|P_{t_0}|_{pp} < 1$ for some $t_0 > 0$ and some $1 < p < +\infty$, then $|P_t|_{pp} \leq K_p\, e^{-\lambda_p\, t}$ for some K_p and $\lambda_p > 0$. Applying the Riesz-Thorin interpolation theorem in an appropriate way (see [8]) one deduces that the same holds for all $1 < p < +\infty$. It follows the following alternative

$$\text{either } |P_t|_{pp} = 1 \text{ for all } 1 < p < +\infty, \text{ or } |P_t|_{pp} \leq K_p\, e^{-\lambda_p\, t} \text{ for all } 1 < p < +\infty. \tag{2.3}$$

Remark 2.2. If there exist c and $\lambda > 0$ such that $\alpha(t) \leq c\, e^{-\lambda t}$, then $|P_t|_{22} \leq e^{-\lambda t}$. The last statement is equivalent to the fact that μ satisfies a Poincaré inequality

$$\text{for all smooth } f \in \mathbb{L}^2_0(\mu)\,, \quad \int |f|^2 d\mu \leq \frac{1}{\lambda} \int |\nabla f|^2 d\mu\,. \tag{2.4}$$

As it is well known, (2.4) implies the existence of some exponential moment for μ. Once this property is not satisfied, for instance if $\omega = \omega_\beta$ is the Barenblatt profile (called a generalized Cauchy distribution in probability), one cannot expect some exponential convergence.

We shall come back later to the rate of convergence α which is of key importance for our problem.

But first, let us indicate a complementary result,

Proposition 2.3. *Consider the equation (2.1). Assume that (H) is satisfied and, for simplicity that $C = 1$ in (H1), and that $f_0 \geq 0$ is such that $\int f_0 dv = 1$.*
If $f_0\, \omega \in \mathbb{L}^r(\omega^{-1} dv)$ for some $r > 1$, then the solution $f_t = P_t(f_0\omega)\, \omega^{-1}$ of (2.1) converges as $t \to +\infty$ towards ω^{-1} in the following sense: for all $1 \leq p < r$ there exists some $\alpha(r,p,t) \to 0$ as $t \to +\infty$, such that

$$\left(\int |P_t(f_0\omega) - 1|^p \omega^{-1}\, dv\right)^{\frac{1}{p}} \leq C(p,r)\, \alpha(r,p,t) \left(\int |(f_0\omega) - 1|^r \omega^{-1}\, dv\right)^{\frac{1}{r}}.$$

In other words for all $r > p \geq 1$, if $f_0 \in \mathbb{L}^r(\omega^{r-1}\, dv)$, $f_t \to \omega^{-1}$ in $\mathbb{L}^p(\omega^{p-1}\, dv)$.

A simple application of Riesz-Thorin interpolation theorem to T_t defined by $T_t g = P_t g - \int g\, d\mu$ with the pairs $(\infty, 2)$ and (q,q) furnishes

$$\text{for } r > p \geq 2, \quad \alpha(r,p,t) \leq c(p,r)\, \alpha^{2(\frac{1}{p} - \frac{1}{r})}(t)\,. \tag{2.5}$$

Another simple proof is contained in [4] lemma 5.1. For $1 \leq p < r \leq 2$ we obtain the result by duality and for $1 \leq p \leq 2 < r$ by a simple combination, thus

$$\begin{aligned} \text{for } 2 \geq r > p \geq 1,\ \alpha(r,p,t) &\leq c(r,p)\, \alpha^{2(\frac{1}{p} - \frac{1}{r})}(t)\,, \\ \text{for } r > 2 \geq p \geq 1,\ \alpha(r,p,t) &\leq c(r,p)\, \alpha^{2(\frac{1}{p} - \frac{1}{r})}(t/2)\,. \end{aligned} \tag{2.6}$$

In the sequel, we will use α.

2.2. **Additive functional and the central limit theorem.**

As we said in the introduction, the propagation of chaos (in time) property explained in [5] allows us to look separately at $v_{t/\theta(\varepsilon)}$ and $\varepsilon \int_0^{t/\theta(\varepsilon)} v_s\, ds$ to get their asymptotic joint law. In the sequel we add a new assumption

$$\int v\, d\mu \,=\, 0\,. \tag{2.7}$$

The asymptotic behavior of such additive functionals is well understood when $v \in \mathbb{L}^2(\mu)$. It is much less understood when $v \in \mathbb{L}^p(\mu)$ for some $p < 2$ (and not in $\mathbb{L}^2(\mu)$ of course).

For the latter situation nothing is known in the continuous time setting of this paper. In a discrete time setting some (in a sense not really satisfactory) results have been obtained in [13, 10].

For the former one, we shall recall here the essential results explained in [4] (with of course plenty of previous results, see the bibliography in [4]). Notice that we are facing here an additional difficulty since the integrand is vector valued (and not real valued).

From now on, we will thus assume that

$$\int |v|^2\, d\mu \,<\, +\infty\,. \tag{2.8}$$

Denote by

$$S_t^i = \int_0^t v_s^i\, ds \quad \text{and} \quad s_t^{ij} = \mathbb{E}_\mu(S_t^i\, S_t^j)\,.$$

The asymptotic behavior of S_t is given by the so called *central limit theorem for additive functionals* (a stronger form is called Donsker invariance principle or functional central limit principle). This principle tells us how to normalize $S_.$ to ensure the convergence of its *law* (or probability distribution) to a gaussian law.

Definition 2.4. *We say that $S_.$ satisfies a multi-times central limit theorem (MCLT) at equilibrium with rate ζ and asymptotic covariance matrix Γ, if for every finite sequence $0 < t_1 \leq ... \leq t_n < +\infty$,*

$$\zeta(\eta)\, \left(S_{t_1/\eta}, ..., S_{t_n/\eta}\right) \to (B_{t_1}, ..., B_{t_n})$$

in law as $\eta \to 0$, where $(B_.)$ is a Brownian motion on $\mathbb{R}^d$ with covariance matrix Γ.
If the previous holds only for $n = 1$ (one time) but all t, we say that (CLT) is satisfied (the limit being then a gaussian vector).

Notice that in the previous definition, we assumed that the initial distribution of v_0 is the invariant distribution μ. We shall similarly use the terminology (MCLT) *out of equilibrium* when we can replace μ by some other initial distribution. Also remark that the definition of (MCLT) in [4] is not correctly stated.

There are mainly three approaches to get (MCLT) in our situation: the Kipnis-Varadhan theorem, mixing and a martingale approach. We start with the first one.

