

# LYAPUNOV CONDITIONS FOR SUPER POINCARÉ INEQUALITIES

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ABSTRACT. We show how to use Lyapunov functions to obtain functional inequalities which are stronger than Poincaré inequality (for instance logarithmic Sobolev or  $F$ -Sobolev). The case of Poincaré and weak Poincaré inequalities was studied in [3]. This approach allows us to recover and extend in a unified way some known criteria in the euclidean case (Bakry-Emery, Wang, Kusuoka-Stroock ...).

*Key words* : Ergodic processes, Lyapunov functions, Poincaré inequalities, Super Poincaré inequalities, logarithmic Sobolev inequalities.

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## 1. Introduction.

During the last thirty years, a lot of attention has been devoted to the study of various functional inequalities and among them a lot of efforts were consecrated to the logarithmic Sobolev inequality. Our goal here will be to give a new and practical condition to prove logarithmic Sobolev inequality in a general setting. Our method being general, we will be able to get also conditions for Super-Poincaré, and in particular to various inequalities as  $F$ -Sobolev or general Beckner inequalities. Our assumptions are based mainly on a Lyapunov type condition as well as a Nash inequality (for example valid in  $\mathbb{R}^d$ ). But let us make precise the objects and inequalities we are interested in.

Let  $(\mathcal{X}, \mathcal{F}, \mu)$  be a probability space and  $\mathcal{L}$  a self adjoint operator on  $L^2(\mu)$ , with domain  $\mathbb{D}_2(\mathcal{L})$ , such that  $P_t = e^{t\mathcal{L}}$  is a Markov semigroup. Consider then the Dirichlet form associated to  $\mathcal{L}$

$$\mathcal{E}(f, f) := \langle -\mathcal{L}f, f \rangle_\mu \quad f \in \mathbb{D}_2(\mathcal{L})$$

with domain  $\mathbb{D}(\mathcal{E})$ . Throughout the paper, all test functions in an inequality will belong to  $\mathbb{D}(\mathcal{L})$ . By definition,  $\mathcal{L}$  possesses a spectral gap if and only if the following Poincaré inequality

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holds (for all nice  $f$ 's)

$$(1.1) \quad \text{Var}_\mu(f) := \int f^2 d\mu - \left( \int f d\mu \right)^2 \leq C_P \mathcal{E}(f, f)$$

where  $C_P^{-1}$  is the spectral gap. Note that such an inequality is also equivalent to the exponential decay in  $L^2(\mu)$  of  $P_t$ .

A defective logarithmic Sobolev inequality (say DLSI) is satisfied if for all nice  $f$ 's

$$(1.2) \quad \text{Ent}_\mu(f^2) := \int f^2 \log f^2 d\mu - \int f^2 d\mu \log \left( \int f^2 d\mu \right) \leq C_{LS} \mathcal{E}(f, f) + D_{LS} \int f^2 d\mu.$$

When  $D_{LS} = 0$  the inequality is said to be tight or we simply say that a logarithmic Sobolev inequality is verified (for short (LSI)). Dimension free gaussian concentration, hypercontractivity and exponential decay of entropy are directly deduced from such an inequality explaining the huge interest in it. Note that if a Poincaré inequality is valid, a defective DLSI, via Rothaus's lemma, can be transformed into a (tight) LSI. For all this we refer to [1] or [32].

Recently, Wang [29] introduced so called Super-Poincaré inequality (say SPI) to study the essential spectrum: there exists a non-increasing  $\beta \in C(0, \infty)$ , all nice  $f$  and all  $r > 0$

$$(1.3) \quad \mu(f^2) \leq r \mathcal{E}(f, f) + \beta(r) \mu(|f|)^2.$$

Wang moreover establishes a correspondence between this SPI and defective  $F$ -Sobolev inequality (F-Sob) for a proper choice of increasing  $F \in [0, \infty[$  with  $\lim_{\infty} F = \infty$ , i.e. for all nice  $f$  with  $\mu(f^2) = 1$

$$(1.4) \quad \mu(f^2 F(f^2)) \leq c_1 \mathcal{E}(f, f) + c_2.$$

More precisely, if (1.4) holds for some increasing function  $F$  satisfying  $\lim_{u \rightarrow +\infty} F(u) = +\infty$  and  $\sup_{0 < u < 1} |uF(u)| < +\infty$ , then (SPI) holds with  $\beta(u) = C_1 F^{-1}(C_2(1 + \frac{1}{u}))$  for some well chosen  $C_1$  and  $C_2$ . Conversely if a (SPI) (1.3) holds, defining

$$\xi(t) = \sup_{u > 0} \left( \frac{1}{u} - \frac{\beta(u)}{ut} \right),$$

a (F-Sob) inequality holds with

$$F(u) = \frac{C_1}{u} \int_0^u \xi(t/2) dt - C_2$$

for some well chosen  $C_1$  and  $C_2$ . For details see [32] Theorem 3.3.1 and Theorem 3.3.3. Note that these results are still available when  $\mu$  is a non-negative possibly non-bounded measure.

In particular an inequality (DLSI) is equivalent to a (SPI) inequality with  $\beta(u) = ce^{c'/u}$ .

These inequalities and their consequences (concentration of measure, isoperimetry, rate of convergence to equilibrium) have been studied for diffusions and jump processes by various authors [29, 4, 5, 25, 11] under various conditions.

In this paper we shall use Lyapunov type conditions. These conditions are well known to furnish some results on the long time behavior of the laws of Markov processes (see e.g. [16, 18, 15]). The relationship between Lyapunov conditions and functional inequalities of Poincaré type (ordinary or weak Poincaré introduced in [24]) is studied in details in the recent work [3]. The present paper is thus a complement of [3] for the study of stronger

inequalities than Poincaré inequality. Let us also mention the paper [2] which is a companion paper of the present one (actually written immediately after the present one) dealing with the (simpler) ordinary Poincaré inequality, and actually with the stronger  $\mathbb{L}^1$  Poincaré inequality, also called Cheeger inequality. The main idea of use of a Lyapunov function is similar in [2] and in the present work, however we have here to face much more technical difficulties when handling the “local term” in the proof of Theorem 2.1 below. In addition we provide a method allowing us to deal with general Markov processes (including jump processes), giving some simple example of application.

We will therefore suppose that  $(\mathcal{X}, d)$  is a Polish space (actually a Riemannian manifold). Namely we will assume

(L) there is a function  $W \geq 1$ , a positive function  $\phi > \phi_0 > 0$ ,  $b > 0$  and  $r_0 > 0$  such that

$$(1.5) \quad \frac{\mathcal{L}W}{W} \leq -\phi + b 1_{B(o, r_0)}$$

where  $B(o, r_0)$  is a ball, w.r.t.  $d$ , with center  $o$  and radius  $r_0$ .

The main idea of the paper (in fact of the use of such Lyapunov functions) is the following one: in order to get some super Poincaré inequality for  $\mu$  it is enough that  $\mu$  satisfies some (SPI) locally and that there exists some Lyapunov function. In other words the Lyapunov function is useful to extend (SPI) on (say) balls to the whole space. General statements are given in section 2.

In particular on nice manifolds the Riemannian measure satisfies locally some (SPI), so that an absolutely continuous probability measure will also satisfy a local (SPI) in most cases. The existence of a Lyapunov function allows us to get some (SPI) on the whole manifold.

The aim of sections 3 and 4 is to show how one can build such Lyapunov functions, either as a function of the log-density or as a function of the Riemannian distance. In the first case we improve upon previous results in [22, 10, 4, 5] among others. In the second case we (partly) recover and extend some celebrated results: Bakry-Emery criterion for the log-Sobolev inequality, Wang’s result on the converse Herbst argument. In particular we thus obtain similar results as Wang’s one, but for measures satisfying sub-gaussian concentration phenomenon. This kind of new result can be compared to the recent [6].

The main interest of this approach (despite the new results we obtain) is that it provides us with a drastically simple method of proof for many results. The price to pay is that the explicit constants we obtain are far to be optimal.

## 2. A general result.

**2.1. Diffusion case.** To simplify we will deal here with the diffusion case: we assume that  $\mathcal{X} = M$  is a  $d$ -dimensional connected complete Riemannian manifold, possibly with boundary  $\partial M$ . When  $\partial M \neq \emptyset$ , we assume that  $W$  in (1.5) satisfies the boundary condition that  $NW|_{\partial M} \leq 0$ , where  $N$  is the inward unit normal vector on the boundary. We denote by  $dx$  the Riemannian volume element and  $\rho(x) = \rho(x, o)$  the Riemannian distance function from a fixed point  $o$ . Let  $\mathcal{L} = \Delta - \nabla V \cdot \nabla$  for some  $V \in W_{loc}^{1,2}$  such that  $Z = \int e^{-V} d\lambda < \infty$ , and  $L$  is self adjoint in  $L^2(\mu)$  where  $d\mu = Z^{-1}e^{-V} dx$ . Note that in this case, we are in the symmetric

diffusion case and the Dirichlet form is given by

$$\mathcal{E}(f, f) = \int |\nabla f|^2 d\mu, \quad \mathcal{D}(\mathcal{E}) = W^{2,1}(\mu).$$

We shall obtain (SPI) by perturbing a known super Poincaré inequality.

