FUNCTIONAL INEQUALITIES VIA LYAPUNOV CONDITIONS

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Abstract. We review here some recent results of the authors, and various coauthors, on (weak,super) Poincaré inequalities, transportation-information inequalities or logarithmic Sobolev inequality via a rather simple and efficient technique: Lyapunov conditions.

Key words : Lyapunov condition, Poincaré inequality, transportation information inequality, logarithmic Sobolev inequality.

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1. Introduction and main concepts

Lyapunov conditions have been around for a long time. They were particularly well fitted to deal with the problem of convergence to equilibrium for Markov processes, see [39, 40, 41, 24] and references therein. They also appeared earlier in the study of large and moderate deviations for empirical functionals of Markov processes (see for examples Donsker-Varadhan [22, 23], Kontoyannis-Meyn [34, 35], Wu [48, 49], Guillin [29, 30],...), for solving Poisson equation [25],...

Their use to obtain functional inequalities is however quite recent, even if one may afterwards find hint of such an approach in Deuschel and Stroock [20] or Kusuocka and Stroock [36]. The present authors and coauthors have developed a methodology that has been successful for various inequalities: Lyapunov-Poincaré inequalities in Bakry & *al.* [4], Poincaré inequalities in Barthe & *al.* [3], transportation inequalities for Kullback information in Cattiaux & *al.* [17] or Fisher information in Guillin & *al.* [33], super Poincaré inequalities in Cattiaux & *al.* [16], weighted and weak Poincaré inequalitiesin Cattiaux & *al.* [13] or the forthcoming paper of Cattiaux & *al.* [18] for super weighted Poincaré inequalities. We refer the reader to the forthcoming book [15] by the authors for a complete review, while for more references on the various inequalities introduced here we refer to books by Bakry [2], Ané & *al.* [1], Ledoux [37] or Wang [47]. The goal of this short review is to explain the methodology used in these papers and to present general sets of conditions for these various functional inequalities. The proofs will, of course, be only scketched and we will refer to the original papers for complete statements.

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Let us first describe our framework. Let E be some Polish state space, μ a probability measure, and a μ-symmetric operator L. The main assumption on L is that there exists some algebra $\mathcal{A}$ of bounded functions, containing constant functions, which is everywhere dense (in the $\mathbb{L}_2(\mu)$ norm) in the domain of L. It enables us to define a "carré du champ" Γ, *i.e.* for $f, g \in \mathcal{A}$, $L(fg) = fLg + gLf + 2\Gamma(f,g)$. We will also assume that Γ is a derivation (in each component), *i.e.* $\Gamma(fg,h) = f\Gamma(g,h) + g\Gamma(f,h)$, i.e. we are in the standard "diffusion" case in [2] and we refer to the introduction of [12] for more details. For simplicity we set $\Gamma(f) = \Gamma(f,f)$. Also, since L is a diffusion, we have the following chain rule formula $\Gamma(\Psi(f), \Phi(g)) = \Psi'(f)\Phi'(g)\Gamma(f,g)$.
In particular if $E = \mathbb{R}^n$, $\mu(dx) = e^{-V(x)}dx$, where V is smooth and $L = \Delta - \nabla V.\nabla$, we may consider the compactly supported C^∞ functions (plus the constant functions) as the interesting subalgebra $\mathcal{A}$, and then $\Gamma(f,g) = \nabla f \cdot \nabla g$.

Now we define the notion of ϕ-Lyapunov function. Let $W \geq 1$ be a smooth enough function on E and ϕ be a C^1 positive increasing function defined on $\mathbb{R}^+$. We say that W is a ϕ-Lyapunov function if there is an increasing family of sets $(A_r)_{r\geq 0} \subset E$, with $\cup_r A_r = E$ and some $b \geq 0$ such that for some $r_0 > 0$

