SINGULAR DIFFUSION PROCESSES AND APPLICATIONS.

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ADVERTISEMENT

These notes are very preliminary. In particular the Bibliography is rather incomplete. They have to be used with caution. Please do not make them circulate.

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INTRODUCTION

These lectures are devoted to some aspects of singular perturbations of diffusion processes. One of the main motivation is an attempt to understand Nelson's approach of the Schrödinger equation (what is now called stochastic mechanics). Another one is to link stochastic mechanics with statistical mechanics, following Föllmer's remark on an old Schrödinger question.
Though very popular in the 80 th's, the topic did not deserve many attention during ten years. One reason is that Nelson himself killed his child by introducing a physical contradiction with the model. In a private conversation, some times ago, P.A. Meyer told me that "if the model is not so interesting for quantum mechanics, it is certainly very interesting for diffusion theory." The spirit of these lectures will be the one indicated by Meyer: we shall not develop the physical counterpart of the mathematical contents.
However, it seems that the community of physicists has some new interests in the subject. Moreover, some of them are now saying that Nelson's (negative) argument is not a contradiction. In addition, a different but neighboring approach, developed by Zambrini and his coauthors, has led to new physycal predictions and interpretations (see Thieullen and Zambrini results on Noether theorem).
As H. Föllmer said recently: "the topic is less celebrated than statistical mechanics is, but it is still fascinating and rich of further developments."
We will try to convince the reader that Föllmer's statement is true.

Before to give the organization of these lectures, let me say that, if I will try to give an account of the theory, it will be definitely incomplete. Not only because I will forget many contributions (with my apologies to all contributors), particularly in the so much explored stationary case, but also because, except in very few cases, no complete proofs will be given. In general indeed, proofs are rather technical, sometimes difficult. I preferred to indicate, when it is possible, the route and the main difficulties.
The main tools we shall use are stochastic calculus and large deviations theory. Remarkable textbooks are available, like [44], [43] and [30]. However, as we shall see, the stochastic calculus approach is sometimes close to the border of our understanding, that is, is sometimes really intricate.

Organization of the lectures
Section 1. Schrödinger equation and a possible probabilistic counterpart.
Section 2. An overview of the stationary (reversible) case.
Section 3. Stochastic quantization via stochastic calculus.
Section 4. Time reversal and applications.
Section 5. Back to Schrödinger equation.
Section 6. The large deviations approach.
Section 7. Miscellaneous.

1. Schrödinger equation and a possible probabilistic counterpart.

To start with, consider the classical Schrödinger equation (without Planck's constant)

$$i\,\frac{\partial\psi}{\partial t} = -\frac{1}{2}\,\Delta\psi + V\psi = H\psi \quad ; \quad \psi(0,.) = \psi_0\,, \tag{1.1}$$

where V (the potential) is a time independent real valued function, belonging to the Kato class

$$\lim_{\alpha\downarrow 0}\ \sup_{x\in\mathbb{R}^d} \int_{|x-y|\leq\alpha} |V(y)|\,G(x,y)\,dy = 0\,,$$

G being the usual Green kernel on $\mathbb{R}^d$. Part of what follows can be extended to the Rellich class (see [11]).
H is a bounded below self adjoint operator on $\mathbb{L}^2(\mathbb{R}^d, dx)$ generating an unitary semi-group e^{-itH} (Stone's theorem), and the solution ψ (wave function) of (1.1) is given as $\psi(t,.) = e^{-itH}\,\psi_0$. We shall give regularity results on ψ taken from [11] and [12]. These results allow to make the calculations meaningful.

Lemma 1.2. *If $\psi_0 \in D(H^{\frac{1}{2}})$ (i.e. the domain of the standard Dirichlet form), then so does ψ_t (i.e. $\nabla\psi_t \in \mathbb{L}^2$), and one can find a jointly measurable version of ψ and $\nabla\psi$. Furthermore $t \mapsto \parallel \nabla\psi_t \parallel_2$ is continuous.*

Accordingly we may define

$$u = Re\,\frac{\nabla\psi}{\psi}\,{}_{\psi\neq 0}\,,\ v = Im\,\frac{\nabla\psi}{\psi}\,{}_{\psi\neq 0}\,,\ \rho = |\psi|^3. \tag{1.3}$$

If $\parallel \nabla\psi_0 \parallel_2 = 1$, then $\rho(t,.)$ is a probability density for all t, and for $0 \leq t \leq T$

$$\int_{\mathbb{R}^d} [|u(t,x)|^2 + |v(t,x)|^2]\,\rho(t,x)\,dx \leq \sup_{0\leq t\leq T}\ \parallel \nabla\psi_t \parallel_2^2 < +\infty\,, \tag{1.4}$$

thanks to 1.2.
The quantity in (1.4), or its time average over $[0,T]$ is called the energy, since it is the quadratic mean of velocities, u is the current velocity and v the osmotic velocity, and (1.4) is referred to as the FINITE ENERGY CONDITION.
The following is due to [12]