2.2.1. *Kipnis-Varadhan approach.* Using reversibility it is not difficult to see that,

$$s_t^{ij} = 2 \int_{0 \leq u \leq s \leq t} \left(\int P_{u/2}v^i\, P_{u/2}v^j\, d\mu\right) duds\,. \tag{2.9}$$

We shall say that the Kipnis-Varadhan condition is satisfied if

$$V := \int_0^{+\infty} \parallel P_t v \parallel_2^2 \, dt < +\infty \, . \tag{2.10}$$

In this situation, using Cesaro rule, it is immediate that

$$\frac{s^{ij}(t)}{t} \to 4V^{ij} = 4 \int_0^{+\infty} \left(\int P_t v^i \, P_t v^j \, d\mu \right) \, dt < +\infty \text{ as } t \to +\infty. \tag{2.11}$$

Kipnis-Varadhan theorem [14], revisited in [4] Theorem 3.3. and Remark 3.6. (which immediately extend to the multi-dimensional setting) then tells us:

Theorem 2.5. *Assume that (H), (2.7) and (2.8) are satisfied. If (2.10) is satisfied, then $S_.$ satisfies the (MCLT) at equilibrium, with rate $\zeta(\eta) = \sqrt{\eta}$ and asymptotic covariance matrix (or effective diffusion tensor) $\Gamma^{ij} = 4V^{ij}$.*

As discussed in Remark 3.6 of [4] a sufficient condition for (2.10) to be satisfied is the following: let $H_0^1 = \mathbb{L}_0^2 \cap \{g \, ; \, \nabla g \in L^2(\mu)\}$. Then (2.10) is satisfied as soon as

$$\left(\int v^i \, g \, d\mu \right)^2 \leq c_i \int |\nabla g|^2 \, d\mu \quad \text{ for all } i = 1, ..., d \text{ and all } g \in H_0^1. \tag{2.12}$$

Now we may apply all the results of section 8 in [4], since our model fulfills all assumptions therein. This allows us to obtain (MCLT) out of equilibrium (see Theorem 8.6 in [4]).

Theorem 2.6. *The conclusion of Theorem 2.5 still holds true out of equilibrium provided the law of the initial condition is either a Dirac mass δ_{v_0} or is absolutely continuous w.r.t. μ.*

As a corollary we obtain (since $x + B_t$ is still a Brownian motion with mean x)

Corollary 2.7. *Assume that (H) holds true (with $C = 1$ for simplicity). Consider (1.4) with $f_0 \geq 0$ such that $\int f_0(x, v) \, dxdv = 1$. Assume in addition that (2.7) is satisfied and that $v \in \mathbb{L}^p(\omega^{-1} dv)$ for some $p \geq 2$.*
Then, provided (2.10) is satisfied, choosing $\theta(\varepsilon) = \varepsilon^2$, the solution f^ε of (1.4) weakly converges as $\varepsilon \to 0$ towards $\rho(t, x) \, \omega^{-1}(v)$ where ρ is the unique solution of the heat equation

$$\partial_t \rho - \nabla_x . (\Gamma \, \nabla_x) \rho = 0$$

with initial datum $\rho_0(x) = \int f_0(x, v) dv$. Here weak convergence means

$$\lim_{\varepsilon \to 0} \int F(x, v) \, f^\varepsilon(t, x, v) \, dxdv = \int F(x, v) \, \rho(t, x) \, \omega^{-1}(v) \, dxdv \, ,$$

for all n, all t and all F which is continuous and bounded.
In particular the result holds true as soon as $\int_0^{+\infty} \alpha^2(p, 2, t) \, dt < +\infty$.

Notice that we can also assume that the initial condition is a Dirac mass δ_{x_0, v_0}. The type of convergence we obtain is different from the one in [20]. Furthermore it can be extended to a multi-time convergence. Of course the framework here is more general.

2.2.2. *Martingale approach.* If one can obtain the Kipnis-Varadhan theorem by using an approximate martingale method (see [4] Theorem 3.3), the (true) martingale method is the most popular method for studying additive functionals, and is actually used in [5]. The method is based on the following idea: assume that we can find a nice solution to the Poisson equation (which here is vectorial)

$$LH = v\,. \tag{2.13}$$

Then applying Ito's formula we have

$$S_t = H(v_t) - H(v_0) - \sqrt{2}\int_0^t \nabla H(v_s).dB_s\,,$$

so that, provided the boundary terms are in a sense neglectable, the asymptotic behavior of S_t is equivalent to the one of the martingale term $M_t = \sqrt{2}\int_0^t \nabla H(v_s).dB_s$ for which (MCLT) is known for a long time.

Formally the solution of (2.13) satisfying $\int H d\mu = 0$ (still assuming (2.7)) is given by

$$H = -\int_0^{+\infty} P_t v\, dt$$

which exists in $\mathbb{L}^2(\mu)$ if and only if

$$\int_0^{+\infty} s \parallel P_s v \parallel_2^2\, ds < +\infty$$

according to [4] corollary 3.2.

This condition is stronger than (2.10) so that, from a general point of view, there is no possible gain by using this strategy, except the following: provided the martingale term is in $\mathbb{L}^2(\mu)$, we only need that $H \in \mathbb{L}^1(\mu)$. For instance it is enough that $\int_0^\infty \parallel \mathbb{P}_t v \parallel_1\, dt < +\infty$, which holds in particular when $v \in \mathbb{L}^p(\mu)$ for some $p > 1$ and $\int_0^t \alpha(p,1,t)\, dt < +\infty$. But in this situation we need $\nabla H \in \mathbb{L}^2(\mu)$ to ensure that the martingale term is squared integrable. It is very hard in general to explicitly control $P_t v$ since even if we know some bound for $\alpha(t)$, which is the case in many situations, this bound only furnishes upper bounds. It turns out, that in some specific cases, one can directly solve (2.13). As shown in [20], it can be done for $\omega = \omega_\beta$. In addition, in this situation one obtains an explicit expression for the effective diffusion tensor. We gather all this in the following theorem (proofs are contained in [4])

Theorem 2.8. *In the situation of Corollary 2.7, assume that a solution H of (2.13) exists and satisfies : $H \in \mathbb{L}_0^1(\omega^{-1}dv)$ and $\nabla H \in \mathbb{L}^2(\omega^{-1}dv)$.*
Then the conclusion of Corollary 2.7 still holds true, with $\theta(\varepsilon) = \varepsilon^2$ and

$$\Gamma^{ij} = 2\int < \nabla H^i, \nabla H^j > \omega^{-1}\, dv\,.$$

2.2.3. *Mixing and possibly anomalous rates of convergence.*

We come now to the third method using mixing. Roughly speaking mixing describes the rate of decay of the correlations of the process, i.e. is some kind of quantitative propagation of chaos. We refer to section 4 in [4] for the details of how mixing results can be used to partly recover the results of the previous subsection.

For simplicity we shall reduce our study to one coordinate, say S_t^1 of S_t. Analytically it corresponds to looking at $f^{\varepsilon,1}(t,x_1,v) = \int f^\varepsilon(t,x_1,y_2...y_d,v)\,dy_2...dy_d$.

The choice of $\theta(\varepsilon)=\varepsilon^2$ in the previous subsection is of course due to the fact that $(1/\sqrt{t})\,S_t^1$ converges, as $t\to+\infty$ in probability distribution to a centered gaussian distribution with variance $4V^{11}$. The normalization by $\sqrt{t}$ is itself due to the asymptotic behavior of s_t^{11} described by (2.11).