$$LW \leq -\phi(W) + b\,\mathbb{1}_{A_{r_0}}\,. \tag{1.1}$$

One has very different behavior depending on ϕ: if ϕ is linear then results of [40, 41] assert that the associated semigroup converges to equilibrium with exponential speed (in total variation or with respect to some weighted norm) so that it is legitimate to hope for a Poincaré inequality to be valid.
When ϕ is superlinear (or more generally in the form $\phi \times W$ where ϕ tends to infinity) we may hope for stronger inequalities (super Poincaré, ultracontractivity...).
Finally if ϕ is sublinear, as asserted in Douc & *al.* [24], only subexponential convergence to equilibrium is valid, so we should be in the regime of weak Poincaré inequalities. We will see class of examples in the next section.
Note however that the result of Meyn-Tweedie (and Douc & *al.*) [40, 41, 24] are valid even in a fully degenerate hypoelliptic setting (such as for the kinetic Fokker-Planck equation (see Wu or Bakry & *al.* [48, 4])) whereas we cannot hope for a (weak, normal or super) Poincaré inequality, due to the "degeneracy" of the Dirichlet form.
We thus have to impose a further condition which will often be a "local inequality" such as local Poincaré inequality (i.e. a Poincaré inequality restricted to a ball, or to a particular set) or a local super Poincaré inequality, preventing degeneracy cases but quite easy to verify for general locally bounded measures.

We now present a main lemma showing how the Lyapunov condition is used in our setting.

Lemma 1.1. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a $\mathcal{C}^1$ increasing function. Then, for any $f \in \mathcal{A}$ and any positive $h \in D(\mathcal{E})$,*

$$\int \frac{-Lh}{\psi(h)}\, f^2\, d\mu \leq \int \frac{\Gamma(f)}{\psi'(h)} d\mu.$$

In particular,

$$\int \frac{-Lh}{h}\, f^2\, d\mu \leq \int \Gamma(f) d\mu.$$

Proof. Since L is μ-symmetric, using that Γ is a derivation and the chain rule formula, we have

$$\int \frac{-Lh}{\psi(h)}\, f^2\, d\mu \;=\; \int \Gamma\left(h, \frac{f^2}{\psi(h)}\right) d\mu = \int \left(\frac{2\, f\, \Gamma(f,h)}{\psi(h)} - \frac{f^2 \psi'(h)\Gamma(h)}{\psi^2(h)}\right) d\mu\,.$$

Since ψ is increasing and according to the Cauchy-Schwarz inequality we get

$$\begin{aligned}\frac{f\,\Gamma(f,h)}{\psi(h)} &\leq \frac{f\sqrt{\Gamma(f)\Gamma(h)}}{\psi(h)} = \frac{\sqrt{\Gamma(f)}}{\sqrt{\psi'(h)}} \cdot \frac{f\sqrt{\psi'(h)\Gamma(h)}}{\psi(h)} \\ &\leq \frac{1}{2}\frac{\Gamma(f)}{\psi'(h)} + \frac{1}{2}\,\frac{f^2\psi'(h)\,\Gamma(h)}{\psi^2(h)}.\end{aligned}$$

The result follows. □

Remark 1.2. In fact the conclusion of the preceding lemma, in the case where ψ is the identity, holds true in a more general setting and requires only the reversibility assumption. It is thus valid for some Markov jump processes ($M/M/\infty$, Lévy processes,...), see Guillin & *al*. [33] where the corresponding proof follows from a large deviations argument, or on more general Riemanian manifolds.

2. Examples of Lyapunov conditions

Before stating the results which can be obtained by our method, let us present some examples of Lyapunov conditions.

We will restrict ourselves to the framework described above: $E = \mathbb{R}^n$, $\mu(dx) = Ze^{-V(x)}dx$ and $L = \Delta - \nabla V.\nabla$. The Lyapunov conditions may be quite different: firstly because of the very nature of V itself, secondly because of the choice of the Lyapunov function W. Let us illustrate it in the Gaussian case $V(x) = |x|^2$

(1) Choose first $U_1(x) = 1 + |x|^2$, so that

$$LU_1(x) = \Delta U_1(x) - 2x.\nabla U_1(x) = 2n - 2|x|^2 \leq -U_1(x) + (2n+1)1_{|x|^2 \leq 2n+1}.$$

(2) Choose now $U_2(x) = e^{a|x|^2}$ for $0 < a < 1$,

$$LU_2(x) = (2n + 4a(a-1)|x|^2)U_2(x) \leq -\lambda|x|^2 U_2(x) + b1_{|x|^2 \leq R}$$

for some λ, b, R.

We can now consider the usual examples: let $T > c > 0$ be convex (convexity and positivity outside a large compact set is sufficient if the measure is properly defined)

- Exponential type measures: $V(x) = T^p$ for some positive p. Then, there exists $b, c, R > 0$ and $U \geq 1$ such that

$$LU \leq -\phi(U) + b1_{B(0,R)}$$

with $\phi(u) = u\, \log^{2(p-1)/p}(c+u)$ increasing. Furthermore, one can choose $U(x) = e^{\gamma|x|^p}$ for x large and γ small enough.