Lemma 1.5. *If $\psi_0 \in D(H^\alpha)$ for some $\alpha > 1 + \frac{d}{4}$, one can find a jointly continuous version of ψ such that $\frac{\partial\psi}{\partial t}$ exists and is also jointly continuous.*

With some extra work one can define good versions of the complex logarithm and write

$$\psi(t,x) = \exp(R(t,x) + iS(t,x))$$

in such a way that

$$u = \nabla R\,,\ v = \nabla S.$$

Using all the previous regularity results, it is not difficult to derive some evolution equations satisfied by the flow $t \mapsto \rho(t,.)$ of probability densities,

$$\begin{cases} i) \quad \frac{\partial\rho}{\partial t} = (\frac{1}{2}\,\Delta + \beta\nabla)^*\,\rho = \frac{1}{2}\,\Delta\rho - \nabla(\beta\,\rho)\,, \\ ii) - \frac{\partial\rho}{\partial t} = (\frac{1}{2}\,\Delta - \overline{\beta}\nabla)^*\,\rho = \frac{1}{2}\,\Delta\rho + \nabla(\overline{\beta}\,\rho)\,, \end{cases} \tag{1.6}$$

with $\beta = u + v$ and $\overline{\beta} = u - v$.

Of course (1.6) i) is nothing else but a Fokker-Planck equation, and ii) is a time reversed Fokker-Planck equation. The starting point of Nelson's theory is to associate to (1.6) a pair of processes in duality. Indeed suppose you can build a solution of the stochastic differential equation

$$dX_t = dw_t + \beta(t, X_t)\, dt \quad , \quad X_0 \overset{\text{def}}{=} \rho_0\, dx \,, \tag{1.7}$$

with w a Wiener process, then thanks to the finite energy condition, it can be shown that

$$\begin{cases} \beta(t, X_t) & = D_t X \overset{\text{def}}{=} \lim_{h \downarrow 0} \mathbb{E}\,[h^{-1}\,(X_{t+h} - X_t)/\mathscr{F}_t] \,, \\ \\ \overline{\beta}(t, X_t) & = \overline{D}_t X \overset{\text{def}}{=} \lim_{h \downarrow 0} \mathbb{E}\,[h^{-1}\,(X_t - X_{t-h})/\mathscr{B}_t] \,, \end{cases}$$

where $\mathscr{F}_t$ and $\mathscr{B}_t$ are respectively the forward and the backward filtrations of the process. D_t and $\overline{D}_t$ are the forward and the backward derivatives of the process.

Now if you define

$$D_t^2 = \frac{1}{2}\,(D_t\,\overline{D}_t + \overline{D}_t\,D_t) \,,$$

which is the stochastic acceleration, easy but formal manipulations yield the stochastic Newton equation

$$D_t^2\, X_t = -\nabla\, V(X_t)$$

i.e. a formal classical formulation, along the paths of the process, of the main equation of classical mechanics.

Nelson tried to push forward the analogy with classical mechanics, by introducing a least action principle associated to some Lagrangian, in order to define critical diffusions (see [55]).

As we said all this derivation is formal, and many attempts to rigorously justify Nelson'theory failed. However, if we assume that both β and $\overline{\beta}$ are smooth, one get a one to one correspondence between critical diffusions and wave functions.

Notice that we may expect that the time marginal laws of the process are $\rho_t\, dx$, thus the finite energy condition becomes

$$\mathbb{E}\,[|\beta(t, X_t)|^2 + |\overline{\beta}(t, X_t)|^2] < +\infty \,, \tag{1.8}$$

which was used to justify the existence of the forward and backward derivatives.

The first main problem concerns the existence of a solution to (1.7). This existence is often referred to as the STOCHASTIC QUANTIZATION problem. It was solved in the flat case (Brownian motion) we have just discussed, first by Carlen ([11]) in 1984, by using a semi-group perturbation approach. Meyer and Zheng ([51]) proposed another approach, but in the symmetric (stationary) case, and Carmona ([12]) proposed, one year later, an alternate and more probabilistic construction.