A more natural normalization is given by the variance itself, i.e. it is natural to look at $S_t^{11}/\sqrt{s^{11}(t)}$ whose $\mathbb{L}^2(\mu)$ norm is constant equal to 1. In our situation (under hypothesis (H) and (2.7)) the Central Limit Theorem (CLT) at equilibrium is known as Denker's theorem, recalled in Theorem 4.5 of [4]

Theorem 2.9. *Assume that (H), (2.7) and (2.8) are satisfied. Also assume that v_0 is distributed according to the invariant measure μ and that $s_t^{11}\to+\infty$ as $t\to+\infty$. Then the following two conditions are equivalent*

(1) $\left(\frac{(S_t^{11})^2}{s_t^{11}}\right)_{t\geq 1}$ *is uniformly integrable,*

(2) $\frac{S_t^{11}}{\sqrt{s_t^{11}}}$ *converges in probability distribution to a standard gaussian law, as $t\to+\infty$.*

As explained in sections 6 and 7 of [4], the first statement is presumably only possible for $s_t^{11}=th(t)$ where h is a *slowly varying* function. Section 6 of [4] gives an example of this situation where one can use Denker's result.

In addition, if s^{11} is as before, we may use again the results of section 8 in [4]. We may thus deduce

Corollary 2.10. *In the situation of Corollary 2.7, assume that $s_t^{11}=th(t)$ where h is a slowly varying function and that condition (1) in Theorem 2.9 is satisfied.*
Then for $\theta(\varepsilon)$ such that $\frac{\theta(\varepsilon)}{h(1/\theta(\varepsilon))}=\varepsilon^2$, the integrated solution $f^{\varepsilon,1}$ of (1.4) converges towards $\rho^1(t,x^1)\omega^{-1}(v)$ where ρ^1 is the solution of the ordinary 1-dimensional heat equation; the meaning of the convergence being the same as the one in Corollary 2.7.

2.3. Application to Barenblatt/Cauchy profiles.

The rate of convergence $\alpha(t)$ is now known for a lot of models. Nevertheless, here we shall only look at the case $\omega^{-1}=C_\beta\,\omega_\beta^{-1}$ i.e. the general Cauchy distribution also known as Barenblatt profile. This case is partly discussed in subsection 5.4.1 of [4], but we shall here give more detailed results.

Lemma 2.11. *Recall that $\alpha(t)=|P_t|_{\infty,2}$, P_t being the semigroup associated to the operator L given by (1.5), we have the following estimate*

$$\alpha(t)\leq\frac{C(\beta,d)}{t^{(\beta-d)/4}}.$$

Proof. In order to calculate $\alpha(t)$ we shall use the optimal *weak Poincaré inequality* obtained in [6] (improving on [21]): there exists some constant $C(d,\beta)$ such that for all nice f with

$\int f\, d\mu = 0$ it holds for all $s > 0$,

$$\int f^2\, d\mu \, \leq \, C\, s^{-2/(\beta-d)} \int |\nabla f|^2\, d\mu + \, s \, \parallel f \parallel_\infty^2 \, . \tag{2.14}$$

An easy optimization in s furnishes for these f's, the *Nash type inequality*

$$\int f^2\, d\mu \, \leq \, C \left(\int |\nabla f|^2\, d\mu \right)^{\frac{\beta-d}{\beta-d+2}} \left(\parallel f \parallel_\infty^2 \right)^{\frac{2}{\beta-d+2}} \, . \tag{2.15}$$

In order to stay self-contained, we recall Theorem 2.2 in [15]

Theorem 2.12. *Let L be a linear operator generating a Markov Semi-group P_t. Define $\mathcal{E}(f,f) = -E_\mu(f(Lf)$, if a Nash type inequality*

$$E_\mu(f^2) \leq \mathcal{E}(f,f)^{1/p} \Phi(f)^{1/q} \qquad \text{with } p^{-1} + q^{-1} = 1$$

holds, with Φ satisfying $\Phi(P_t f) \leq \Phi(f)$, then there exists $c > 0$ such that

$$E_\mu((P_t f)^2) \leq \Phi(f) t^{1-q}, \quad t > 0, \quad f \in L^2(\mu), \quad E_\mu(f) = 0.$$

It follows that for all $t \geq 1$,

$$\alpha^2(t) \leq \frac{c(\beta,d)}{t^{(\beta-d)/2}} \, , \text{ hence } \alpha^2(p,2,t) \leq \frac{c(\beta,d,p)}{t^{(\beta-d)(\frac{1}{2}-\frac{1}{p})}} \, . \tag{2.16}$$

□

Note, that the latter is integrable if and only if

$$(\beta - d) \left(\frac{1}{2} - \frac{1}{p} \right) > 1 \, .$$

Recall that $v \in \mathbb{L}^p(\mu)$ if and only if $\beta - d > p$, so that we may apply Corollary 2.7 provided $\beta > d + 4$.

Remark 2.13. The general trick used to obtain $\alpha(t)$ starting from (2.14), explained in [21] Theorem 2.1 and recalled in [4], is not optimal, since some extra logarithmic factor appears. As said in [4], this trick, starting from the optimal weak Poincaré inequality furnishes the optimal mixing rate *up to a slowly varying* extra factor.

It is also useful at this point to clarify the "mixing" property we are speaking about. Denote by $\mathcal{F}_t$ the filtration generated by v_s for $s \leq t$ (or equivalently here generated by the Brownian motion $B_.$) and by $\mathcal{G}_u$ the σ-field generated by v_s for $s \geq u$. If F and G are bounded, non-negative and respectively $\mathcal{F}_t$ and $\mathcal{G}_u$ measurable for some $u > t$, then

$$\mathbb{E}_\mu(FG) \leq \alpha^2 \left(\frac{u-t}{2} \right) \parallel F \parallel_\infty \parallel G \parallel_\infty \, .$$

Indeed, using first the Markov property, then conditional expectation w.r.t. v_t and finally stationarity and symmetry we have

$$\begin{aligned}
\mathbb{E}_\mu(FG) &= \mathbb{E}_\mu(F\, \mathbb{E}_\mu(G|v_u)) = \mathbb{E}_\mu(F\, g(v_u)) \\
&= \mathbb{E}_\mu(F\, (P_{u-t} g)(v_t)) = \mathbb{E}_\mu(\mathbb{E}_\mu(F|v_t)\, (P_{u-t} g)(v_t)) = \mathbb{E}_\mu(f(v_t)\, (P_{u-t} g)(v_t)) \\
&= \int f\, P_{u-t} g\, d\mu = \int P_{\frac{u-t}{2}} f\, P_{\frac{u-t}{2}} g\, d\mu
\end{aligned}$$

where f and g are bounded respectively by $\parallel F \parallel_\infty$ and $\parallel G \parallel_\infty$, so that using Cauchy-Schwarz inequality and the decay of the semi-group we have the desired result. $\diamondsuit$

As we said in the previous section, we can here explicitly solve the Poisson equation

$$LH = v\,.$$

Inspired by the calculation in [20] we search for

$$H^i = v^i\,(a|v|^2 + b)\,.$$

Notice that the v^i's are exchangeable, so that a and b are the same for all components. As in [20] we get

$$a = \frac{1}{4+2d-3\beta} \quad , \quad b = \frac{3}{4+2d-3\beta}\,,$$

except if $\beta = \frac{2d+4}{3}$ which is impossible if $v \in \mathbb{L}^2$, i.e. $\beta > d+2$.
Now $|\nabla H|^2$ behaves like $|v|^4$, so that it is integrable if and only if $\beta > d+4$. In this situation $H \in \mathbb{L}^1_0(\mu)$. We may thus apply Theorem 2.8, which furnishes an explicit expression for Γ^{ij}, the one obtained in [20].
Notice that

$$\Gamma^{ij} = \int \, < \nabla H^i, \nabla H^j > \, \omega^{-1}\, dv \, = 0\,,$$

for $i \neq j$, and $\Gamma^{ii} = \gamma$ does not depend on i, all these properties being easy consequences of symmetries.
We thus have

Theorem 2.14. *For $\omega^{-1} = C_\beta\, \omega_\beta^{-1}$ and $\beta > d+4$, Corollary 2.7 holds true with*

$$\Gamma \, = \left(\int \, |\nabla H|^2\, \omega^{-1}\, dv\right) \, Id\,,$$

where $H^i = v^i\,(a|v|^2 + b)$ with $a = \frac{1}{4+2d-3\beta}$ and $b = \frac{3}{4+2d-3\beta}$.