**Theorem 2.1.** *Suppose that the Lyapunov condition (L) is verified for some function  $\phi$  such that  $\phi(x) \rightarrow \infty$  as  $\rho(x, o) \rightarrow \infty$  and  $NW|_{\partial M} \leq 0$  if  $\partial M \neq \emptyset$ . Assume also that there exists  $T$  locally Lipschitz continuous on  $M$  such that  $d\lambda = \exp(-T) dx$  satisfies a (SPI) (1.3) with function  $\beta$ .*

*Then (SPI) holds for  $\mu$  and some  $\alpha : (0, \infty) \rightarrow (0, \infty)$  in place of  $\beta$ .*

*More precisely, for a family of compact sets  $\{A_r \supset B(o, r_0)\}_{r \geq 0}$  such that  $A_r \uparrow M$  as  $r \uparrow \infty$ , define for  $r > 0$  :*

$$\Phi(r) := \inf_{A_r^c} \phi, \quad \Phi^{-1}(r) := \inf\{s \geq 0 : \Phi(s) \geq r\},$$

$$g(r) := \sup_{\rho(\cdot, A_r) \leq 2} |V - T|, \quad G(r) := \sup_{\rho(\cdot, A_r) \leq 2} |\nabla(V - T)|^2, \quad H(r) = \text{Osc}_{\rho(\cdot, A_r) \leq 2}(V - T).$$

*Then we may choose for  $s > 0$ , either*

$$(1) \quad \alpha(s) := \inf_{\varepsilon \in (0, 1)} \left\{ \frac{5}{2\varepsilon} \beta \left( \frac{\varepsilon s}{10} \wedge \frac{\varepsilon}{16} \wedge \frac{2(1-\varepsilon)}{G \circ \Phi^{-1}(\frac{4b}{\varepsilon} \vee \frac{4}{s\varepsilon})} \right) \exp \left( g \circ \Phi^{-1}(\frac{4b}{\varepsilon} \vee \frac{4}{s\varepsilon}) \right) \right\},$$

*or*

$$(2) \quad \alpha(s) := 2 \exp \left[ 2H \left( r_0 \vee \Phi^{-1} \left( \frac{4}{s} \vee \frac{bs}{2} \right) \right) \right] \beta \left( \frac{s}{8} e^{-H \circ \Phi^{-1}(\frac{4}{s} \vee \frac{bs}{2})} \right).$$

*Proof.* Let  $f \in C_0^\infty(M)$ . For  $r > r_0$  it holds

$$\begin{aligned} \int f^2 d\mu &= \int_{A_r^c} f^2 d\mu + \int_{A_r} f^2 d\mu \\ &= \int_{A_r^c} \frac{f^2 \phi}{\phi} d\mu + \int_{A_r} f^2 d\mu \\ &\leq \frac{1}{\Phi(r)} \int f^2 \phi d\mu + \int_{A_r} f^2 d\mu \\ &\leq \frac{1}{\Phi(r)} \int f^2 \left( \frac{-\mathcal{L}W}{W} \right) d\mu + \left( \frac{b}{\Phi(r)} + 1 \right) \int_{A_r} f^2 d\mu \end{aligned}$$

using our assumption (L).

The proof turns then to the estimation of the two terms in the right hand side of the latter inequality, a global term and a local one. Let  $\mu_\partial$  be the measure on  $\partial M$  induced by  $\mu$ , which of course vanishes if the boundary is empty. For the first term remark, by our assumption

on  $\mathcal{L}$  and  $W$  that

$$\begin{aligned} \int f^2 \left( \frac{-\mathcal{L}W}{W} \right) d\mu &= \int \nabla \left( \frac{f^2}{W} \right) \cdot \nabla W d\mu + \int_{\partial M} (NW) \frac{f^2}{W} d\mu_{\partial} \\ &\leq 2 \int \frac{f}{W} \nabla f \cdot \nabla W d\mu - \int \frac{f^2 |\nabla W|^2}{W^2} d\mu \\ &= \int |\nabla f|^2 d\mu - \int \left| \nabla f - \frac{f}{W} \nabla W \right|^2 d\mu \end{aligned}$$

which leads to

$$(2.2) \quad \int f^2 \left( \frac{-\mathcal{L}W}{W} \right) d\mu \leq \int |\nabla f|^2 d\mu.$$

For the local term we will localize the (SPI) for the measure  $\lambda$ . To this end, let  $\psi$  be a Lipschitz function defined on  $M$  such that  $\mathbb{1}_{A_r} \leq \psi(u) \leq \mathbb{1}_{\rho(\cdot, A_r) \leq 2}$  and  $|\nabla \psi| \leq 1$ . Writing (SPI) for the function  $f\psi$  we get that for all  $s > 0$

$$(2.3) \quad \begin{aligned} \int_{A_r} f^2 d\lambda &\leq \int f^2 \psi^2 d\lambda \\ &\leq 2s \int |\nabla f|^2 \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\lambda + 2s \int f^2 \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\lambda \\ &\quad + \beta(s) \left( \int |f| \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\lambda \right)^2. \end{aligned}$$

To deduce a similar local inequality for  $\mu$  we have two methods. For the first one we apply this inequality to  $f e^{-V/2+T/2}$ . It yields

$$\begin{aligned} \int_{A_r} f^2 d\mu &= \int_{A_r} f^2 e^{-V+T} d\lambda \\ &\leq 2s \int |\nabla f|^2 \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\mu + \frac{s}{2} \int f^2 |\nabla(V-T)|^2 \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\mu \\ &\quad + 2s \int f^2 \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\mu + \beta(s) \left( \int |f| e^{(V-T)/2} \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\mu \right)^2 \end{aligned}$$

so that if we choose  $s$  small enough so that  $sG(r) \leq 2(1-\varepsilon)$ , we get

$$(2.4) \quad \begin{aligned} \int_{A_r} f^2 d\mu &\leq \int 2s |\nabla f|^2 d\mu + (1-\varepsilon) \int f^2 d\mu + 2s \int f^2 d\mu \\ &\quad + \beta(s) \exp(g(r)) \left( \int |f| d\mu \right)^2. \end{aligned}$$

Now combine (2.2) and (2.4). On the left hand side we get

$$\left( 1 - \left( \frac{b}{\Phi(r)} + 1 \right) (1-\varepsilon + 2s) \right) \int f^2 d\mu.$$

For the coefficient to be larger than  $\varepsilon/2$  it is enough that  $s \leq \varepsilon/16$  and  $\Phi(r) \geq 4b/\varepsilon$ . Assuming this in addition to  $sG(r) \leq 2(1 - \varepsilon)$  we obtain that for such  $s > 0$  and  $r$ ,

$$\mu(f^2) \leq \frac{2}{\varepsilon} \left( \frac{1}{\Phi(r)} + 5s/2 \right) \mu(|\nabla f|^2) + \frac{5}{2\varepsilon} \beta(s) \exp(g(r)) \mu(|f|)^2.$$

If  $t$  is given, it remains to choose first

$$r = \Phi^{-1} \left( \frac{4b}{\varepsilon} \vee \frac{4}{\varepsilon t} \right),$$

and then

$$s = \frac{\varepsilon t}{10} \wedge \frac{\varepsilon}{16} \wedge \frac{2(1 - \varepsilon)}{G(r)},$$

to get the first  $\alpha(t)$ .

The second method is more naive but do not introduce any condition on the gradient of  $V$ .