Extending stochastic quantization to more general operators (no more Laplace operator) is not only a mathematical challenge. It has a clear physical interest (bounded domains, infinite dimensional spaces, manifolds, string theory). Substantial progresses have been made in these directions by Zheng, Nagasawa, Mikami, Léonard and the author, in the non stationary case. The stationary case, in relationship with Dirichlet forms theory has been impressively explored by people like Albeverio, Ma, Röckner, Takeda, Song, Wu, Stannat, Eberle, Oshima, Fitzsimmons, Chen, Fradon and many others.

It is certainly useful to understand why this problem is really a difficult one.
Looking at (1.7), a probabilist will immediately write that the (weak) solution $\mathbb{Q}$ is given via a Girsanov transformation of drift i.e.

$$\frac{d\mathbb{Q}}{d\mathbb{P}_{\rho_0}}_{|\mathscr{F}_T} = \exp\big(\int_0^T \beta(t,w_t).dw_t - \frac{1}{2}\int_0^T |\beta(t,w_t)|^2\,dt\big) = G_T\,, \tag{1.9}$$

where $\mathbb{P}_{\rho_0}$ denotes Wiener measure with initial law $\rho(0,.)dx$, on the path space whose generic element is denoted by w.
This of course would be the case for a bounded β. But β is not a priori so regular. To see what happens, look at the stationary case

$$\frac{1}{2}\Delta\psi = V\psi \quad , \quad \beta = \frac{\nabla\psi}{\psi}\,{}_{\psi\neq 0}\,.$$

(or similarly for the ground state replacing V by the bottom of the spectrum)
Even if ψ is regular, problems will certainly occur on the nodal set $\psi = 0$. So, not only usual criteria (Novikov or Kazamaki) ensuring that G_T is a martingale cannot be checked, but G_T itself is not well defined.
One first has to define G in a correct manner. Following [44] we put

$$\begin{cases} G_T & = \exp\big(\int_0^T \beta(t,w_t).dw_t - \frac{1}{2}\int_0^T |\beta(t,w_t)|^2\,dt\big)\,, \\ & \text{if } T \in \cup_n[0,T_n]\,, \quad T_n = \inf\{t\geq 0,\ \int_0^t |\beta(s,w_s)|^2\,ds \geq n\}\,, \\ G_T & = \liminf G_{T_n}\,,\ \text{otherwise.} \end{cases} \tag{1.10}$$

One can then define $\mathbb{Q}$ as the associated Föllmer measure defined on the set of exploding trajectories i.e.

$$\mathbb{E}^{\mathbb{Q}}[F\,{}_{\tau<\xi}] = \mathbb{E}^{\mathbb{P}_{\rho_0}}[F\,G_\tau]$$

for any bounded stopping time τ, any $\mathscr{F}_\tau$ measurable F, where ξ is the explosion time.
The problem is then to know whether $\mathbb{Q}(\xi = +\infty) = 1$ or not, i.e. is the transformed process conservative.
Remember that in addition, we have to check that the time marginal laws

$$\mathbb{Q}_t = \mathbb{Q}\circ X_t^{-1}$$

are equal to $\rho(t,.)\,dx$, while we do not know a priori uniqueness for the Fokker-Planck equation (1.6) i).
The fact that the only "a posteriori" information we know, i.e. the finite energy condition

$$\sup_{0\leq t\leq T}\int |\beta(t,x)|^2\,\rho(t,x)\,dx < +\infty$$

is actually enough to show conservativeness, demonstrates that the diffusion process certainly has something to tell about Schrödinger equation.
The reader certainly noticed that we have slightly changed the finite energy condition, since

the latest one does not involve the backward drift. That duality is a priori "built in" was suggested by Föllmer ([35]) as a consequence of time reversal results. The first construction of Nelson's processes involving only the forward finite energy condition seems to be the one in [17], and a complete study of time reversal is done in [22].

But we are just at the beginning of the story. In 1932, Schrödinger asked the following question (free translation of the french version):

" Imagine you observe a system of free particles such that, at time 0 their distribution is rather uniform, and at time 1 their distribution is far to be uniform. What is the most probable way to explain this deviation."

We may formulate this question in modern mathematical words, as seen by Föllmer in [36]. To this end, pick an infinite sample of Brownian motions $(X_i)_{i\in\mathbb{N}}$ with initial law μ_0 (which is dx in Schrödinger statement but we prefer to work with probability measures). Observations of the system are given at time 0 and 1 by the empirical measures

$$L_n^0 = \frac{1}{n}\sum_{i=1}^n \delta_{X_i(0)} \quad , \quad L_n^1 = \frac{1}{n}\sum_{i=1}^n \delta_{X_i(1)}.$$

The question is:
what is (at least asymptotically) the conditional law of X_1 knowing that the pair (L_n^0, L_n^1) is close to a given pair (μ_0, μ_1) of probability measures, when μ_1 is not the law at time 1 of a Brownian motion with initial law μ_0 ?