- Cauchy type measures: $V(x) = (n+\beta)\log(T)$ for some positive β. Then, there exists $k > 2$, $b, R > 0$ and $U \geq 1$ such that

$$LU \leq -\phi(U) + b1_{B(0,R)}$$

with $\phi(u) = cu^{(k-2)/k}$ for some constant $c > 0$. Moreover, one can choose $U(x) = |x|^k$ for x large. k has to be choosen so that there exists $\epsilon > 0$ such that $k + n\epsilon - 2 - \beta(1-\epsilon) < 0$.

The details can be found in Cattiaux & *al*. [13] for example.

3. Poincaré's like inequalities

The prototype of inequalities we will consider in this section is the following Poincaré inequality: for every nice function f there exists $C > 0$ such that

$$Var_\mu(f) := \int f^2 d\mu - \left(\int f d\mu\right)^2 \leq C \int |\nabla f|^2 d\mu.$$

Poincaré inequalities have attracted a lot of attention due to their many properties: they are equivalent to the exponential L_2 decay of the associated semigroup, they give exponential dimension free concentration of measure,... We refer to [1, 37] for historical and mathematical references. Weak and Super Poincaré inequalities will be variant (weaker or stronger) of this inequality. As we will see, this inequality may be proved very quickly using Lyapunov conditions and local inequalities.

3.1. Poincaré inequality.

Let us begin by

Theorem 3.1. *Assume that the following Lyapunov condition holds: there exists $W \geq 1$ in the domain of L, $\lambda > 0$, $b > 0$, $R > 0$ such that*

$$LW \leq -\lambda W + b 1_{\{|x| \leq R\}}\,. \tag{3.1}$$

Assume in addition that the following local Poincaré inequality holds: there exists κ_R such that for all nice functions f

$$\int_{|x|\leq R} f^2 d\mu \leq \kappa_R \int \Gamma(f) d\mu + \mu(\{|x| \leq R\})^{-1} \left(\int_{|x|\leq R} f d\mu\right)^2. \tag{3.2}$$

Then we have the following Poincaré inequality: for all nice f

$$Var_\mu(f) \leq \frac{b\kappa_R + 1}{\lambda} \int \Gamma(f) d\mu.$$

As the proof is very simple, it will be quite the only one we will write completely:

Proof. Denote $c_R = \int_{|x|\leq R} f d\mu$. Remark now that we may rewrite the Lyapunov condition as

$$1 \leq -\frac{LW}{\lambda W} + \frac{b}{\lambda} 1_{\{|x|\leq R\}}$$

so that by Lemma 1.1 and the local Poincaré inequality, we have

$$\begin{aligned} Var_\mu(f) &\leq \int (f - c_R)^2 d\mu \\ &\leq \int (f - c_R)^2 \frac{-LW}{\lambda W} d\mu + \frac{b}{\lambda} \int_{|x|\leq R} (f - c_R)^2 d\mu \\ &\leq \frac{1}{\lambda} \int \Gamma(f) d\mu + \frac{b}{\lambda} \kappa_R \int |\nabla f|^2 d\mu \end{aligned}$$

which is the desired result. □

Remark 3.2. Using this theorem combined with the examples provided above, we recover simply the nice result of Bobkov [8] asserting that every log-concave measure (i.e. V convex) satisfies a Poincaré inequality ([3]).
Consider the two following conditions.:

(1) there exist $0 < a < 1$, $c > 0$ and $R > 0$ such that for all $|x| \geq R$, we have $(1-a)|\nabla V|^2 - \Delta V \geq c$;
(2) there exist $c > 0$ and $R > 0$ such that for all $|x| \geq R$, we have $x.\nabla V(x) \geq c|x|$.

Then under 1) or 2), a Poincaré inequality holds true. Indeed, the first one implies a Lyapunov condition for $W = e^{aV}$ and the second one for $W(x) = e^{a|x|}$. Note that the first one was known with $a = 1/2$ for a long time but with quite harder proof.

Remark 3.3. Using a variant of this approach, Barthe & *al*. [3] have proved that the same Lyapunov condition implies a L^1 Poincaré inequality, also called Cheeger inequality.