Remark 2.15. If $\beta > d+4$, $H \in \mathbb{L}^q(\mu)$ for $q < (\beta - d)/3$. Since all zero mean harmonic function (harmonic w.r.t. L) are constant functions, H is the unique solution of $LH = v$. Recall that $G = -\int_0^{+\infty} P_t v\, dt$ also satisfies $LG = v$ so that $G = H$. It is interesting to notice that we directly get that $G \in \mathbb{L}^q(\mu)$ provided $\alpha(p,q,t)$ is integrable, for some $p < \beta - d$. This leads to $q < (\beta - d)/3$. This means that the decay to 0 of $P_t v$ is more or less equal to the uniform rate we recalled in (2.16). $\diamondsuit$

3. Anomalous rate of convergence: a general method.

We shall look now at the case $\beta - d \leq 4$, for which $|\nabla H|$ does no more belong to $\mathbb{L}^2(\mu)$.
So we are facing several problems

(1) When $\beta > d+2$, $v \in \mathbb{L}^2(\mu)$ so that S_t has a finite variance. But what is the long time behavior of $\mathbb{E}_\mu(S_t^2)$ since ∇H is no more square integrable ?

(2) What is the "good" normalization s_t for $S^i_t/\sqrt{s_t}$ to converge in distribution ? The natural choice would thus be $s_t = \mathrm{Var}_\mu(S^i_t)$, but several arguments in [4] and an old result by Lamperti for the invariance principle indicate that this will be the case only if $\mathrm{Var}_\mu(S^i_t)$ behaves like t times a slowly varying function.
(3) If such an s_t exists, what is the limiting distribution ?
(4) What happens with the joint distribution, i.e. with the random *vector* S_t ?

In order to answer to all these questions we shall adopt a very simple and general strategy:

(1) Use some cut-off functions $K(t)$ directly on H. To this end, for $K > 0$, we define

$$H_K(v) = bv + \, av|v|^2 \, \mathbf{1}_{|v|\leq K} \, + \, a \, \left(3K^2 \, v - 2K^3 \, \frac{v}{|v|}\right) \, \mathbf{1}_{|v|>K} \, . \tag{3.1}$$

Note that H_K is of class C^1 and also is in L^1_μ, and that the second derivatives exist and are continuous for $|v| \neq K$. If we define $v_K = LH_K$, it is not difficult to see, that for $|v| > K$,

$$v_K(v) = v \, \left(\frac{2a(d-1)K^3}{|v|^3} \, - \, \frac{3a\beta \, K^2}{1+|v|^2} \, - \, \frac{b\beta}{1+|v|^2}\right) \, ,$$

so that there exists $C > 0$ such that

$$|v_K(v) - v| \, \leq \, C \, |v| \, \mathbf{1}_{|v|\geq K} \, . \tag{3.2}$$

Actually it is easier to use a cut-off on H rather than on v, introducing some bounded v_K, since then we do not know the explicit solution of the Poisson equation $LH'_K = v_K$.

From now on we assume that $K(t)$ growths to infinity as t goes to infinity.

Note the following properties satisfied by H_K that will be useful later on. First of all, $|\nabla H_K| \leq CK^2$. Moreover

$$\text{if } \beta - d = 4, \; \mu(|\nabla H_K|^2) \approx C \ln K \tag{3.3}$$

and, once $p > 2$,

$$\mu(|\nabla H_K|^p) \leq CK^{2p+d-\beta} \, . \tag{3.4}$$

(2) Define $S^K_t = \int_0^t v_K(v_s)\, ds$. Since $\nabla H^i_K \in \mathbb{L}^2(\mu)$ for all i, we may compute the covariance matrix of S^K_t. In particular if we define for $\theta \in \mathbb{S}^d$, $s^K_t = \mathrm{Var}_\mu(<\theta, S^K_t>)$, it is not hard to see that s^K_t does not depend on θ.
(3) Choose some $K_j(t)$ $(j = 1, 2)$ growing to infinity with t such that on one hand $(S_t - S^{K_2(t)}_t)/\sqrt{s^{K_2(t)}_t}$ goes to 0 as $t \to +\infty$ in $\mathbb{L}^2(\mu)$, so that $\mathrm{Var}_\mu(S_t)$ will be asymptotically equal to $s^{K_2(t)}_t$, on the other hand $(S_t - S^{K_1(t)}_t)/\sqrt{s^{K_1(t)}_t}$ goes to 0 in *Probability* (for instance in $\mathbb{L}^1(\mu)$).
(4) Prove some Central Limit Theorem for $S^{K_1(t)}_t/\sqrt{s^{K_1(t)}_t}$, so that the same Central Limit Theorem will be available for $S_t/\sqrt{s^{K_1(t)}_t}$ thanks to Slutsky's theorem. The difference between $K_1(t)$ and $K_2(t)$ will thus explain the *anomalous rate of convergence* since the normalization *will not be the asymptotic square root of the variance.*

It is worth noticing that $K_1(t)$ has to be chosen in order to satisfy two conditions: a good cut-off property and some central limit theorem for $S_t^{K_1(t)}/\sqrt{s_t^{K_1(t)}}$. Not any cut-off can do the job.

3.1. Computation of $\mathbb{E}_\mu((S_t^{i,K})^2)$.

We start with the previous point (2). In the sequel we shall sometimes simply write K instead of $K(t)$ to simplify the notation. Let us prove

Lemma 3.1. *The following holds true : if $\beta - d = 4$, then provided $K(t) \ll \sqrt{t \, \ln K(t)}$, there exists $\kappa' > 0$ such that for all $i = 1, ..., d$*

$$\frac{\mathbb{E}_\mu((S_t^{i,K(t)})^2)}{t \, \ln K(t)} \to \kappa' \quad \text{as } t \to +\infty .$$

Proof. Though H_K^i is not C^2, but $\partial^2 H_K^i$ is piecewise continuous, we may apply the extended Ito's formula (sometimes called Meyer-Ito formula) and write

$$S_t^{i,K} = H_K^i(v_t) - H_K^i(v_0) - \sqrt{2} \int_0^t < \nabla H_K^i(v_s), dB_s > \, . \tag{3.5}$$

We denote by $M_.^{i,K}$ the martingale $\sqrt{2} \int_0^. < \nabla H_K^i(v_s), dB_s >$ with brackets

$$\langle M^{i,K} \rangle_u = 2 \int_0^u |\nabla H_K^i|^2(v_s) \, ds.$$

Since all S^i have the same distribution, from now on we skip the superscript i.