This kind of formulation is well known in statistical mechanics, and the answer is given by the GIBBS CONDITIONING PRINCIPLE. To be precise, let us introduce some notation

Notation 1.11. $\mathbb{P}_{\mu_0}$ will denote the Wiener measure with initial law μ_0, and for $\alpha > 0$ we denote by $\mathbb{P}^\alpha_{n,\mu}$ the conditional law defined as

$$\mathbb{P}^\alpha_{n,\mu}(A) = \mathbb{P}^{\otimes n}(X_1 \in A \,/\, (L_n^0, L_n^1) \in B(\mu, \alpha))$$

where $\mu = (\mu_0, \mu_1)$ and $B(\mu, \alpha)$ denotes the open ball centered at μ with radius α for a metric compatible with weak convergence of probability measures (for example Lévy metric or Kantorovitch one).

Then, if

$$H(\mathbb{Q}, \mathbb{P})$$

denotes the relative entropy (or Kullback information, or I-divergence) of $\mathbb{Q}$ with respect to $\mathbb{P}$, the following is known (see [30])

Theorem 1.12 (**Gibbs conditioning principle.**). *If*

$$A^\alpha_\mu = \{\, \mathbb{Q}\,,\ s.t.\ H(\mathbb{Q}, \mathbb{P}) < +\infty \ and\ (\mathbb{Q}_0, \mathbb{Q}_1) \in B(\mu, \alpha) \,\}$$

is not empty, then $\mathbb{P}^\alpha_{n,\mu}$ converges in variation distance, when n goes to $+\infty$, to the probability measure $\mathbb{Q}^\alpha_\mu$ satisfying

$$\mathbb{Q}^\alpha_\mu = \arg\inf \{\, H(\mathbb{Q}, \mathbb{P})\,,\ \mathbb{Q} \in A^\alpha_\mu \,\}.$$

In other words, $\mathbb{Q}_\mu^\alpha$ minimizes relative entropy among all probability measures on the path space with marginal laws at times 0 and 1 α close to (μ_0, μ_1).

Since we want to build some $\mathbb{Q}$ with marginals (μ_0, μ_1), one can then look at the behaviour of the $\mathbb{Q}_\mu^\alpha$'s when α goes to 0, or try to improve this theorem by taking $\alpha(n)$ going to 0 as n goes to $+\infty$. This is not so easy and we shall come back later to this point.
Nevertheless, answer to Schrödinger question is given by this principle.
Actually more can be shown, i.e. for all n

$$H(\mathbb{P}_{n,\mu}^\alpha, \mathbb{Q}_\mu^\alpha) \leq -\frac{1}{n} \log \mathbb{P}^{\otimes n}((L_n^0, L_n^1) \in B(\mu, \alpha)) - H(\mathbb{Q}_\mu^\alpha, \mathbb{P}). \tag{1.13}$$

Theorem 1.12 follows from (1.13) by applying Sanov large deviations theorem, and the well known Pinsker inequality

$$\text{variation distance } (\mathbb{Q}, \mathbb{P}) \leq (2\, H(\mathbb{Q}, \mathbb{P}))^{\frac{1}{2}}.$$

Inequality (1.13) is due to Csiszar [28].

If we replace the observations (L_n^0, L_n^1) by the full empirical process

$$t \mapsto L_n^t = \frac{1}{n} \sum_{i=1}^n \delta_{X_i(t)}\, , \; t \in [0, T]\, , \tag{1.14}$$

one can formulate a similar result just replacing

i) (μ_0, μ_1) by a flow $t \mapsto \mu_t$,
ii) the ball $B(\mu, \alpha)$ by an open blowup of this flow in $\mathscr{C}^0([0, T], \mathbb{M}_1(\mathbb{R}^d))$,
iii) A_μ^α by a similar $A_\mu^{\alpha, T}$.

To understand the relationship between this result, and the stochastic quantization problem we have discussed earlier, remember the definition of relative entropy.

Definition 1.15. *If $\mathbb{Q}$ and $\mathbb{P}$ are probability measures on a metric space Γ,*

$$H(\mathbb{Q}, \mathbb{P}) \stackrel{\text{def}}{=} \sup_{F \in \mathscr{B}_b(\Gamma, \mathbb{R})} \Big(\int F\, d\mathbb{Q} - \log \int e^F\, d\mathbb{P} \Big)\, ,$$

where $\mathscr{B}_b$ denotes the set of bounded measurable functions (when Γ is Polish one can take $\mathscr{C}_b$), and the following holds

$$H(\mathbb{Q}, \mathbb{P}) = \int \frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\, d\mathbb{P}$$

if $\mathbb{Q} \ll \mathbb{P}$, and $+\infty$ otherwise.