3.2. **Weighted and weak Poincaré inequality.** We will now consider weaker inequalities: weighted Poincaré inequalities as introduced recently by Bobkov and Ledoux [11] or [13], i.e. with an additional weight in the Dirichlet form or in the variance, or weak Poincaré inequalities introduced by Röckner and Wang [43] (see also Barthe & *al*. [6] or Cattiaux & *al*. [13]), useful to establish sub exponential concentration inequalities or algebraic rate of decay to equilibrium for the associated Markov process. We shall state here

Theorem 3.4. *Let the following ϕ-Lyapunov condition hold: there exist some sublinear $\phi : [1,\infty[\to \mathbb{R}^+$ and $W \geq 1$, $b > 0$, $R > 0$ such that*

$$LW \leq -\phi(W) + b1_{\{|x|\leq R\}}. \tag{3.3}$$

Let also μ satisfy a local Poincaré inequality (3.2) then

(1) *for all nice f, the following weighted Poincaré inequality holds*

$$Var_\mu(f) \leq \max\left(\frac{b\kappa_R}{\phi(1)}, 1\right) \int \left(1 + \frac{1}{\phi'(W)}\right) \Gamma(f)d\mu;$$

(2) *for all nice f, the following converse weighted Poincaré inequality holds*

$$\inf_c \int (f-c)^2 \frac{\phi(W)}{W} d\mu \leq (1 + b\kappa_R) \int \Gamma(f)d\mu;$$

(3) *define $F(u) = \mu(\phi(W) < uW)$ and for $s < 1$, $F^{-1}(s) := \inf\{u; F(u) > s\}$ then the following weak Poincaré inequality holds:*

$$Var_\mu(f) \leq \frac{C}{F^{-1}(s)} \int \Gamma(f)d\mu + s\, Osc_\mu(f)^2.$$

Proof. The proof of the first two parts can be easily derived as in the proof for the usual Poincaré inequality. For the weak Poincaré inequality, start with the variance, divide the integral with respect to large or small values of $\phi(W)/W$ and use the converse Poincaré inequality previously established, see Cattiaux & *al*. [13]. □

Remark 3.5. Using $V(x) = 1 + |x|^2$ in the examples of the previous section, one gets a weighted inequality with weight $1 + |x|^2$, and converse inequality with weight $(1 + |x|^2)^{-1}$ recovering results of Bobkov and Ledoux [11] (with worse constants however). Note also that it enables us to get the correct order for the weak Poincaré inequality (as seen in dimension 1 in Barthe & *al*. [6]). One finds in Bakry & *al*. [4], another approach based on weak Lyapunov-Poincaré inequality.

3.3. **Super Poincaré inequality.** Our next inequality has been considered first by Wang [46] to study the essential spectrum of Markov operators. It is also useful for concentration of measures (see Wang [46]) or isoperimetric inequalities (see Barthe & *al*. [7]). Wang also showed that they are, under Poincaré inequalities, equivalent to F-Sobolev inequality (in particular one specific super Poincaré inequality is equivalent to the logarithmic Sobolev inequality), so that the results we will present now enables us to consider a very large class of inequalities stronger than Poincaré inequality.

Theorem 3.6. *Assume that there is an increasing family of increasing sets $(A_r)_{r\geq 0}$ with $\cup_r A_r = E$, an $r_0 > 0$ and a superlinear ϕ such that for some $b > 0$ the following Lyapunov condition holds*

$$LW \leq -\phi(W) + b\mathbb{1}_{A_{r_0}} . \tag{3.4}$$

Assume in addition that a local super Poincaré inequality holds, i.e. there exists β_{loc} decreasing in s (for all r) such that for all s and nice f

$$\int_{A_r} f^2 d\mu \leq s \int \Gamma(f) d\mu + \beta_{loc}(r,s) \left(\int_{A_r} |f| d\mu\right)^2 . \tag{3.5}$$

Then, if $G(r) := (\inf_{A_r^c} \phi(W)/W)^{-1})$ tends to 0 as $r \to \infty$, μ satisfies for all positive s

$$\int f^2 d\mu \leq 2s \int \Gamma(f) d\mu + \tilde{\beta}(s) \left(\int |f| d\mu\right)^2 \tag{3.6}$$

where

$$\tilde{\beta}(s) = c_{r0} \beta_{loc}(G^{-1}(s), s/c_{r0})$$

and $c_{r_0} = 1 + b\frac{\sup_{A_{r_0}} W}{\inf_{A_{r_0}^c} \phi(W)/W}$.