The key point is that, since μ is reversible, if v_0 is distributed according to μ, $s \mapsto v_{t-s}$ has the same distribution (on the path space up to time t) as $s \mapsto v_s$. We may thus write

$$S_t^K = \int_0^t v_K(v_{t-s}) \, ds = H_K(v_0) - H_K(v_t) - \hat{M}_t^K \, ,$$

where $\hat{M}_.^K$ is a *backward* martingale with brackets

$$\langle \hat{M}^K \rangle_u = 2 \int_0^u |\nabla H_K|^2(v_{t-s}) \, ds.$$

In particular we have the following decomposition, known as Lyons-Zheng decomposition

$$S_t^K = -\frac{1}{2} \left(M_t^K + \hat{M}_t^K \right) .$$

Another easy application of the reversibility property is the following: provided S_t^K and H_K are square integrable, i.e. for $2 < \beta + d$,

$$\mathbb{E}_\mu(S_t^K \, (H_K(v_t) - H_K(v_0))) = 0.$$

It follows

$$\mathbb{E}_\mu((S_t^K)^2) + \mathbb{E}_\mu((H_K(v_t) - H_K(v_0))^2) = \mathbb{E}_\mu((M_t^K)^2) = \mathbb{E}_\mu((\hat{M}_t^K)^2) \, .$$

Now thanks to stationarity

$$\mathbb{E}_\mu((M_t^K)^2) = 2t \, \mu(|\nabla H_K|^2).$$

It follows if $\beta - d = 4$, by (3.3), $\mathbb{E}_\mu((M_t^K)^2) = C(d,\beta)\, t\, \ln K + t\, o(\ln K)$.
At the same time,

$$\begin{aligned}\mathbb{E}_\mu((H_K(v_t) - H_K(v_0))^2) &= 2\,\mu(|H_K|^2) - 2\,\mathbb{E}_\mu(H_K(v_0)H_K(v_t)) \\ &= 2\,\mu(|H_K|^2) - 2\,\mu(|P_{t/2}H_K|^2).\end{aligned}$$

Since $\mu(|H_K|^2) \approx C'(d,\beta)\, K^{6+d-\beta}$, we get

$$\mathbb{E}_\mu((H_K(v_t) - H_K(v_0))^2) \leq C K^{6+d-\beta}$$

and thus, if we choose $K(t) \ll \sqrt{t\, \ln K(t)}$,

$$\mathbb{E}_\mu((H_K(v_t) - H_K(v_0))^2) \ll \mathbb{E}_\mu((M_t^K)^2).$$

□

Notice that, since $\mu(< \nabla H_K^i, \nabla H_K^j >) = 0$ for $j \neq i$, the martingales $M^{i,K}$ and $M^{j,K}$ are orthogonal, yielding

$$\frac{1}{t\, \ln K(t)}\, \mathrm{Cov}_\mu(S_t^K) \to \kappa'\, Id.$$

3.2. Computation of $\mathbb{E}_\mu((S_t^i)^2)$.

We may decompose $S_t = (S_t - S_t^K) + S_t^K$ so that $\mathbb{E}_\mu((S_t^i)^2)$ will behave like $\mathbb{E}_\mu((S_t^{i,K})^2)$ provided $\mathbb{E}_\mu((S_t^{i,K})^2) \gg \mathbb{E}_\mu((S_t^i - S_t^{i,K})^2)$.

Now

$$\begin{aligned}\mathbb{E}_\mu(|S_t - S_t^K|^2) &= 2\,\mathbb{E}_\mu\left(\int_0^t \int_0^s (v_u - v_K(v_u))(v_s - v_K(v_s))\, du\, ds\right) \\ &= 2\int_0^t \int_0^s \mathbb{E}_\mu\left(g\, P_{s-u} g\right)\, du\, ds \\ &\leq C\, t \int_0^{t/2} \parallel P_s g \parallel_2^2\, ds\,, \end{aligned} \tag{3.6}$$

where $g = v - v_K$ has zero μ mean and $|g(v)| \leq C|v|\, \mathbf{1}_{|v|\geq K}$ according to (3.2). Recall that $g \in \mathbb{L}^p(\mu)$ for $p < \beta - d$, and that,

$$\parallel g \parallel_p^p \leq \frac{C}{\beta - d - p}\, K^{p+d-\beta}\,.$$

Actually, there is no need to be clever, because all choices of p will give the same rough bounds. So just take $p = 2$ and apply the contraction property of the semi-group, which yields

$$\mathbb{E}_\mu(|S_t - S_t^K|^2) \leq C\, (t^2/K^{\beta-d-2})\,. \tag{3.7}$$

Lemma 3.2. *In the case where $\beta - d = 4$, there exists κ such that*

$$\frac{1}{t\, \ln t}\, E_\mu((S_t^i)^2) \to \kappa \;\; \text{as } t \to +\infty. \tag{3.8}$$

Hence with our previous notations one good choice for $K_2(t)$ is $\sqrt{t}$, yielding the correct asymptotic behavior for the variance of S_t and $\kappa = \frac{1}{2}\,\kappa'$ for the κ' in Lemma 3.1.

Proof. According to Lemma 3.1 (2) and to (3.7), choosing

$$K^2 \,\ln K \gg t \gg K^2/\ln K \quad \text{by taking } \ K(t) = \sqrt{t} \tag{3.9}$$

yields

$$\frac{1}{t\,\ln K(t)}\,\mathrm{Cov}_\mu(S_t) \to \kappa'\, Id \ \text{ as } t \to +\infty. \tag{3.10}$$

□

Remark 3.3. In this situation, since

$$2t \int_0^{t/4} \parallel P_s v \parallel_2^2 \, ds \le s_t \le 4t \int_0^{t/2} \parallel P_s v \parallel_2^2 \, ds \,,$$

which easily follows from (2.9) (see [4] lemma 2.3), we obtain using (3.10) that

$$C\,\ln t \le \int_0^t \parallel P_s v \parallel_2^2 \, ds \le C'\,\ln t \,, \tag{3.11}$$

which is exactly the uniform bound we obtain for functions in $\mathbb{L}^4$, but $v \notin \mathbb{L}^4$. ♢

3.3. Approximation of S_t in Probability, i.e. finding $K_1(t)$.

The title of this subsection is clear, we want to find $K_1(t)$ such that

$$(S_t - S_t^{K_1(t)})/\sqrt{\mathrm{Var}_\mu(S_t^{K_1(t)})}$$

goes to 0 in Probability, as $t \to +\infty$. As we have seen, convergence in $\mathbb{L}^2(\mu)$ holds for $K^2(t) \gg t/\ln t$, but convergence in $\mathbb{L}^1(\mu)$ will hold in more general situations.