Start with

$$\begin{aligned} \int f^2 \mathbb{1}_{A_r} d\mu &= \int f^2 e^{-V+T} \mathbb{1}_{A_r} d\lambda \leq e^{-\inf_{A_r}(V-T)} \int f^2 \mathbb{1}_{A_r} d\lambda \\ &\leq e^{-\inf_{A_r}(V-T)} \left( 2s \int |\nabla f|^2 \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\lambda + \right. \\ &\quad \left. + 2s \int f^2 \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\lambda + \beta(s) \left( \int |f| \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\lambda \right)^2 \right) \\ &\leq e^{-\inf_{A_r}(V-T)} e^{\sup_{\rho(\cdot, A_r) \leq 2}(V-T)} \left( 2s \int |\nabla f|^2 \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\mu + \right. \\ &\quad \left. + 2s \int f^2 \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\mu + \beta(s) e^{\sup_{\rho(\cdot, A_r) \leq 2}(V-T)} \left( \int |f| \mathbb{1}_{\rho(\cdot, A_r) \leq 2} d\mu \right)^2 \right) \\ &\leq e^{Osc_{\rho(\cdot, A_r) \leq 2}(V-T)} \left( 2s \int |\nabla f|^2 d\mu + 2s \int f^2 d\mu \right) + \\ &\quad + e^{2Osc_{\rho(\cdot, A_r) \leq 2}(V-T)} \beta(s) \left( \int |f| d\mu \right)^2. \end{aligned}$$

If we combine the latter inequality with (2.2) and denote  $s' = 2s e^{Osc_{\rho(\cdot, A_r) \leq 2}(V-T)}$  we obtain

$$\begin{aligned} &\left( 1 - \frac{bs'}{\Phi(r)} \right) \int f^2 d\mu \leq \\ &\leq \left( s' + \frac{1}{\Phi(r)} \right) \int |\nabla f|^2 d\mu + e^{2Osc_{\rho(\cdot, A_r) \leq 2}(V-T)} \beta(s' e^{-Osc_{\rho(\cdot, A_r) \leq 2}(V-T)}/2) \left( \int |f| d\mu \right)^2. \end{aligned}$$

Hence, if we choose  $r = \Phi^{-1}(\frac{4}{s} \vee \frac{bs}{2})$  and  $s' = s/4$  we obtain the second possible  $\alpha(s)$ .  $\square$

**Remark 2.5.** (1) The previous proof extends immediately to the general case of a “diffusion” process with a “carré du champ” which is a derivation, i.e. if  $\mathcal{E}(f, f) = \int \Gamma(f, f) d\mu$  for a symmetric  $\Gamma$  such that  $\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h)$  (see [3] for more details on this framework).

(2) For a general diffusion process, say with a non constant diffusion term, as noted in the previous remark we have to modify the energy term so that it is no further difficulty and

there are numerous examples where condition **(L)** is verified, i.e. consider  $L = a(x)\Delta - x \cdot \nabla$  where  $a$  is uniformly elliptic and bounded (consider  $W = e^{a|x|^2}$  so  $\phi(x) = c|x|^2$ ). But our method as expressed here relies crucially on the explicit knowledge of  $V$ . Note however, that for the second approach, only an upper bound on the behavior of  $V$  over, say, balls is needed, which can be made explicit in some cases.  $\diamond$

**Remark 2.6.** We may for instance choose  $A_r = \bar{V}_r := \{x; |V - T|(x) < r\}$  (i.e. a level set of  $|V - T|$ ) provided  $|V - T|(x) \rightarrow +\infty$  as  $\rho(o, x) \rightarrow +\infty$ . However we have to look at an enlargement  $\bar{V}^{r+2} = \{x; \rho(x, \bar{V}_r) < 2\}$  (not the level set of level  $r + 2$ ).

If we want to replace  $\bar{V}^{r+2}$  by the level set  $\bar{V}_{r+2}$  we have to modify the proof, choosing some ad-hoc function  $\psi$  which is no more 1-Lipschitz. It is not difficult to see that we have to modify (2.3) and what follows, replacing 1 (the 1 of 1-Lipschitz) by  $\sup_{\bar{V}_{r+2}} |\nabla(V - T)|^2$ . So we have to modify the condition on  $s$  in (1) of the previous theorem, i.e.

$$(2.7) \quad \frac{2}{\inf_{(\bar{V}_r)^c} \phi} \leq \varepsilon s \leq \frac{2}{\inf_{(\bar{V}_r)^c} \phi} + \frac{2(1 - \varepsilon)}{\sup_{\bar{V}_{r+2}} |\nabla(V - T)|^2},$$

i.e. we get the same result as (1) but with  $\Phi(r) = \inf_{(\bar{V}_r)^c} \phi$ ,  $g(r) = r + 2$  and  $G(r) = \sup_{\bar{V}_{r+2}} |\nabla(V - T)|^2$ .

The second case (2) cannot (easily) be extended in this direction.  $\diamond$

Actually one can derive a lot of results following the lines of the proof, provided some “local” (SPI) is satisfied. Here is the more general result in this direction.

**Theorem 2.8.** *In theorem 2.1 define  $\lambda_{A_r}(f) = \lambda(f \mathbf{1}_{A_r})$  where  $A_r$  is an increasing family of open sets such that  $\bigcup_r A_r = M$ . Given two such families  $A_r \subseteq B_r$ , assume that for all  $r$  large enough the following local (SPI) holds,*

$$(2.9) \quad \lambda_{A_r}(f^2) \leq s \lambda_{B_r}(|\nabla f|^2) + \beta_r(s) (\lambda_{B_r}(|f|))^2.$$

*Then the conclusions of theorem 2.1 are still true if we replace  $\rho(\cdot, A_r) \leq 2$  by  $B_r$  and  $\beta(s)$  by  $\beta_{r(s)}(s)$  with  $r(s) = \Phi^{-1}\left(\frac{4b}{\varepsilon} \vee \frac{4}{\varepsilon s}\right)$  for each given  $\varepsilon$  in case (1) and  $r(s) = \Phi^{-1}\left(\frac{4}{s} \vee \frac{bs}{2}\right)$  in case (2).*

**2.2. General case.** We consider here the case of general Markov processes on a Manifold  $M$ , with a particular care to jump processes. Indeed, a crucial step in the previous proof is to prove (2.2) and it has been made directly taking profit of the gradient structure, but it can be proved in greater generality. However the second part relying on a perturbation approach seems more difficult. We therefore introduce a local Super-Poincaré inequality.

Let  $\mathcal{D}_w(\mathcal{L})$  be the weak domain of  $\mathcal{L}$  for the martingale problem; i.e.  $f \in \mathcal{D}_w(\mathcal{L})$  if and only if  $t \mapsto f(X_t) - \int_0^t \mathcal{L}f(X_s) ds$  is a local martingale for a Markov process  $X_t$  generated by  $\mathcal{L}$ .

**Theorem 2.10.** *Suppose that  $W \in \mathcal{D}_w(\mathcal{L})$  satisfies the Lyapunov condition (L) for some function  $\phi$  such that  $\phi(x) \rightarrow \infty$  as  $\rho(x, o) \rightarrow \infty$ . Assume also the following family of local Super Poincaré inequality holds for  $\mu$ : for a family of compact sets  $\{A_r \supset B(o, r_0)\}_{r \geq 0}$  such that  $A_r \uparrow M$  as  $r \uparrow \infty$ , there exists  $\beta(r, \cdot)$  such that for all nice  $f$  and  $s > 0$*

$$(2.11) \quad \mu(f^2 1_{A_r}) \leq s \mathcal{E}(f, f) + \beta(r, s) \mu(|f|)^2.$$

Then, denoting

$$\Phi(r) := \inf_{A_r^c} \phi, \quad \Phi^{-1}(r) := \inf\{s \geq 0 : \Phi(s) \geq r\},$$

$\mu$  verifies a Super Poincaré inequality with function for small enough  $s > 0$

$$\alpha(s) = \beta(\Phi^{-1}(2/s), s/2) \left( \frac{b}{2}s + 1 \right).$$

*Proof.* The proof relies on a simple optimization procedure between the weighted energy term and the local Super Poincaré inequality. Namely for all positive  $s$

$$\begin{aligned} \mu(f^2) &= \mu(f^2 1_{A_r^c}) + \mu(f^2 1_{A_r}) \\ &\leq \frac{1}{\Phi(r)} \mu \left( f^2 \frac{-\mathcal{L}W}{W} \right) + \left( \frac{b}{\Phi(r)} + 1 \right) \mu(f^2 1_{A_r}) \\ &\leq \left( \frac{1}{\Phi(r)} + s \right) \mathcal{E}(f, f) + \beta(r, s) \left( \frac{b}{\Phi(r)} + 1 \right) \mu(|f|)^2 \end{aligned}$$

where in the last line we use a generalization of (2.2) which is done by a large deviations argument in Lemma 2.12. Conclude now by setting  $s = 1/\Phi(r)$ , possible for small enough  $s$ , and changing  $2s$  by  $s$ .  $\square$

**Lemma 2.12.** *For every continuous function  $U \geq 1$  such that  $U \in \mathcal{D}_w(\mathcal{L})$  and  $-\mathcal{L}U/U$  is bounded from below,*

$$(2.13) \quad \int -\frac{\mathcal{L}U}{U} f^2 d\mu \leq \mathcal{E}(f, f), \quad \forall f \in \mathbb{D}(\mathcal{E}).$$

*Proof.* Remark that

$$N_t = U(X_t) \exp \left( - \int_0^t \frac{\mathcal{L}U}{U}(X_s) ds \right)$$

is a  $\mathbb{P}_\mu$ -local martingale. Indeed, let  $A_t := \exp \left( - \int_0^t \frac{\mathcal{L}U}{U}(X_s) ds \right)$ , we have by Ito's formula,

$$dN_t = A_t [dM_t(U) + \mathcal{L}U(X_t) dt] - \frac{\mathcal{L}U}{U}(X_t) A_t U(X_t) dt = A_t dM_t(U).$$