Proof. In fact, one just has to play with the extra strength provided by the Lyapunov condition (i.e. superlinearity) and the set A_r, i.e.

$$\int f^2 d\mu \leq \int_{A_r} f^2 d\mu + \int_{A_r^c} f^2 d\mu \leq \int_{A_r} f^2 d\mu + \frac{1}{\inf_{A_r^c} \phi(W)/W} \int f^2 \frac{\phi(W)}{W} d\mu.$$

The first term is tackled by using the local inequality and for the second one, use the Lyapunov condition, the crucial Lemma 1.1, and once again the local super Poincaré inequality. Optimize in r to get the conclusion, see the details in Cattiaux & *al*. [16]. □

Remark 3.7. Note that if the Boltzmann measure μ is locally bounded, using Nash inequality for Lebesgue measures on balls, it is quite easy to find a local Super Poincaré inequality, see discussion in Cattiaux & *al*. [16].

Remark 3.8. Using this approach, one may recover famous criteria for logarithmic Sobolev inequality: Bakry-Emery convexity criterion [5] (with worse constants), Kusuocka and Stroock conditions [36], or pointwise Wang's criterion, see [47].

Remark 3.9. The link with generalized entropy or Poincaré inequalities introduced by Cordero-Erausquin & *al*. [19] will be developped in Cattiaux & *al*. [18].

4. Transportation's inequalities

We will consider here another type of inequalities linking Wasserstein distance to various information form, namely Kullback information or Fisher information defined respectively by: if f is a density of probability with respect to μ

$$H(fd\mu, d\mu) := \mathrm{Ent}_\mu(f) := \int f\log(f)d\mu$$

$$I(fd\mu, d\mu) := \int \frac{|\nabla f|^2}{f}d\mu.$$

The Wasserstein distance is defined by: for all measure ν and μ

$$W_p(\nu,\mu) := \inf\left\{\mathbb{E}(d^p(X,Y))^{1/p};\, X\sim\nu, Y\sim\mu\right\}.$$

4.1. **Transportation and Kullback information.** Firstly, let us consider the usual transportation inequalities: for all probability density f wrt μ

$$W_p(\nu,\mu) \le \sqrt{c\,H(fd\mu, d\mu)}.$$

These types of inequalities were introduced by Marton [38] as they imply straightforwardly concentration of measure Later they were shown to be equivalent to deviation inequality by a beautiful characterization of Bobkov-Goetze [10]. The case $p = 1$ was proved to be equivalent to Gaussian integrability in Djellout & *al*. [21].

The case $p = 2$ is much more difficult: Talagrand established the inequality for Gaussian measure [44], whereas Otto-Villani [42] and Bobkov-Gentil-Ledoux [9] proved that a logarithmic Sobolev inequaliy is a sufficient condition. More recently (see [14] by the authors), the authors proved that the logarithmic Sobolev inequality is strictly stronger, and provided such an example in dimension one. We will prove here that one may give a nice Lyapunov condition to verify this transportation inequality. Let us finish by the beautiful characterization obtained by Gozlan [26] proving that the case $p = 2$ is in fact equivalent to the Gaussian dimension free concentration of measure, see Villani [45] or Gozlan and Léonard [27] for more on the subject. We will prove here

Theorem 4.1. *Suppose that there exists* $W \ge 1$*, some point* x_0 *and constants* b, c *such that*

$$LW \le (-cd^2(x,x_0) + b)W \tag{4.1}$$

then there exists $C > 0$ *such that for all density* f *w.r.t.* μ

$$W_2(fd\mu,\mu) \le \sqrt{K\,H(fd\mu, d\mu)}.$$

Proof. Refining arguments of Bobkov-Gentil-Ledoux [9], the authors proved that it is in fact sufficient to get a logarithmic Sobolev inequality for a restricted class of function, i.e. functions f such that

$$\log(f^2) \leq \log\left(f^2 d\mu\right) + 2\eta\left(d^2(x,x_0) + \int d^2(x,x_0)d\mu\right).$$

Using this property and truncation arguments, one sees how Lemma 1.1 comes into play. We refer to Cattiaux & *al*. [17] for the tedious technical details. □

Remark 4.2. It is not difficult to remark that for $V(x) = x^3 + 3x^2\sin(x) + x$ near infinity, the Lyapunov condition is verified. However the logarithmic Sobolev inequality does not hold in this case as shown in [14].