Lemma 3.4. *If $K(t) \gg (t/\ln t)^{1/6}$, then*

$$\frac{|S_t - S_t^{K(t)}|}{\sqrt{t \ln K(t)}} \to 0 \quad in \quad \mathbb{L}^1(\mathbb{P}_\mu).$$

Proof. Indeed,

$$\mathbb{E}_\mu \left(\frac{|S_t - S_t^{K(t)}|}{\sqrt{t \ln K(t)}} \right) \le \frac{t}{\sqrt{t \ln K(t)}}\, \mu(|v - v_{K(t)}(v)|) \le C(d)\, \frac{t}{\sqrt{t\,\ln K(t)}\, K^3(t)} \,,$$

will go to 0 as soon as

$$K(t) \gg (t/\ln t)^{1/6} \,. \tag{3.12}$$

□

This lemma shows that if we choose $K(t) = t^\nu$, then $\nu \geq \frac{1}{6}$. So we may take $K_1(t) = t^{1/6}$, while $K_2(t) = t^{1/2}$. Why choosing $K_1(t) = t^{1/6}$? Recall that $K_1(t)$ has to be such that the truncated $S_t^{K_1(t)}$ will satisfy some CLT with an appropriate rate. It is thus natural to think that the smaller $K_1(t)$ is, the better for the CLT, since all involved quantities will be as small as possible. Since one can also think that an additional slowly varying function will not change the situation, this choice seems to be the best one. We shall see in section 3.4 that this choice is not only appropriate but is the good polynomial order to prove the required CLT.

Remark 3.5. Notice that as $t \to +\infty$, $(\mathrm{Cov}_\mu(S_t)/t \ln t) \to \kappa\, Id$ for some $\kappa > 0$, while $(\mathrm{Cov}_\mu(S_t^{t^{1/6}})/t \ln t) \to \frac{1}{3}\kappa\, Id$. Since $S_t/\sqrt{t \ln t}$ and $S_t^{t^{1/6}}/\sqrt{t \ln t}$ have the same behaviour in distribution, if $S_t^i/\sqrt{\mathrm{Var}_\mu(S_t^i)}$ converges in distribution to some limiting distribution, this limiting distribution will have a variance less than or equal to 1/3, i.e. there is a *variance breaking.* ♢

Remark 3.6. Since $(S_t - S_t^{t^{1/6}})/\sqrt{t\, \ln t}$ is bounded in $\mathbb{L}^2$, convergence to 0 in Probability or in $\mathbb{L}^1$ are equivalent thanks to Vitali's integrability theorem. So the power 1/6 is actually the best we can obtain for applying Slutsky's theorem. ♢

3.4. **Central Limit Theorem.**

In order to prove the convergence in distribution of $S_t^{K(t)}/\sqrt{s_t}$ with $K(t) = t^\nu$ ($\nu \geq \frac{1}{6}$ according to subsection 3.3), and $s_t = t\, \ln t$, what is required is a Central Limit Theorem (CLT) for *triangular arrays.* Such results go back to Lindeberg for triangular arrays of independent variables, and have been extended by many authors for weakly dependent variables.

In the sequel $K(t) = t^\nu$ will be abridged in K when no confusions are possible. In addition as we previously did *we skip the index i in all quantities*, and finally $\kappa > 0$ denotes the limit as $t \to +\infty$ of $\mathrm{Var}_\mu(S_t)/t\, \ln t$ as in (3.10).

The main part of this section is the proof of the following Lemma :

Lemma 3.7. *If $\beta - d = 4$, for $i = 1, ..., d$ define $s_t = \mathbb{E}_\mu((S_t^i)^2)$.*

(1) *there exists $\kappa > 0$ such that $s_t/(t\, \ln t) \to \kappa$ as $t \to +\infty$.*
(2) *$S_t^i/\sqrt{t\, \ln t}$ converges in distribution to a centered gaussian random variable with variance equal to $\kappa/3$.*

Proof. Come back to (3.5). Since $H_{K(t)}(v_s)/\sqrt{s_t}$ goes to 0 in $\mathbb{L}^1(\mu)$ for $s = 0$ and $s = t$, the convergence we are looking for amounts to the one of $M_t^{K(t)}/\sqrt{s_t}$ where $M_.^K$ is the martingale

$$M_.^K = \int_0^. \nabla H_K(v_s)\, dB_s$$

(we have skipped the $\sqrt{2}$).

Since (CLT) are written for mixing *sequences* we introduce some notations. For $N=[t]$, and $n\in\mathbb{N}$, we define

$$Z_{n,N} = \frac{1}{\sqrt{N\,\ln N}}\int_n^{n+1} \nabla H_K(v_s)\;dB_s\,.$$

Hence, $(1/\sqrt{t\,\ln t})\,S_t^K = S_N + R(t)$ with $S_N=\sum_{n=0}^N Z_{n,N}$ and $R(t)$ goes to 0 in $\mathbb{L}^2(\mu)$.
Of course, under $\mathbb{P}_\mu$ (i.e. starting from equilibrium) the sequence $Z_{.,N}$ is stationary and since the $Z_{j,N}$'s are martingale increments, their correlations are equal to 0. This will be a key point in the proof and explains why we are using these variables instead of directly look at the increments of $S_.$. We skip the subscript μ in what follows, when there is no possible confusion.

According to Lemma 3.1 (2), $\kappa_N := \text{Var}_\mu(S_N)\to 2\nu\,\kappa$ as $N\to+\infty$.
Let γ be a standard gaussian r.v.. For our purpose it is thus enough to show that

$$\lim_{n\to\infty}\Delta_n(h)=0,$$

where we set,

$$\Delta_N(h)=\mathbb{E}_\mu\left(h(S_N)-h(\sqrt{\kappa_N}\,\gamma)\right)$$

and where h denotes some complex exponential function $h(x)=e^{i\,\lambda\,x}$, $\lambda\in\mathbb{R}$.

Now we follow Lindeberg-Rio method to study the convergence in distribution of S_N to a centered normal distribution with variance $2\nu\,\kappa$.
The idea is to decompose Δ_N into the sum of small increments using the hierarchical structure of the triangular array.

Denote, for $j\geq 0$,

$$S_{j,N}=\sum_{n=0}^{j} Z_{n,N} \;=\; \frac{1}{\sqrt{N\,\ln N}}\int_0^{j+1}\nabla H_K(v_s)\,dB_s\,.$$

We have to look at $v_{j,N}=\text{Var}_\mu(S_{j,N})-\text{Var}_\mu(S_{j-1,N})$, where of course $S_{-1,N}=0$. But thanks to the martingale property

$$v_{j,N}=\mathbb{E}(Z^2_{j,N})=\frac{1}{N\,\ln N}\,\mathbb{E}\left(\int_0^1 |\nabla H_K(v_s)|^2\,ds\right)=v_N=\frac{\kappa_N}{N+1}\leq C/N\,.$$

Introduce gaussian random variables $N_{j,N}\sim\mathcal{N}(0,v_N)$ (of course $v_N>0$). The sequence $(N_{j,N})_{1\leq j\leq N+1\,,\,N\geq 1}$ is assumed to be independent and independent of the sequence $(Z_{j,N})$. For $1\leq j\leq N$, we set $T_{j,N}=\sum_{k=j+1}^{N+1} N_{k,N}$, empty sums are, as usual, set equal to 0. In particular $T_{0,N}$ has the same distribution as $\sqrt{\kappa_N}\gamma$.
We are in position to use Rio's decomposition

$$\Delta_N(h)=\sum_{j=0}^{N}\Delta_{j,N}(h), \tag{3.13}$$

with $\Delta_{j,N}(h)=\mathbb{E}\left(h(S_{j-1,N}+Z_{j,N}+T_{j+1,N})-h(S_{j-1,N}+N_{j+1,N}+T_{j+1,N})\right)$.