Now let  $\beta := (1 + U)^{-1} d\mu/Z$  ( $Z$  being the normalization constant).  $(N_t)$  is also local martingale, then a super-martingale w.r.t.  $\mathbb{P}_\beta$ . We so get

$$\mathbb{E}^\beta \exp \left( - \int_0^t \frac{\mathcal{L}U}{U}(X_s) ds \right) \leq \mathbb{E}^\beta N_t \leq \beta(U) < +\infty.$$

Let  $u_n := \min\{-\mathcal{L}U/U, n\} \leq -\mathcal{L}U/U$ . Since  $u_n \leq -\mathcal{L}U/U$  the above estimate implies

$$F(u_n) := \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^\beta \exp \left( \int_0^t u_n(X_s) ds \right) \leq 0.$$

We cannot use this argument if we also truncate  $-\mathcal{L}U/U$  from below. We may now apply the lower bound of large deviation in [34, Theorem B.1, Corollary B.11] and Varadhan's Laplace principle, for which we need  $u_n$  to be bounded. This requires  $-\mathcal{L}U/U$  to be bounded below, so that  $u_n$  is clearly bounded. Define now  $I(\nu|\mu) = \mathcal{E}(\sqrt{d\nu/d\mu}, \sqrt{d\nu/d\mu})$ , we get

$$F(u_n) \geq \sup\{\nu(u_n) - I(\nu|\mu); \nu \in M_1(E)\}.$$



Thus  $\int u_n d\nu \leq I(\nu|\mu)$ , which yields to (by letting  $n \rightarrow \infty$  and monotone convergence)

$$(2.14) \quad \int -\frac{\mathcal{L}U}{U} d\nu \leq I(\nu|\mu), \quad \forall \nu \in M_1(E).$$

That is equivalent to (2.13) by the fact that  $\mathcal{E}(|f|, |f|) \leq \mathcal{E}(f, f)$  for all  $f \in \mathbb{D}(\mathcal{E})$ .  $\square$

We will fully discuss examples on diffusion processes in the next sections. Let us just give a simple example on jump processes, see also [32, Th. 3.4.2] for results in this direction. Remark that in full generality, one of the main difficulty is to find a local Super Poincaré inequality. However if the state space is discrete, this difficulty mainly disappears.

*Lyapunov conditions for discrete valued pure jump process.* Let us consider here a pure jump process with values in a countable space  $E$ , symmetric with respect to the invariant measure  $\mu$ , such that for all  $x \in E$ ,  $\mu(x) > 0$ . The important remark is that a local Super Poincaré inequality in the sense of (2.11) is easy to obtain, indeed for  $B \subset E$  with finite cardinal

$$\mu(|f|1_B)^2 = \left( \sum_{x \in B} |f(x)|\mu(x) \right)^2 \geq \sum_{x \in B} f^2(x)\mu^2(x) \geq \left( \inf_{x \in B} \mu(x) \right) \mu(f^2 1_B)$$

so that

$$\mu(f^2 1_B) \leq \left( \inf_{x \in B} \mu(x) \right)^{-1} \mu(|f|)^2.$$

Let us apply the previous theorem for the birth and death process on  $\mathbb{N}$  with birth rate  $b_i$  and death rate  $d_i$  both equal to  $i^a \log^\alpha(1+i)$  for  $a \geq 2$  and  $\alpha \in \mathbb{R}$  and  $d_0 = 0$ ,  $b_1 = 1$ . Remark that the invariant measure is given by  $\mu(i) = (1/Z) d_i$  (where  $Z$  is the normalizing constant) so that we have some local Super Poincaré inequality

$$\mu(f^2 1_{x \leq n}) \leq Z n^s \log^\alpha(1+n) \mu(|f|)^2.$$

Choose now  $W(n) = 1 + n^\gamma$  with  $0 < \gamma < 1$ , for which we derive that for  $n \geq n_0$ , there exists  $\lambda > 0$  such that

$$\mathcal{L}W \leq -\lambda n^{s-2} \log^\alpha(s) W$$

which is our Lyapunov condition. By Theorem 2.10, and for  $a = 2$ , if  $\alpha > 0$  then  $\mu$  verifies a Super Poincaré inequality: for all positive  $s$ , there exist  $c, C > 0$

$$\mu(f^2) \leq s\mathcal{E}(f, f) + ce^{Cs^{1/\alpha}} \mu(|f|)^2.$$

Note that we find back the (optimal) results of [30] in a simple way.

### 3. Examples in $\mathbb{R}^n$ .

We use the setting of the subsection 2.1 (or of the remark 2.5) but in the euclidean case  $M = \mathbb{R}^n$  for simplicity. Hence in this section  $\lambda$  is the Lebesgue measure, i.e we have  $T = 0$ . Recall that  $d\mu = Z^{-1} e^{-V} dx$ . It is well known that  $\lambda$  satisfies a (SPI) with  $\beta(s) = c_1 + c_2 s^{-n/2}$ . However it is interesting to have some hints on the constants (in particular dimension dependence). It is also interesting (in view of Theorem 2.8) to prove (SPI) for subsets of  $\mathbb{R}^n$ .

Hence we shall first discuss the (SPI) for  $\lambda$  and its restriction to subsets. Since we want to show that the Lyapunov method is also quite quick and simple in many cases, we shall also recall the quickest way to recover these (SPI) results.

**3.1. Nash inequalities for the Lebesgue measure.** Let  $A$  be an open connected domain with a smooth boundary. For simplicity we assume that  $A = \{\psi(x) \leq 0\}$  for some  $C^2$  function  $\psi$  such that  $|\nabla\psi|^2(x) \geq a > 0$  for  $x \in \partial A = \{\psi = 0\}$ . It is then known that one can build a Brownian motion reflected at  $\partial A$ , corresponding to the heat semi-group with Neumann condition. Let  $P_t^N$  denote this semi-group, and denote by  $p_t^N$  its kernel. Recall the following

**Proposition 3.1.** *The following statements are equivalent*

(3.1.1) *for all  $0 < t \leq 1$  and all  $f \in \mathbb{L}^2(A, dx)$ ,*

$$\| P_t^N f \|_\infty \leq C_1 t^{-n/4} \| f \|_{\mathbb{L}^2(A, dx)},$$

(3.1.2) *(provided  $n > 2$ ) for all  $f \in C^\infty(\bar{A})$ ,*

$$\| f \|_{\mathbb{L}^{2n/n-2}(A, dx)}^2 \leq C_2 \left( \int_A |\nabla f|^2 dx + \int_A f^2 dx \right),$$

(3.1.3) *for all  $f \in C^\infty(\bar{A})$ ,*

$$\| f \|_{\mathbb{L}^{2+4/n}(A, dx)}^{2+4/n} \leq C_3 \left( \int_A |\nabla f|^2 dx + \int_A f^2 dx \right) \| f \|_{\mathbb{L}^1(A, dx)}^{4/n},$$

(3.1.4) *the (SPI) inequality*

$$\int_A f^2 dx \leq s \int_A |\nabla f|^2 dx + \beta(s) \left( \int_A |f| dx \right)^2$$

*holds with  $\beta(s) = C_4(s^{-n/2} + 1)$ .*

Furthermore any constant  $C_i$  can be expressed in terms of any other  $C_j$  and the dimension  $n$ .

These results are well known. they are due to Nash, Carlen-Kusuoka-Stroock ([9]) and Davies, and can be found in [14] section 2.4 or [26]. See also Varopolous [28] for the equivalence between (3.1.1) and (3.1.2) in a very general setting. Generalizations to other situations (including general forms of rate functions  $\beta$ ) can be found in [32] section 3.3.

If  $A = \mathbb{R}^n$  (3.1.1) holds (for all  $t$ ) with  $C = (2\pi)^{-n/2}$  and  $\alpha = n/2$ , yielding a (SPI) inequality with

$$(3.2) \quad \beta(s) = c_n s^{-n/2} = \left( \frac{1}{4\pi} \right)^{n/2} s^{-n/2},$$

which is equivalent, after optimizing in  $s$ , to the Nash inequality

$$(3.3) \quad \| f \|_2^{2+4/n} \leq C_n \left( \int |\nabla f|^2 dx \right) \| f \|_1^{4/n},$$

with  $C_n = 2(1 + 2/n)(1 + n/2)^{2/n} (1/8\pi)^{n/4}$ .

For nice open bounded domains in  $\mathbb{R}^n$ , as we consider here, (3.1.2) is a well known consequence of the Sobolev inequality in  $\mathbb{R}^n$  (see e.g. [14] Lemma 1.7.11 and note that the particular cases

$n = 1, 2$  can be treated by extending the dimension (see [14] theorem 2.4.4). But we want here to get some information on the constants. In particular, when  $A$  is the level set  $\bar{V}_r$  we would like to know how  $\beta_r$  depends on  $r$ .