4.2. **Transportation and Fisher Information.** Transportation-information inequalities with Fisher inequalities have been very recently studied in Guillin & *al*. [33, 32, 31], because of their equivalence with deviation inequalities for Markov processes due to large deviations estimation. The two main interesting ones are for $p=1$ and $p=2$: for all probability density f w.r.t. μ

$$W_p(fd\mu, d\mu) \leq \sqrt{C\, I(fd\mu, d\mu)}.$$

In Guillin & *al*. [33], various criteria were studied, such as Lipschitz spectral gap. In particular if $p=1$ and the distance is the trivial one, this inequality is in fact equivalent to a Poincaré inequality. These authors also proved:

Theorem 4.3. *Suppose that a Poincaré inequality holds, and that the following Lyapunov condition holds: there exists* $W \geq 1, x_0$, $c, b > 0$ *such that*

$$LW \leq -cd^2(x, x0)W + b\,. \tag{4.2}$$

Then we have for all probability density f *w.r.t.* μ

$$W_1(fd\mu, d\mu) \leq \sqrt{C\, I(fd\mu, d\mu)}.$$

Proof. Let us sketch the proof: by [45]

$$\begin{aligned} W_1(fd\mu, d\mu) &\leq \int d(x,x_0)|f-1|d\mu \\ &\leq \sqrt{\int |f-1|d\mu}\sqrt{\int d^2(x,x_0)|f-1|d\mu}. \end{aligned}$$

For the first term, we use the fact that the Poincaré inequality is equivalent to a control of the total variation distance by the square of the Fisher Information, and for the second one the Lyapunov condition. One has of course to be careful as $|f-1|$ is not in the domain of L, so that an approximation argument has to be developped. We refer to Guillin & *al*. [33] for details. □

Remark 4.4. It is quite easy to remark that a logarithmic Sobolev inequality implies a transportation information inequality with Fisher information in the case $p=2$, but it is unknown if it is strictly weaker. *A fortiori*, no Lyapunov condition is known in the case $p=2$.

5. Logarithmic Sobolev inequalities under curvature

Recall the classical Logarithmic Sobolev inequality, i.e. for all nice f

$$Ent_\mu(f^2) \leq c \int \Gamma(f) d\mu \,.$$

This inequality has a long history.
First proved by Gross [28] to study hypercontractivity, it has been extensively studied by many authors due to its relationship with the study of decay to equilibrium, concentration of measure property, efficacity in spin systems study, see [2, 1, 37, 47, 45] for further references. A breakthrough condition was the Bakry-Emery one: namely if $Hess(V) + Ric \geq \delta > 0$ then a logarithmic Sobolev holds. Kusuoka and Stroock gave a Lyapunov-type condition (recovered by the study given in the super-Poincaré case), while using Harnack inequalities, Wang proved that in the lower bounded curvature case, i.e.

$$Hess(V) + Ric \geq \delta \tag{5.1}$$

with δ possibly negative, a sufficient Gaussian integrability, i.e. $\int e^{((-\delta)_+/2+\epsilon)|x|^2} d\mu < \infty$, is enough to prove a logarithmic Sobolev inequality. We will prove here

Theorem 5.1. *Assume that (4.1) and (5.1) hold, then μ satisfies a logarithmic Sobolev inequality.*

Proof. Remark first that by (4.1), a Poincaré inequality holds due to the effort of Section 3. Remark also that by Lyapunov conditions (and maximization arguments)

$$W_2^2(f d\mu, d\mu) \leq 2 \int d^2(x, x_0)|f-1| d\mu \leq I(f d\mu, d\mu) + C.$$

Use now the HWI inequality of Otto-Villani [42] (see also Bobkov & *al*. [9]):

$$H(f d\mu, d\mu) \leq 2\sqrt{I(f d\mu, d\mu)}W_2(f d\mu, d\mu) - \frac{\delta}{2} W_2^2(f d\mu, d\mu).$$

With the previous bound on W_2^2, it implies a defective logarithmic Sobolev inequality holds, that may be tightened as usual via Rothaus' Lemma due to the Poincaré inequality. We refer to Cattiaux & *al*. [17] for details. □

Remark 5.2. Let us give here an example not covered by Wang's condition. On $\mathbb{R}^2$, take $V(x, y) = r^2 g(\theta)$ in polar coordinates with $g(\theta) = 2 + \sin(k\theta)$. It is not hard to remark that $Hess(V)$ is bounded and that the Lyapunov condition (4.1) is verified. However Wang's integrability condition is not verified, despite the fact that a logarithmic Sobolev inequality does hold, from our theorem.

Remark 5.3. One may also give Lyapunov conditions when δ is replaced by some unbounded function of the distance.

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