Again we decompose $\Delta_{j,N}(h) = \Delta^{(1)}_{j,N}(h) - \Delta^{(2)}_{j,N}(h)$, with

$$\Delta^{(1)}_{j,N}(h) = \mathbb{E}(h(S_{j-1,N}+Z_{j,N}+T_{j+1,N}))-\mathbb{E}(h(S_{j-1,N}+T_{j+1,N}))-\frac{v_{j,N}}{2}\mathbb{E}(h''(S_{j-1,N}+T_{j+1,N})), \tag{3.14}$$

$$\Delta^{(2)}_{j,N}(h) = \mathbb{E}(h(S_{j-1,N}+N_{j+1,N}+T_{j+1,N}))-\mathbb{E}(h(S_{j-1,N}+T_{j+1,N}))-\frac{v_{j,N}}{2}\mathbb{E}(h''(S_{j-1,N}+T_{j+1,N})). \tag{3.15}$$

Define the functions

$$x \to h_{j,N}(x) = \mathbb{E}(h(x+T_{j+1,N})) = e^{-\lambda^2 \kappa_N \, ((N-j+1)/2(N+1))}\, h(x)\,.$$

Using independence (recall the definition of $T_{j,N}$) it is not difficult to see that one can write

$$\Delta^{(1)}_{j,N}(h) = \mathbb{E}(h_{j,N}(S_{j-1,N}+Z_{j,N})) - \mathbb{E}(h_{j,N}(S_{j-1,N})) - \frac{v_{j,N}}{2}\mathbb{E}(h''_{j,N}(S_{j-1,N})),$$

$$\Delta^{(2)}_{j,N}(h) = \mathbb{E}(h_{j,N}(S_{j-1,N}+N_{j+1,N})) - \mathbb{E}(h_{j,N}(S_{j-1,N})) - \frac{v_{j,N}}{2}\mathbb{E}(h''_{j,N}(S_{j-1,N}))\,.$$

• *Bound for* $\Delta^{(2)}_{j,N}(h)$.
Taylor expansion yields the existence of some random variable $\tau_{j,N} \in (0,1)$:

$$\begin{aligned}\Delta^{(2)}_{j,N}(h) \;&=\; \mathbb{E}(h'_{j,N}(S_{j-1,N})N_{j+1,N}) + \frac{1}{2}\mathbb{E}(h''_{j,N}(S_{j-1,N})(N^2_{j+1,N}-v_{j,N}))\\ &\quad+\frac{1}{6}\mathbb{E}(h^{(3)}_{j,N}(S_{j-1,N}+\tau_{j,N}N_{j+1,N})N^3_{j+1,N})\,.\end{aligned}$$

Using independence, we see that the first two terms vanish. In addition since the third derivative of h is bounded we get $|\Delta^{(2)}_{j,N}(h)| \le C\,\mathbb{E}(|N_{j+1,N}|^3)$, hence, since N is gaussian,

$$|\Delta^{(2)}_{j,N}(h)| \le \, C\, v^{3/2}_{j,N} \le \, C\, N^{-(3/2)}\,.$$

It follows that $\Delta^{(2)}_N(h) = \sum_{j=0}^N \Delta^{(2)}_{j,N}(h) \, \le C\, N^{-(1/2)}$ goes to zero.

• *Bound for* $\Delta^{(1)}_{j,N}(h)$. Set $\Delta^{(1)}_{j,N}(h) = \mathbb{E}(\delta^{(1)}_{j,N}(h))$.

Then, using Taylor formula again (with some random $\tau_{j,N} \in (0,1)$), we may write

$$\delta^{(1)}_{j,N}(h) = h'_{j,N}(S_{j-1,N})Z_{j,N} + \frac{1}{2}h''_{j,N}(S_{j-1,N})(Z^2_{j,N}-v_{j,N}) + \frac{1}{6}\left(h^{(3)}_{j,N}(S_{j-1,N}+\tau_{j,N}Z_{j,N})Z^3_{j,N}\right).$$

We analyze separately the terms in the previous expression.

The first term vanishes thanks to the martingale property of Z.

The last term can be bounded in the following way

$$|\mathbb{E}(h^{(3)}_{j,N}(S_{j-1,N}+\tau_{j,N}Z_{j,N})Z^3_{j,N})| \le C\,\mathbb{E}(|Z^3_{j,N}|)\,.$$

We use $K = t^\nu$, Burkholder-Davis-Gundy inequality and Jensen's inequality to get

$$\begin{aligned} \mathbb{E}(|Z^3_{j,N}|) &\leq C\,(N\,\ln N)^{-(3/2)}\,\mathbb{E}\left(\left(\int_0^1 |\nabla H_K(v_s)|^2\,ds\right)^{\frac{3}{2}}\right) \\ &\leq C\,(N\,\ln N)^{-(3/2)}\,\mu(|\nabla H_K|^3) \leq C\,(N\,\ln N)^{-(3/2)}\,K^2 \\ &\leq C\,N^{-(3/2)+2\nu}\,(\ln N)^{-(3/2)}\,, \end{aligned} \tag{3.16}$$

so that summing up from $j=0$ to $j=N$ we obtain a term going to 0 if $\nu \leq \frac{1}{4}$.

It remains to study the second term, i.e.

$$\sum_{j=0}^{N} \mathbb{E}_\mu(h''_{j,N}(S_{j-1,N})(Z^2_{j,N} - v_{j,N}))$$

and to show that it goes to 0. To this end, we split the sum in two terms: $\sum_{j\leq N'}$ and $\sum_{N'<j\leq N}$.
Since $|A_j| = |\mathbb{E}_\mu(h''_{j,N}(S_{j-1,N})(Z^2_{j,N} - v_{j,N}))| \leq C/N$, the summation up to N' is less or equal to CN'/N and will go to 0 provided $N \gg N'$.

For $j \geq N'$, A_j can be written

$$\begin{aligned} A_j &= \mathbb{E}_\mu((h''_{j,N}(S_{j-1,N}) - h''_{j,N}(S_{k,N}))\,(Z^2_{j,N} - v_N)) + \\ &\quad + \mathbb{E}_\mu(h''_{j,N}(S_{k,N})\,(Z^2_{j,N} - v_N)) \\ &= A^1_j + A^2_j\,. \end{aligned}$$

To control the second term we may use the mixing property. Indeed

$$\begin{aligned} A^2_j &= \frac{1}{N\,\ln N}\,\mathbb{E}_\mu\left(h''_{j,N}(S_{k,N})\,\left(\int_j^{j+1} (|\nabla H_K(v_s)|^2 - v_N)\,ds\right)\right) \\ &= \frac{1}{N\,\ln N}\int_j^{j+1} \mathrm{Cov}(h''_{j,N}(S_{k,N}), |\nabla H_K(v_s)|^2)\,ds \\ &= \frac{1}{N\,\ln N}\int_j^{j+1} E_\mu([h''_{j,N}(S_{k,N}) - E_\mu(h''_{j,N}(S_{k,N}))][|\nabla H_K(v_s)|^2 - E_\mu(|\nabla H_K(v_s)|^2]) \end{aligned}$$

Then by the Liggett theorem and remark 2.13, we get since $s > j$

$$\begin{aligned} A^2_j &\leq \frac{C}{N\,\ln N}\,\alpha^2((j-k)/2)\,\parallel |\nabla H_K|^2 \parallel_\infty ||h''_{j,N}||_\infty \\ &\leq \frac{C}{N\,\ln N}\,\frac{K^4}{(j-k)^2}\,. \end{aligned}$$

Hence choosing $j - k = K^2$, i.e. $k = j - N^{2\nu}$ and $N' = N^{2\nu}$, the sum of all these terms for $j \geq N'$, will go to 0.