**Remark 3.4.** If  $n = 1$ , we have an explicit expression for  $p_t^N$  when  $A = (0, r)$ , namely

$$p_t^N(x, y) = (2\pi t)^{-n/2} \sum_{k \geq 0} \left( \exp\left(-\frac{(x-y-2kr)^2}{2t}\right) + \exp\left(-\frac{(x+y+2kr)^2}{2t}\right) \right).$$

It immediately follows that

$$(3.5) \quad \sup_{x, y \in (0, r)} p_t^N(x, y) \leq (2\pi t)^{-n/2} \left( 2 + \sum_{k \geq 1} \left( \exp\left(-\frac{((2k-1)r)^2}{2t}\right) + \exp\left(-\frac{(2kr)^2}{2t}\right) \right) \right),$$

so that, using translation invariance, for any interval  $A$  of length  $r > r_0$  and for  $0 < t \leq 1$ ,  $\sup_{x, y \in A} p_t^N(x, y) \leq c(r_0) (2\pi t)^{-n/2}$ . Hence (3.1.1) is satisfied, and a (SPI) inequality holds in  $A$  with the same function  $\beta_r(s) = c_B (s^{-n/2} + 1)$  independently on  $r > r_0$ . By tensorization, the result extends to any cube or parallelepiped in  $\mathbb{R}^n$  with edges of length larger than  $r_0$ .  $\diamond$

If we replace cubes by other domains, the situation is more intricate. However in some cases one can use some homogeneity property. For instance, for  $n > 2$  we know that (3.1.2) holds for the unit ball with a constant  $C_2$  (for  $n = 2$  we may add a dimension and consider a cylinder  $B_2(0, 1) \otimes \mathbb{R}$  as in [14] theorem 2.4.4). But a change of variables yields

$$\|f\|_{\mathbb{L}^{2n/n-2}(B(0, r), dx)}^2 \leq C_2 \left( \int_{B(0, r)} |\nabla f|^2 dx + r^{-2} \int_{B(0, r)} f^2 dx \right),$$

so that for  $r \geq 1$  (3.1.2) holds in the ball of radius  $r$  with a constant  $C_2$  independent of  $r$ .

The previous argument extends to  $A = \bar{V}_r$  provided for  $r \geq r_0$ ,  $\bar{V}_r$  is star-shaped, in particular it holds if  $V$  is convex at infinity. This is a direct consequence of the coarea formula (see e.g [17] proposition 3 p.118). Indeed if  $f$  has his support in an annulus  $r_0 < r_1 < V(x) < r_2$  the surface measure on the level sets  $\bar{V}_r$  is an image of the surface measure on the unit sphere. This is immediate since the application  $x \mapsto (V(x), \frac{x}{|x|})$  is a diffeomorphism in this annulus. Hence for such  $f$ 's the previous homogeneity property can be used. For a given  $r > r_0$  large enough, it remains to cover  $\bar{V}_r$  by such an annulus and a large ball (such that the ball contains  $\bar{V}_{r_0}$  and is included in  $\bar{V}_r$ ) and to use a partition of unity related to this recovering. We thus get as before that for  $r$  large enough,  $C_2$  can be chosen independent of  $r$ .

For general domains  $A$ , recall that (3.1.2) holds true if  $A$  satisfies the ‘‘extension property’’ of the boundary, i.e. the existence of a continuous extension operator  $E : \mathbb{W}^{1,2}(A) \rightarrow \mathbb{W}^{1,2}(\mathbb{R}^n)$ . If this extension property is true, (3.1.2) is true in  $A$  with a constant  $C_2$  depending only on  $n$  and the operator norm of  $E$  (see [14] proposition 1.7.11).

If  $A = \bar{V}_r$  is bounded, as soon as  $\nabla V$  does not vanish on  $\partial A$ , the implicit function theorem tells us that for all  $x \in \partial A$  one can find an open neighborhood  $v_x$  of  $x$ , an index  $i_x$  and a 2-Lipschitz function  $\phi_x$  defined on  $v_x$  such that

$$v_x \cap A = v_x \cap \{\phi(y_1, \dots, y_{i_x-1}, y_{i_x+1}, \dots, y_n) < y_{i_x}\}.$$

To this end choose  $i_x$  such that  $|\partial_{i_x} V|(x) \geq |\partial_j V|(x)$  for all  $j = 1, \dots, n$ , so that, for  $y \in \partial A$  neighboring  $x$ ,  $2|\partial_{i_x} V|(y) \geq |\partial_j V|(y)$ , and the partial derivative of the implicit function  $\phi$  given by the ratio  $\partial_j V(y)/\partial_{i_x} V(y)$  is less than 2 in absolute value.

By compactness we may choose a finite number  $Q$  of points such that  $\bigcup_{j=1, \dots, Q} v_{x_j} \supset \partial A$ . Hence we are in the situation of [14] proposition 1.7.9. This property implies the extension property but with some extension operator  $E$  whose norm depends on two quantities : first the maximal  $\varepsilon > 0$  such that for all  $x \in \partial A$ ,  $B(x, \varepsilon) \subseteq v_{x_j}$  for some  $j = 1, \dots, Q$ ; second, the maximal integer  $N$  such that any  $x \in \partial A$  belongs to at most  $N$  such  $v_{x_j}$ 's. This is shown in [27] p.180-192.

Actually an accurate study of Stein's proof (p. 190 and 191) shows that  $\|E\| \leq C(n) (N/\varepsilon)$  (recall that we have chosen  $\phi$  2-Lipschitz).

Now assume that

$$(3.6) \quad \text{there exist } R > 0, v > 0, k \in \mathbb{N} \text{ such that for } |x| \geq R, |\nabla V(x)| \geq v > 0.$$

Then it is easy to check that for  $A = \bar{V}_r$  it holds  $\varepsilon \leq \varepsilon_0 = c(v, R, n) \theta^{-1}(r)$  with

$$(3.7) \quad \theta(r) = \sup_{x \in \partial \bar{V}_r} \max_{i, j=1, \dots, n} \left| \frac{\partial^2 V}{\partial x_i \partial x_j}(x) \right|.$$

But  $\varepsilon_0$  being given, it is well known that one can find a covering of  $A$  by balls of radius  $\varepsilon_0/2$  such that each  $x \in \bar{V}_r$  belongs to at most  $N = c^n$  such balls for some universal  $c$  large enough. Hence  $N$  can be chosen as a constant depending on the dimension only. It follows that

**Proposition 3.8.** *If (3.6) is satisfied, the (SPI) (3.1.4) holds with  $A = \bar{V}_r$ ,  $\theta$  defined by (3.7) and*

$$\beta_r(s) = C(n) \theta^n(r) (1 + s^{-n/2}).$$

For the computation of  $\beta_r$  we used [14] lemma 1.7.11 which says that  $C_2 = c(n) \|E\|^2$  and [14] proof of theorem 2.4.2 p.77 which yields a logarithmic Sobolev inequality with  $\beta(\varepsilon) = -(n/4) \log \varepsilon + (n/4) \log(C_2 n/4)$  together with [14] corollary 2.2.8 which gives  $C_1 = c(n) C_2^{n/4}$ . Finally the proof of [14] theorem 2.4.6 gives  $\beta(s) = C_1^2 (1 + (s/2)^{-n/2})$ .

Proposition 3.8 gives of course the worse result and in many cases one can expect a much better behavior of  $\beta_r$  as a function of  $r$ . In particular in the homogeneous case we know the result with a constant independent of  $r$ .

**Remark 3.9.** Another possibility to get (SPI) in some domain  $A$ , is to directly prove the Nash inequality (3.1.3). One possible way to get such a Nash inequality is to prove some Poincaré-Sobolev inequality. The case of euclidean balls is well known.

According to [26] theorem 1.5.2, for  $n > 2$ , with  $p = 2$  and  $s = 2n/(n-2) = 2^*$  therein, for all  $r > 0$  and all ball  $B_r$  with radius  $r$ , if  $\lambda_r$  is the Lebesgue measure on  $B_r$  and  $\bar{f}_r = (1/\text{Vol}(B_r)) \int_{B_r} f dx$  we have

$$(3.10) \quad \lambda_r \left( |f - \bar{f}_r|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \leq C_n \lambda_r (|\nabla f|^2)^{\frac{1}{2}},$$

so that using first Minkowski, we have

$$(3.11) \quad \lambda_r \left( |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \leq C_n \lambda_r (|\nabla f|^2)^{\frac{1}{2}} + \frac{1}{\text{Vol}(B_r)} \lambda_r(|f|),$$

and finally using Hölder inequality and Cauchy-Schwarz inequality we get the local Nash inequality

$$(3.12) \quad \begin{aligned} \lambda_r(|f|^2) &\leq (\lambda_r(|f|))^{4/(n+2)} \left( C_n \lambda_r (|\nabla f|^2)^{\frac{1}{2}} + \frac{1}{\text{Vol}(B_r)} \lambda_r(|f|) \right)^{2n/(n+2)}, \\ &\leq (\lambda_r(|f|))^{4/(n+2)} \left( C_n \lambda_r (|\nabla f|^2)^{\frac{1}{2}} + \frac{1}{\sqrt{\text{Vol}(B_r)}} \lambda_r(|f|^2)^{\frac{1}{2}} \right)^{2n/(n+2)}. \end{aligned}$$

Again, for  $r > r_0$  we get a Nash inequality hence a (SPI) inequality independent of  $r$  with  $\beta_r(s) = c_n(1 + s^{-n/2})$ .