If $\mathbb{Q} \ll \mathbb{P}_{\mu_0}$ and $\mathbb{Q}_0 = \mu_0$, it is known (see [44]) that there exists an adapted process β_t such that

$$\begin{cases} \frac{d\mathbb{Q}}{d\mathbb{P}_{\mu_0}}_{|\mathscr{F}_T} = \exp\left(\int_0^T \beta_t . dw_t - \frac{1}{2} \int_0^T |\beta_t|^2\, dt \right), \\ \qquad \text{if } T \in \cup_n [0, T_n]\, , \quad T_n = \inf\{t \geq 0,\, \int_0^t |\beta_s|^2\, ds \geq n\}\, , \\ \frac{d\mathbb{Q}}{d\mathbb{P}_{\mu_0}}_{|\mathscr{F}_T} = \liminf \frac{d\mathbb{Q}}{d\mathbb{P}_{\mu_0}}_{|\mathscr{F}_{T_n}}\, , \text{ otherwise.} \end{cases} \tag{1.16}$$

An easy calculation yields

$$\text{(1.17)} \qquad \text{if } \mathbb{Q}_0 = \mu_0 \text{ and } \mathbb{Q} \ll \mathbb{P}_{\mu_0} \text{ then } H(\mathbb{Q}, \mathbb{P}_{\mu_0}) = \frac{1}{2}\,\mathbb{E}^{\mathbb{Q}}\,[\int_0^T |\beta_t|^2\,dt\,],$$

and we recognize in (1.17) the energy condition, i.e.

if $\mu_t = \rho_t\,dx$, where ρ_t is the amplitude of the wave function, the stochastic quantization problem is equivalent to the non emptiness of $A_\mu^{0,T}$ (with $\alpha = 0$).

As a conclusion, we see that Nelson's formalism has a statistical interpretation in which finite energy is transformed into finite entropy. Introducing an entropy in quantum mechanics will certainly appear to most physicists as an heretic point of view. Nevertheless the above relationship cannot be a hazard.
We shall now focus on mathematical objects we have introduced.

Remark 1.18. If we deal with A_μ^α instead of $A_\mu^{\alpha,T}$ the stochastic quantization problem is much more simple. Indeed

$$\mathbb{Q}_\mu = \int \mathbb{P}_x^y\,\mu^*(dx, dy)$$

where $\mathbb{P}_x^y$ is the law of the Brownian bridge between x and y, and

$$\mu^* = \arg\inf\{\,H(\nu, p) \text{ where } p = \mathbb{P}_{\mu_0} \circ (X(0), X(1))^{-1},\, (\nu_0, \nu_1) = (\mu_0, \mu_1)\,\}.$$

The problem thus reduces to a minimization problem in finite dimension which was first tackled by Beurling (see [16] for a bibliography). It turns out that the minimizing μ^* has a splitting property which allows to make the link with markov reciprocal processes introduced by Jamison.
$\mathbb{Q}_\mu$ is often called a Schrödinger bridge and is the relevant process in the euclidean version of stochastic mechanics. For more information on this approach see [36] and [26].

2. **An overview of the stationary (reversible) case.**

As we already said, the stationary case is particular, not only because time dependence is the origin of annoying problems, but because of its relationship with DIRICHLET FORMS theory (see the textbooks [40] and [49]).

In this section we shall give the flavor of stochastic quantization in the flat reversible case, i.e. Meyer-Zheng result. Some references to extensions will be given at the end of this section, as well as references to related and specific problems in this context.

We are given

$$\text{(2.1)} \qquad \rho = \psi^2 \text{ , a density of probability, and } \beta = \frac{\nabla\psi}{\psi}\,{}_{\psi\neq 0} = \frac{1}{2}\,\nabla\log\rho\,{}_{\rho\neq 0}.$$

We may assume that $\psi \geq 0$. The finite energy condition is then

$$\int |\beta|^2\,\rho\,dx = \int |\nabla\psi|^2\,dx \,< +\infty$$

i.e.

$$\psi \in H^1(\mathbb{R}^d)$$

the usual Sobolev space. For f and g belonging to $\mathscr{C}_0^\infty$ one thus has

$$\int (\frac{1}{2}\,\Delta + \beta\nabla)\, f\, g\, \rho\, dx = -\frac{1}{2} \int \nabla f \,.\, \nabla g\, \rho\, dx \tag{2.2}$$

just using integration by parts. In particular ρ satisfies the weak stationary Fokker-Planck equation

$$(\frac{1}{2}\,\Delta + \beta\nabla)^*\, \rho = 0\,.$$

Due to (2.2) we are led to study the Dirichlet form

$$\mathscr{E}\,(f,g) = \int \nabla f \,.\, \nabla g\, \rho\, dx \tag{2.3}$$

with domain $H_0^1(\rho)$ which is the completion of $\mathscr{C}_0^\infty$ equipped with the semi norm $(\mathscr{E}_1(f))^{\frac{1}{2}}$ where $\mathscr{E}_1(f) = \parallel f \parallel^2_{\mathbb{L}^2(\rho)} + \mathscr{E}\,(f,f)$.