The first term can be written

$$\begin{aligned} A_j^1 &= \mathbb{E}_\mu(h^{(3)}_{j,N}(S_{k,N}+\tau_{j,N})\,(S_{j-1,N}-S_{k,N})\,(Z^2_{j,N}-v_N))) \\ &\le C\,\left(\mathbb{E}_\mu(|S_{j-1,N}-S_{k,N}|\,Z^2_{j,N}) + \ v_N\,\mathbb{E}_\mu(|S_{j-1,N}-S_{k,N}|)\right) \\ &\le \frac{C}{\sqrt{N\,\ln N}}\,\mathbb{E}_\mu\left(Z^2_{j,N}\left|H_K(v_j)-H_K(v_{k+1})-\int_{k+1}^j v_K(v_s)ds\right|\right) + C\,N^\nu N^{-3/2}\,, \end{aligned}$$

since

$$\mathbb{E}_\mu(|S_{j-1,N}-S_{k,N}|) \le \left(\mathbb{E}_\mu\left(\frac{1}{N\,\ln N}\int_{k+2}^{j+1}|\nabla H_K(v_s)|^2\,ds\right)\right)^{\frac12} \le \sqrt{(j-k)\,v_N} = C\,N^\nu N^{-1/2}\,.$$

Now the first term in the previous sum can be written

$$\begin{aligned} A_j^{1,1} &= \frac{C}{\sqrt{N\,\ln N}}\,\mathbb{E}_\mu\left(Z^2_{j,N}\left|H_K(v_j)-H_K(v_{k+1})-\int_{k+1}^j v_K(v_s)ds\right|\right) \\ &= \frac{C}{(N\,\ln N)^{3/2}}\,\mathbb{E}_\mu\left(\left(\int_j^{j+1}|\nabla H_K(v_s)|^2\,ds\right)\left|H_K(v_j)-H_K(v_{k+1})-\int_{k+1}^j v_K(v_u)du\right|\right)\,. \end{aligned}$$

Recall that

$$|v_K(v)| \le C(N^\nu\,\mathbf{1}_{|v|\le K} + |v|\,\mathbf{1}_{|v|\ge K}) \tag{3.17}$$

and similarly

$$|H_K(v)| \le C\left(N^{3\nu}\,\mathbf{1}_{|v|\le K} + N^{2\nu}\,|v|\,\mathbf{1}_{|v|\ge K}\right)\,. \tag{3.18}$$

At j and k fixed, we will now decompose $A_j^{1,1}$ in a first part corresponding to the velocities v_k and v_j less than K, and a second part for the velocities greater than K. This yields

$$\begin{aligned} A_j^{1,1} &= \frac{C}{(N\,\ln N)^{3/2}}\,\mathbb{E}_\mu\left(\left(\int_j^{j+1}|\nabla H_K(v_s)|^2\,ds\right)\left[|H_K(v_j)|\,\mathbf{1}_{|v_j|\le K} + |H_K(v_{k+1})|\,\mathbf{1}_{|v_{k+1}|\le K}\right]\right) \\ &+ \frac{C}{(N\,\ln N)^{3/2}}\int_{k+1}^j \mathbb{E}_\mu\left(\left(\int_j^{j+1}|\nabla H_K(v_s)|^2\,ds\right)|v_K(v_u)|\mathbf{1}_{|v_u|\le K}\right)du \\ &+ \frac{C}{(N\,\ln N)^{3/2}}\,\mathbb{E}_\mu\left(\left(\int_j^{j+1}|\nabla H_K(v_s)|^2\,ds\right)\left[|H_K(v_j)|\,\mathbf{1}_{|v_j|\ge K} + |H_K(v_{k+1})|\,\mathbf{1}_{|v_{k+1}|\ge K}\right]\right) \\ &+ \frac{C}{(N\,\ln N)^{3/2}}\int_{k+1}^j \mathbb{E}_\mu\left(\left(\int_j^{j+1}|\nabla H_K(v_s)|^2\,ds\right)|v_K(v_u)|\mathbf{1}_{|v_u|\ge K}\right)du. \end{aligned}$$

Then

$$\begin{aligned} A_j^{1,1} &\le \frac{C}{(N\,\ln N)^{3/2}}\,\mathbb{E}_\mu\left(\int_j^{j+1}|\nabla H_K(v_s)|^2\,ds\right)[N^{3\nu}+(j-k)N^\nu] \\ &+ \frac{C}{(N\,\ln N)^{3/2}}\,K^4[\mathbb{E}_\mu\left(K^2|v|\mathbf{1}_{|v|\ge K}\right) + (j-k)\mathbb{E}_\mu\left(|v|\mathbf{1}_{|v|\ge K}\right)] \\ &\le C\,\frac{N^{3\nu}}{N^{3/2}}\left(\frac{1}{(\ln N)^{1/2}} + \frac{1}{(\ln N)^{3/2}}\right) \end{aligned}$$

Note that to bound the terms involving velocities greater than K, we have not been very clever. We just used the fact that $|\nabla H_K|^2 \leq C\,K^4$ and that $\int |v|\,\mathbf{1}_{|v|\geq K}\,d\mu \leq C\,K^{-3}$. For the velocities less than K, we used that fact that

$$E_\mu\left(\int_j^{j+1} |\nabla H_K(v_s)|^2\,ds\right) \leq C\ln N.$$

Since we still have to sum up the terms, the constraints on ν are now $\nu \leq \frac{3}{4}$ and $1+3\nu-\frac{3}{2} \leq 0$, that is $\nu \leq \frac{1}{6}$. Since we have to assume that $\nu \geq \frac{1}{6}$, the value $K = t^{1/6}$ is (up to slowly varying perturbation) the only possible one.

Gathering all these intermediate bounds we have obtained $A_j^1 \leq \frac{C}{N\sqrt{\ln N}}$. □

Actually one can generalize Lemma 3.7, replacing $h(x) = e^{i\lambda x}$ defined on $\mathbb{R}$ by $h(x) = e^{i\langle\lambda,x\rangle}$ defined on $\mathbb{R}^d$. The proof above immediately extends to this situation, replacing the gaussian r.v. by a gaussian random vector with independent entries, and using that the correlations between the S_t^i's are vanishing. Details are left to the reader. One can also check that the assumptions in Proposition 8.1 of [4] are satisfied in order to deduce a (MCLT) from the previous (CLT). This allows us to state our main theorem

Theorem 3.8. *For $\omega = \omega_\beta$ with $\beta - d = 4$, defining s_t as in lemma 3.7. Then (MCLT) holds at equilibrium and out of equilibrium with rate $\zeta(\eta) = \sqrt{\eta}/\sqrt{\ln(1/\eta)}$ and asymptotic covariance matrix $\Gamma = (\kappa/3)\,Id$.*
Hence Corollary 2.7 still holds true but with $\theta(\varepsilon) = \varepsilon^2\,\ln(1/\varepsilon)$.

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