Notice that (3.10) is scale invariant, i.e. if it holds for some subset  $A$ , it holds for the homotetic  $rA$  ( $r > 0$ ) with the same constants. That is why the constants do not depend on the radius for balls. If we replace a ball by a convex set, the classical method of proof using Riesz potentials (see e.g. [26] or [14] lemma 1.7.3) yields a similar results but with an additional constant, namely  $\text{diam}^n(A)/\text{Vol}(A)$ , so that if  $V$  is a convex function the constant we obtain with this method in (3.10) for  $\bar{V}_r$  may depend on  $r$ .

Actually the Sobolev-Poincaré inequality (3.10) extends to any John domain with a constant  $C$  depending on the dimension  $n$  and on the John constant of the domain. This result is due to Bojarski [7] (also see [20] for another proof and [8] for a converse statement). Actually a John domain satisfies some chaining (by cubes or balls) condition which is the key for the result (see the quoted papers for the definition of a John domain and the chaining condition). But an explicit calculation of the John constant is not easy.  $\diamond$

**3.2. Typical Lyapunov functions and applications.** We here specify classes of natural Lyapunov function: function of the potential or of the distance. As will be seen, it gives new practical conditions for super-Poincaré inequality and for logarithmic Sobolev inequality.

First, since  $W \geq 1$  we may write  $W = e^U$  so that condition (L) becomes

$$(3.13) \quad \Delta U + |\nabla U|^2 - \nabla U \cdot \nabla V + \phi \leq 0 \quad \text{“at infinity”}.$$

**3.2.1. Lyapunov function  $e^{aV}$ .** Test functions  $e^{aV}$  for  $a < 1$  are quite natural in that they are the limiting case for the spectral gap (see [3]). Indeed,  $\mu(e^{aV})$  is finite if and only if  $a < 1$  and a drift condition such that

$$\mathcal{L}W \leq -\lambda W + b1_C$$

formally implies by integration by  $\mu$ , that  $\mu(W)$  is finite. So in a sense,  $e^{aV}$  are the “largest” possible Lyapunov functions.

Hence, if  $W = e^{aV}$ ,  $\frac{\mathcal{L}W}{W} = a (\Delta V - (1-a)|\nabla V|^2)$ . Introduce the following conditions

$$(3.14.1) \quad V(x) \rightarrow +\infty \text{ as } |x| \rightarrow +\infty,$$

(3.14.2) there exist  $0 < a_0 < 1$ , a non-decreasing function  $\eta$  with  $\eta(u) \rightarrow +\infty$  as  $u \rightarrow +\infty$  and a constant  $b_0$  such that

$$(1 - a_0)|\nabla V|^2 - \Delta V \geq \eta(V) + b_0 \mathbb{1}_{|x| < R},$$

(3.14.3)  $\limsup_{|x| \rightarrow +\infty} (\eta(V(x))/|\nabla V(x)|^2) < +\infty$ .

Then for  $0 < a < a_0$  condition (L) is satisfied with

$$\phi = a(a_0 - a)|\nabla V|^2 + a\eta(V).$$

In addition  $\inf_{(\bar{V}_r)^c} \phi(V) \leq c \sup_{\bar{V}_{r+2}} |\nabla V|^2$ .

Following remark 2.6 (we choose arbitrarily  $\varepsilon = 1/2$  here) we obtain for some constant  $c$

$$(3.15) \quad \int f^2 d\mu \leq s \int |\nabla f|^2 d\mu + c \left( 1 + \sup_{\bar{V}_{2+\eta^{-1}(c/s)}} |\nabla V|^2 \right)^{n/2} e^{\eta^{-1}(c/s)} \left( \int |f| d\mu \right)^2.$$

We thus clearly see that to get an explicit (SPI) we need to control the gradient  $\nabla V$  on the level sets of  $V$ .

If instead of using theorem 2.1.(1) we want to use theorem 2.1.(2) or more precisely theorem 2.8 we have to use proposition 3.8. Hence since (3.6) is satisfied we obtain for  $s$  small enough

$$(3.16) \quad \int f^2 d\mu \leq s \int |\nabla f|^2 d\mu + C \theta^n(\eta^{-1}(c/s)) e^{2\eta^{-1}(c/s)} \left( 1 + s^{-n/2} e^{n\eta^{-1}(c/s)/2} \right) \left( \int |f| d\mu \right)^2.$$

We have obtained

**Theorem 3.17.** *Assume that (3.14.1), (3.14.2), (3.14.3) are satisfied. Then  $\mu$  will satisfy a (SPI) inequality with function  $\beta$  in one of the following cases*

(3.17.1) *for  $|x|$  large enough,  $|\nabla V|(x) \leq \gamma(V(x))$  and  $\beta(s) = C(1 + e^{\eta^{-1}(c/s)} \gamma^n(\eta^{-1}(c/s)))$ ,*

(3.17.2) *for  $|x|$  large enough  $\left| \frac{\partial^2 V}{\partial x_i \partial x_j}(x) \right| \leq \theta(V(x))$  and*

$$\beta(s) = C \left( 1 + \theta^n(\eta^{-1}(c/s)) s^{-n/2} e^{(n+4)\eta^{-1}(c/s)/2} \right).$$

**Remark 3.18.** If  $\eta(u) = u$  we thus obtain that  $\mu$  satisfies a (defective) logarithmic Sobolev inequality provided either  $\gamma(u) \leq e^{Ku}$  or  $\theta(u) \leq e^{Ku}$ . But (3.14.1) and (3.14.2) imply that  $\mu$  satisfies a Poincaré inequality (see e.g. [3] corollary 4.1). Hence using Rothaus lemma we get that  $\mu$  satisfies a (tight) logarithmic Sobolev inequality.

Conditions (3.14.1) and (3.14.2), with  $\eta(u) = u$ , appear in [22] where the authors show that they imply the hypercontractivity of the associated symmetric semi-group, hence a logarithmic Sobolev inequality by using Gross theorem. In particular the additional assumptions on the first or the second derivatives do not seem to be useful. Another approach using Girsanov transformation was proposed in [10] for  $a_0 = 1/2$ , again without the technical assumptions on the derivatives. This approach extends to more general processes with a ‘‘carré du champ’’. Here we directly get the logarithmic Sobolev inequality without using Gross theorem, but with some conditions on  $V$ .

The advantage of theorem 3.17 is that it furnishes an unified approach of various inequalities of  $F$ -Sobolev type. In [4] conditions (3.14.1) and (3.14.2) are used (for particular  $\eta$ 's) to get the Orlicz-hypercontractivity of the semi-group hence a  $F$ -Sobolev inequality thanks to the Gross-Orlicz theorem proved therein. The use of this theorem requires some quite stringent conditions on  $\eta$  but covers the case  $\eta(u) = u^\alpha$  for  $1 < \alpha < 2$ , yielding a  $F$ -Sobolev inequality for  $F(u) = \log_+^\alpha(u)$  (more general  $F$  are also studied in [5] section 7). Note that in theorem 3.17 we do no more need the restriction  $\alpha < 2$ , but we need some control on the growth on  $\gamma$  or  $\theta$ , namely we need again  $\gamma(u) \leq e^{Ku}$  (the same for  $\theta$ ).

We also obtain a larger class of  $F$ -Sobolev inequalities thanks to the correspondence between  $F$ -Sobolev and (SPI) recalled in the introduction. The reader is referred to [32] section 5.7 for related results in the ultracontractive case.  $\diamond$

3.2.2. *Lyapunov function*  $e^{a|x|^b}$ . If we try to use  $W = e^{a|x|^b}$  we are led to choose

$$\phi(x) = ab|x|^{b-2}\psi(x)$$

with

$$\psi(x) = x \cdot \nabla V - \left( n + (b-2) + ab|x|^b \right)$$

provided the latter quantities are bounded from below by a positive constant for  $|x|$  large enough.