The form is easily seen to be local and regular. Hence Fukushima's theory is associating to it a, possibly non conservative, diffusion process. In order to obtain a more precise construction, including conservativeness, one has to work harder. The construction breaks into five steps.

Step 1. Truncation. One chooses a quasi continuous version of ψ (i.e. ψ can be chosen at each point except some polar set for Brownian motion). Next one considers the truncated

$$\psi_n = (\psi \vee \frac{1}{n}) \wedge n$$

which satisfies (in $\mathbb{L}^2$)

$$\nabla\psi_n = \nabla\psi\;{}_{1/n\leq\psi\leq n} \text{ and } \nabla \log \psi_n = \frac{\nabla\psi}{\psi}\;{}_{1/n\leq\psi\leq n} = \beta_n$$

thanks to the chain rule. Notice that, if $\rho_n = \psi_n^2$ is no more a probability measure, it still satisfies a weak Fokker-Planck equation with β_n in place of β.

Step 2. Regularity. We denote by X_t the generic element of the path space (continuous functions) and by $\mathbb{P}$ the Wiener measure with initial measure dx. Since $\log\,\psi_n \in H^1$, one can use FUKUSHIMA-ITO decomposition

$$\log\,\psi_n(X_t) - \log\,\psi_n(X_0) = M_t^n + A_t^n\,,\, \forall t\,, \mathbb{P}\, a.s. \tag{2.4}$$

where M^n is a $\mathbb{P}$ martingale with brackets

$$< M^n >_t = \int_0^t |\beta_n|^2(X_s)\, ds$$

and A^n is of zero energy. This last term is difficult to control but, using reversibility of $\mathbb{P}$, one can write the time reversed Fukushima's decomposition of $\log \psi_n$. Taking the average of both formulas, one obtains the LYONS-ZHENG decomposition (see [48])

$$\log\,\psi_n(X_t) - \log\,\psi_n(X_0) = \frac{1}{2}\, M_t^n + \frac{1}{2}\,(M^n_{T-t}(R_T) - M^n_T(R_T))\,,\, \forall t \in [0,T]\,, \mathbb{P}\, a.s. \tag{2.5}$$

where R_T denotes the time reversal $X_. \mapsto X_{T-.}$ on the path space. The key point is that Lyons-Zheng decomposition only involves the martingale terms, that is the annoying zero energy term disappeared.

Step 2. Drift transformation. We define

$$\mathbb{Q}_n = \left(\exp\left(M_T^n - \frac{1}{2} < M^n >_T\right)\right) \mathbb{P}_{\rho_n\, dx} \,.$$

Standard Girsanov theory tells us that $\mathbb{Q}_n$ is conservative. In addition, $\mathbb{Q}_n$ is symmetric (remember that ρ_n satisfies the Fokker-Planck equation), and defining

$$N_t^n = M_t^n - < M^n >_t$$

one checks that N^n is a $\mathbb{Q}_n$ martingale with the same brackets as M^n. Finally, Lyons-Zheng decomposition is available $\mathbb{Q}_n$ a.s., just replacing M by N.
In particular the finite energy condition implies

$$\sup_n \mathbb{E}^{\mathbb{Q}_n}\left[< N^n >_T\right] \leq T \parallel \psi \parallel_{H^1}^2 < +\infty \,. \tag{2.6}$$

Now introduce the sequence of stopping times

$$\tau_n = \inf\{t > 0\,,\, \psi(X_t) \notin [1/n\,,\, n]\,\} \text{ and } \tau = \sup_n \tau_n \,.$$

We also define

$$M_t = M_t^n \text{ and } < M >_t = < M^n >_t \text{ if } t \leq \tau_n$$

and

$$\mathbb{Q} = \left(\exp\left(M_T - \frac{1}{2} < M >_T\right)\right)_{t<\tau} \mathbb{P}_{\rho\, dx} \,. \tag{2.7}$$

It is not difficult to see that

$$\text{on } t < \tau_n \quad , \quad \mathbb{Q} \text{ and } \mathbb{Q}_n \text{ coincide.}$$

In particular, if ξ denotes the explosion time

$$\mathbb{Q}\,(\tau < \xi) = \lim_n \mathbb{Q}\,(\tau_n < \xi) = \lim_n \mathbb{Q}_n\,(\tau_n < \xi) = 1$$

since $\mathbb{Q}_n$ is conservative.
This means that τ (which is the hitting time of the nodal set) is less than the explosion time. Hence in order to show conservativeness, it is enough to show that $\mathbb{Q}\,(\tau < T) = 0$.