Introduce now the standard curvature assumption

$$(3.19) \quad \text{for all } x, \quad \text{Hess}V(x) \geq c_0 \text{Id}$$

for some  $\rho \in \mathbb{R}$ . This assumption allows to get some control on  $x \cdot \nabla V$  namely

**Lemma 3.20.** *If (3.19) holds,  $x \cdot \nabla V(x) \geq V(x) - V(0) + c_0|x|^2/2$ .*

*Proof.* Introduce the function  $g(t) = tx \cdot \nabla V(tx)$  defined for  $t \in [0, 1]$ . (3.19) implies that  $g'(t) \geq x \cdot \nabla V(tx) + tc_0|x|^2$  and the result follows by integrating the latter inequality between 0 and 1.  $\square$

We may thus state

**Proposition 3.21.** *Assume that (3.19) is satisfied. Then one can find positive constants  $c, C$  such that  $\mu$  satisfies some (SPI) with function  $\beta$  (given below for  $s$  small enough) in the following cases*

$$(3.21.1) \quad c_0 \geq 0, V(x) \geq c'|x|^b \text{ for } |x| \text{ large enough some } c' > 0 \text{ and } b > 1,$$

$$\beta(s) = C e^{c(1/s)^{\frac{b}{2((b-1) \wedge 1)}}}.$$

$$(3.21.2) \quad c_0 \geq 0, d'|x|^{b'} \geq V(x) \geq c'|x|^b \text{ for } |x| \text{ large enough some } d', c' > 0 \text{ and } b' \geq b > 1,$$

$$\beta(s) = C e^{c(1/s)^{\frac{b'}{b'+b-2}}}.$$

$$(3.21.3) \quad c_0 \leq 0, \text{ for } |x| \text{ large enough, } V(x) \geq (\varepsilon - c_0/2)|x|^2 \text{ for some } \varepsilon > 0, \text{ and}$$

$$\beta(s) = C e^{c(1/s)}.$$

$$(3.21.4) \quad c_0 \leq 0, \text{ for } |x| \text{ large enough, } d'|x|^{b'} \geq V(x) \geq c'|x|^b \text{ and } \beta \text{ as in (3.21.2).}$$

*Proof.* In all the proof  $D$  will be an arbitrary positive constant whose value may change from place to place. All the calculations are assuming that  $|x|$  is large enough.

Consider first case 1. Choosing  $a$  small enough and using lemma 3.20, we see that  $\phi(x) \geq D|x|^{b-2}V(x)$ . If  $b \geq 2$  we thus have  $\phi(x) \geq DV(x)$  while for  $b < 2$ ,  $\phi(x) \geq DV^{\frac{2(b-1)}{b}}(x)$  for large  $|x|$  according to the hypothesis. For  $\phi$  to go to infinity at infinity,  $b > 1$  is required. In particular on the level sets  $\bar{V}_r$  we have either  $\phi(x) \geq Dr$  or  $\phi(x) \geq Dr^{2(b-1)/b}$ .

Now since the level sets  $\bar{V}_r$  are convex, we know that some Nash inequality holds on  $\bar{V}_r$  according to the discussion in the previous subsection. We may thus use theorem 2.8 in the situation (2) of theorem 2.1. Choosing  $s = d/r$  or  $s = d/r^{2(b-1)b}$  for some well chosen  $d$  yields the result with an extra factor  $s^{-k}$  for some  $k > 0$ . This extra term can be skipped just changing the constants in the exponential term.

Case 2 is similar but improving the lower bound for  $\phi$ . Indeed since  $D|x| \geq V^{1/b'}(x)$ ,  $\phi(x) \geq DV^{\frac{b'+b-2}{b'}}(x)$ . It allows us to improve  $\beta$ .

Let us now consider Case 3. Since  $b = 2$ , our hypothesis implies that for  $2a < \varepsilon$ ,  $\phi \geq DV$ . But the curvature assumption implies that the level sets of  $x \mapsto H(x) = V(x) + c_0|x|^2/2$  are convex. Since  $V(x) \geq D|x|^2$ , one has  $cr \leq V(x) \leq r$  if  $x \in \bar{H}_r$ . We may thus mimic case 1, just replacing  $\bar{V}_r$  by  $\bar{H}_r$ . Case 4 is similar to the previous one just improving the bound on  $\phi$  as in case 2.  $\square$

- Corollary 3.22.** (1) *If (3.21.3) holds,  $\mu$  satisfies a logarithmic Sobolev inequality.*  
 (2) *If (3.21.1) holds with  $b = 2$ ,  $\mu$  satisfies a logarithmic Sobolev inequality. In particular if  $\rho > 0$ ,  $\mu$  satisfies a logarithmic Sobolev inequality (Bakry-Emery criterion).*  
 (3) *If (3.21.1) holds for some  $1 < b < 2$ ,  $\mu$  satisfies a  $F$ -Sobolev inequality with  $F(u) = \log_+^{2(1-(1/b))}(u)$ .*

The first statement of the theorem is reminiscent to Wang's improvement of the Bakry-Emery criterion, namely if  $\int \int e^{(-\rho+\varepsilon)|x-y|^2} \mu(dx) \mu(dy) < +\infty$ ,  $\mu$  satisfies a logarithmic Sobolev inequality. Our statement is weaker since we are assuming some uniform behavior. The third statement can thus be seen as an extension of Wang's result to the case of  $F$ -Sobolev inequalities interpolating between Poincaré inequality and log-Sobolev inequality. These inequalities are related to the Latala-Oleskiewicz interpolating inequalities [23], see [4] for a complete description.

It should be interesting to improve (3) in the spirit of Wang's concentration result. See [21, 6] for an attempt involving modified log-Sobolev inequalities introduced in [19] and mass transport.

#### 4. THE GENERAL MANIFOLD CASE

In fact as one guesses, the main point is to get the additional Super Poincaré inequality, local as developed in Section 3.1, or global (and then using the localization technique already mentioned). It is of course a fundamental field of research which encompasses the scope of the present paper. We may however use our main results Theorem 2.1 and Theorem 2.8, with the same Lyapunov functionals as developed in Sections 3.2.1 and 3.2.2, replacing of



course the euclidean distance by the Riemannian distance (w.r.t a fixed point), at least in two main cases.

According to [13], if the injectivity radius of  $M$  is positive then (1.3) holds for  $T = 0$  and  $\beta(s) = c_1 + c_2 s^{-d/2}$  for some constants  $c_1, c_2 > 0$ ; if in particular the injectivity is infinite, then one may take  $c_1 = 0$ , [29] page 225.

Next, if the Ricci curvature of  $M$  is bounded below, then by [29] Theorem 7.1, there exists  $c_1, c_2 > 0$  such that (1.3) holds for  $T = c_1 \rho$  and  $\beta(s) = c_2 s^{-d/2}$ . For simplicity, throughout this section we assume that

(H) *The injectivity radius of  $M$  is positive.*

**4.1. Lyapunov condition  $e^{aV}$ .** In this context, one may readily generalize the result of Theorem (3.17) for the first case (3.17.1), with the euclidean distance replaced by the Riemannian one, assuming (3.14.1), (3.14.2) and (3.14.3).

**Theorem 4.1.** *Assume (H) and that (3.14.1), (3.14.2), (3.14.3) are satisfied. Suppose moreover that for large  $\rho$ ,  $|\nabla V|(x) \leq \gamma(V(x))$ . Then  $\mu$  will satisfy a (SPI) inequality with function  $\beta$  given by*

$$\beta(s) = C(1 + e^{\eta^{-1}(c/s)} \gamma^n(\eta^{-1}(c/s))).$$

The second point of Theorem 3.17 is more delicate as it relies on finer conditions on the manifold and the potential, it should however be possible to give mild additional assumptions ensuring such a result (for instance the so called “rolling ball condition”). Remark that it extends to the manifold case Kusuoka-Stroock’s result (giving life to Remark (2.49) in their paper).

**4.2. Lyapunov condition  $e^{a\rho^b}$ .** We suppose moreover here that  $M$  is a Cartan-Hadamard manifold, i.e. a simply connected complete Riemannian manifold with non positive sectional curvature. According to the Cartan-Hadamard theorem,  $M$  is thus diffeomorphic to  $\mathbb{R}^n$ . One certainly should relax this assumption, introducing however some technicalities. In addition we will assume that the Ricci curvature is bounded from below.

If we try to use  $W = e^{a\rho^b}$  for  $\rho \geq 1$ , since  $\Delta\rho$  is bounded above on  $\{\rho \geq 1\}$  (see for example Th.0.4.10 in [32]), (L) holds for

$$\phi := ab\rho^{b-2}\psi$$

with

$$\psi := \langle \nabla \rho^2, \nabla V \rangle - (c + ab\rho^b)$$

for some constant  $c > 0$  provided  $\psi$  is positive for large  $\rho$ .

We may then extend Lemma 3.20 in the manifold context.

**Lemma 4.2.** *If (3.19) holds, then  $\rho \langle \nabla \rho, \nabla V \rangle \geq V - V(o) + c_0 \rho^2/2$ .*

*Proof.* For  $x \in M$ , let  $\xi : [0, \rho(x)] \rightarrow M$  be the minimal geodesic from  $o$  to  $x$ . Let

$$g(t) = t \langle \nabla \rho, \nabla V \rangle (\xi_t), \quad t \geq 0.$$

We have

$$g'(t) = \langle \nabla \rho, \nabla V \rangle(\xi_t) + t \text{Hess}_V(\nabla \rho, \nabla \rho)(\xi_t) \geq c_0 t + \frac{dV(\xi_t)}{dt}.$$

This implies the desired assertion by integrating both sides on  $[0, \rho(x)]$ .  $\square$

We may thus state

**Proposition 4.3.** *Let  $M$  be a Cartan-Hadamard manifold with Ricci curvature bounded below. Let  $V$  satisfy (3.19). Then one can find positive constants  $c, C$  such that  $\mu$  satisfies some (SPI) with function  $\beta$  (given below for  $s$  small enough) in the following cases*

$$(4.3.1) \quad c_0 \geq 0, V(x) \geq c' \rho^b \text{ for } \rho \text{ large enough some } c' > 0 \text{ and } b > 1,$$

$$\beta(s) = C e^{c(1/s)^{\frac{b}{2((b-1) \wedge 1)}}}.$$

$$(4.3.2) \quad c_0 \geq 0, d' \rho^{b'} \geq V(x) \geq c' \rho^b \text{ for } \rho \text{ large enough some } d', c' > 0 \text{ and } b' \geq b > 1,$$

$$\beta(s) = C e^{c(1/s)^{\frac{b'}{b'+b-2}}}.$$

$$(4.3.3) \quad c_0 \leq 0, \text{ for } \rho \text{ large enough, } V(x) \geq (\varepsilon - c_0/2)\rho^2 \text{ for some } \varepsilon > 0, \text{ and}$$

$$\beta(s) = C e^{c(1/s)}.$$

$$(4.3.4) \quad c_0 \leq 0, \text{ for } \rho \text{ large enough, } d' \rho^{b'} \geq V(x) \geq c' \rho^b \text{ and } \beta \text{ as in (3.21.2)}.$$

The first point of this proposition specialized to the case  $c_0 > 0$  enables us to recover [31, Th.1.3] which extends Bakry-Emery criterion to lower bounded Ricci curvature manifold. It then extends the result to various  $F$ -Sobolev.

*Proof.* The proof follows exactly the same line than in the flat case so that case 1 and case 2 follows once it is noted that since  $\text{Hess}_V \geq 0$  implies the convexity of the level sets  $\bar{V}_r$ , we know that some Nash inequality holds on  $\bar{V}_r$  according to the discussion in the previous subsection and the boundedness of these level sets ensured by our hypotheses on  $V$ .

Let us now consider Case 3. Since  $b = 2$ , our hypothesis implies that for  $2a < \varepsilon$ ,  $\phi \geq DV$ . But (3.19) and  $\text{Hess}_{\rho^2} \geq 2$  on Cartan-Hadamard manifolds imply that the level sets of  $x \mapsto H = V + c_0 \rho^2/2$  are convex. Since  $V \geq D\rho^2$ , one has  $cr \leq V \leq r$  on  $\bar{H}_r$ . We may thus mimic case 1, just replacing  $\bar{V}_r$  by  $\bar{H}_r$ . Case 4 is similar to the previous one just improving the bound on  $\phi$  as in case 2.  $\square$

**Remark 4.4.** Remark that in full generality, according to [33] Theorem 1.2 and the recent paper [12], there always exists  $T \in C^\infty(M)$  such that  $d\lambda := e^{-T(x)} dx$  satisfies a logarithmic Sobolev inequality hence (SPI) with  $\beta(s) = e^{s^{-1}}$ . Of course for practical purposes, this very general fact is not completely useful since  $T$  is unknown.

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## REFERENCES

- [1] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses*. Société Mathématique de France, Paris, 2000.
- [2] D. Bakry, F. Barthe, P. Cattiaux, and A. Guillin. A simple proof of the Poincaré inequality for a large class of measures including the logconcave case. *Elec. Comm. Prob.* 13: 60–66, 2008.
- [3] D. Bakry, P. Cattiaux, and A. Guillin. Rate of convergence for ergodic continuous Markov processes : Lyapunov versus Poincaré. To appear in *J. Funct. Anal.*. Available on Mathematics ArXiv.math.PR/0703355, 2007.
- [4] F. Barthe, P. Cattiaux, and C. Roberto. Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry. *Rev. Mat. Iber.*, 22(3):993–1066, 2006.
- [5] F. Barthe, P. Cattiaux, and C. Roberto. Isoperimetry between exponential and Gaussian. *Electronic J. Prob.*, 12:1212–1237, 2007.
- [6] F. Barthe and A. V. Kolesnikov. Mass transport and variants of the logarithmic Sobolev inequality. Preprint. Available on Mathematics ArXiv.math.PR/0709.3890, 2007.
- [7] B. Bojarski. Remarks on Sobolev imbedding inequalities. In *Complex Analysis*, volume 1351 of *Lecture Notes in Math.*, pages 52–68. Springer, Berlin, 1987.
- [8] S. M. Buckley and P. Koskela. Sobolev-Poincaré implies John. *Math. Res. Lett.*, 2:881–901, 1995.
- [9] E. Carlen, S. Kusuoka, and D. Stroock. Upper bounds for symmetric Markov transition functions. *Ann. Inst. Henri Poincaré. Prob. Stat.*, 23:245–287, 1987.
- [10] P. Cattiaux. Hypercontractivity for perturbed diffusion semi-groups. *Ann. Fac. des Sc. de Toulouse*, 14(4):609–628, 2005.
- [11] P. Cattiaux and A. Guillin. Trends to equilibrium in total variation distance. To appear in *Ann. Institut H. Poincaré*. Available on Mathematics ArXiv.math.PR/0703451, 2007.
- [12] X. Chen and F. Y. Wang. Construction of larger Riemannian metrics with bounded sectional curvatures and applications. *Bull. London Math. Soc.* 40(2008), 659–663.
- [13] C. B. Croke. Some isoperimetric inequalities and eigenvalue estimates. *Ann. Sci. Éc. Norm. Super.*, 13:419–435, 1980.
- [14] E. B. Davies. *Heat kernels and spectral theory*. Cambridge University Press, 1989.
- [15] R. Douc, G. Fort, and A. Guillin. Subgeometric rates of convergence of  $f$ -ergodic strong Markov processes. Preprint. Available on Mathematics ArXiv.math.ST/0605791, 2006.
- [16] N. Down, S. P. Meyn, and R. L. Tweedie. Exponential and uniform ergodicity of Markov processes. *Ann. Prob.*, 23(4):1671–1691, 1995.
- [17] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. CRC Press, 1992.
- [18] G. Fort and G. O. Roberts. Subgeometric ergodicity of strong Markov processes. *Ann. Appl. Prob.*, 15(2):1565–1589, 2005.
- [19] I. Gentil, A. Guillin, and L. Miclo. Modified logarithmic Sobolev inequalities and transportation inequalities. *Probab. Theory Related Fields*, 133(3):409–436, 2005.
- [20] P. Hajlasz. Sobolev inequalities, truncation method, and John domains. *Report. Univ. Jyväskylä*, 83:109–126, 2001. also see the Web page of the author.
- [21] A. V. Kolesnikov. Modified log-Sobolev inequalities and isoperimetry. Preprint. Available on Mathematics ArXiv.math.PR/0608681, 2006.
- [22] S. Kusuoka and D. Stroock. Some boundedness properties of certain stationary diffusion semigroups. *J. Func. Anal.*, 60:243–264, 1985.
- [23] R. Latała and K. Oleszkiewicz. Between Sobolev and Poincaré. in geometric aspects of Functional Analysis. *Lect. Notes Math.*, 1745:147–168, 2000.
- [24] M. Röckner and F. Y. Wang. Weak Poincaré inequalities and  $L^2$ -convergence rates of Markov semigroups. *J. Funct. Anal.*, 185(2):564–603, 2001.
- [25] M. Röckner and F. Y. Wang. Supercontractivity and ultracontractivity for (non-symmetric) diffusion semi-groups on manifolds. *Forum Math.*, 15(6):893–921, 2003.
- [26] L. Saloff-Coste. *Aspects of Sobolev type inequalities*. Cambridge University Press, 2002.
- [27] E. Stein. *Singular integrals and differentiability properties of functions*. Princeton University Press, 1970.
- [28] N. Varopolous. Hardy Littlewood theory for semigroups. *J. Funct. Anal.* 63(2):240–260, 1985.

- [29] F. Y. Wang. Functional inequalities for empty essential spectrum. *J. Funct. Anal.*, 170(1):219–245, 2000.
- [30] F. Y. Wang. Sobolev type inequalities for general symmetric forms. *Proc. Amer. Math. Soc.*, 128(12):3675–3682, 2000.
- [31] F. Y. Wang. Logarithmic Sobolev inequalities: conditions and counterexamples. *J. Oper. Th.*, 46:183–197, 2001.
- [32] F. Y. Wang. *Functional inequalities, Markov processes and Spectral theory*. Science Press, Beijing, 2004.
- [33] F. Y. Wang. Functional inequalities on arbitrary Riemannian manifolds. *J. Math. Anal. Appl.*, 30:426–435, 2004.
- [34] L. Wu. Uniformly integrable operators and large deviations for Markov processes. *J. Funct. Anal.*, 172(2):301–376, 2000.

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