This approach, showing that the nodal set is never attained, is specific to the stationary case. It is closely related to the choice of $\mathbb{Q}$ we have made. Indeed note that, contrary to section 1, we introduced a cut-off $_{t<\tau}$ in the Girsanov density.

Step 4. Nelson estimate. We shall now derive an estimate obtained by Nelson for smooth diffusions. Let us calculate

$$\mathbb{Q}_n\left(\sup_{0\leq t\leq T}\left(\frac{\psi_n(X_t)}{\psi_n(X_0)} \vee \frac{\psi_n(X_0)}{\psi_n(X_t)}\right) \geq e^a\right) = \mathbb{Q}_n\left(\sup_{0\leq t\leq T} |\log\, \psi_n(X_t) - \log\, \psi_n(X_0)| \geq a\right).$$

Thanks to Lyons-Zheng decomposition and easy manipulations, this quantity is less than

$$\mathbb{Q}_n\left(\sup_{0\leq t\leq T} |N_t^n| \geq a/2\right) + \mathbb{Q}_n\left(\sup_{0\leq t\leq T} |N_t^n(R_T)| \geq a/2\right).$$

Now using Doob's inequality and (2.6) we obtain that for all n

$$\mathbb{Q}_n \left(\sup_{0\leq t\leq T} \left(\frac{\psi_n(X_t)}{\psi_n(X_0)} \vee \frac{\psi_n(X_0)}{\psi_n(X_t)}\right) \geq e^a \right) \leq \frac{8}{a^2}\, T \, \parallel \psi \parallel_{H^1}^2 \ . \tag{2.8}$$

Step 5. Conclusion.

$$\mathbb{Q}\,(1/k \leq \psi(X_0) \leq k \, , \, \tau_n < T) = \mathbb{Q}\,(1/k \leq \psi(X_0) \leq k \, , \, \tau_n < T)$$

$$\leq \mathbb{Q}_n \left(\sup_{0\leq t\leq T} \left(\frac{\psi_n(X_t)}{\psi_n(X_0)} \vee \frac{\psi_n(X_0)}{\psi_n(X_t)}\right) \geq \frac{n}{k} \right)$$

$$\leq 8\,(\log(n/k))^{-2}\, T \, \parallel \psi \parallel_{H^1}^2 \ .$$

Hence, letting first n, then k go to $+\infty$ we finally get

$$\mathbb{Q}\,(\tau < T) = 0\,.$$

The proof is finished. Let us summarize the result we just proved

Theorem 2.9. *If $\psi \in H^1$, the measure $\mathbb{Q}$ defined in (2.7) is a probability measure (is conservative). Furthermore, this measure is reversible and solves the martingale problem $\mathscr{M}(\frac{1}{2}\,\Delta + \frac{\nabla\psi}{\psi}\,\nabla \, , \, \mathscr{C}_0^\infty \, , \, \rho\, dx)$.*

Remark 2.10. The above proof, in a slightly different setting, was first given by Meyer and Zheng in [51]. The earlier [3] also deals with a similar problem, but with more regularity. Connection with Donsker-Varadhan occupation measure (i.e. another large deviations point of view) was done by Fukushima and Takeda [41], and the above form of the construction is due to Takeda [67] (also see section 6.3 in [40]) in a more general context. In finite dimension, one may replace the Laplace operator by a general second order symmetric operator, with smooth enough coefficients (in order to perform integration by parts) and uniformly elliptic.

This approach has been successfully extended and completed: see e.g. [4], [6], [7], [33], [34] for general finite or infinite dimensional Dirichlet forms.

Remark 2.11. Once existence is shown, the next natural question is the one of uniqueness. There are many notions of uniqueness in this case. We refer to Wu ([70], [71]) for a discussion on these various notions, and to Eberle [32] for an up to date situation.

One notion is connected with the uniqueness of the solution of the associated martingale problem. It is studied e.g. in [5]. Another one is the so called Markov uniqueness, i.e. the uniqueness of a Markovian extension of the Dirichlet form. Relevant references are [61], [68], [8], [62], [64], [65], [38] and [14]. The latest is the only one which proposed a purely analytical proof.

Related topics like the study of invariant measures, are studied in [9], [14], [2] or [10] (and the bibliography therein).

Extensions to time-dependent Dirichlet forms of Oshima and generalized Dirichlet forms are given in [66] and [50]. The case of bounded domains (with the help of Dirichlet forms theory) was studied by Chen ([23], [24], [25]) and by Fradon [39]. This latest case uses deep analysis on the regularity of the boundary (Cacciopoli sets).

As the reader saw, some amount of results have been obtained in the symmetric case. Much less was done in the non stationary case we shall now study.

3. **Stochastic quantization via stochastic calculus.**

In this section we shall study the construction of a singular diffusion process, via a Girsanov's like drift transformation as in (1.10), but in the non flat case. To begin with, we have to introduce some notations and definitions.

Let a be a measurable flow of non negative symmetric matrices, b and β be measurable flows of vector fields. We define:

$$L(t,x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t,x)\ \partial_i\partial_j \ + \sum_i \ b_i(t,x)\ \partial_i, \tag{3.1}$$

and

$$A(t,x) = L(t,x) + a(t,x)\beta(t,x).\nabla \tag{3.2}$$

where . denotes scalar product and ∇ is the space gradient;

$$\sigma(t,x) \text{ a measurable non negative square root of } a(t,x). \tag{3.3}$$

All functions are defined on the whole space $\mathbb{R}\times\mathbb{R}^d$, or possibly on the d-dimensional torus $\mathbb{R}\times\mathbb{T}^d$ if they are space-periodic.

We shall look at A as a perturbation of L, and so build a diffusion process associated to A as a transformation of the one associated to L by some Girsanov's like multiplicative functional. Here, the expression "diffusion process" is understood in a non rigid way which will be explained in the statement of the results. Actually, we ask for more. We want to impose the law of all time marginals of the process. This of course implies that this flow satisfies some Fokker-Planck equation; more precisely:

Definition 3.4. *Let $\nu \stackrel{\text{def}}{=} (\nu_s)_{s\in[0,T]}$, be a flow of Probability measures on $\mathbb{R}^d$, and Λ be a set of Borel functions defined on $\mathbb{R}\times\mathbb{R}^d$. We shall say that ν satisfies the Λ-weak forward equation on $[0,T]$ if, for every $f\in\Lambda$:*

i) $(\frac{\partial}{\partial t}+A)f$ *is defined and belongs to* $\mathbb{L}^1([0,T]\times\mathbb{R}^d, ds\,d\nu_s(x))$*;*

ii) $\forall\, 0\le u\le t\le T$,

$$\int f(t,x)\nu_t(dx) - \int f(u,x)\nu_u(dx) = \int_u^t\int(\tfrac{\partial}{\partial s}+A)f(s,x)\,ds\,d\nu_s(x).$$

In general, Λ will be a nice set, and $\mathscr{C}_0^\infty([0,T]\times\mathbb{R}^d)\subset\Lambda$.

Let us say now what we call a diffusion process.

Definition 3.5. *Let $\mathbb{Q}$ be a Probability measure on $\Omega=\mathscr{C}^0([0,T],\mathbb{R}^d)$. We say that $\mathbb{Q}$ is an A-diffusion with initial measure ν_0 if:*

i) $\mathbb{Q}_0=\nu_0$;

ii) $\forall\, f\in\mathscr{C}_0^\infty(\mathbb{R}\times\mathbb{R}^d)$

$$f(t,X_t)-f(0,X_0)-\textstyle\int_0^t(\frac{\partial}{\partial s}+A)f(s,X_s)ds$$

is a $\mathbb{Q}$-local continuous martingale up to time T, with brackets given by

$$\int_0^t(\nabla f.a\nabla f)(s,X_s)ds\,.$$

Here, $t\mapsto X_t$ is the canonical process on Ω equipped with the natural right continuous and complete filtration.

The statement ii) is equivalent to a similar one replacing the full $\mathscr{C}_0^\infty$ by the coordinate functions of the process, in particular, we are in the situation of Chapter 12 of [44].
In the flat case $a=Id\,,\,b=0$, we are in the situation of section 1, up to the change of notation $\nu_t=\rho_t\,dx$.

Definition 3.6. *We shall say that $\mathbb{Q}$ solves the stochastic quantization problem, if $\mathbb{Q}$ is an A-diffusion such that $\mathbb{Q}_t=\nu_t$ for all $t\in[0,T]$.*

In the rest of this section, we assume the following:

(3.7) There exists a strong Markov family $(\mathbb{P}_{u,x};(u,x)\in\mathbb{R}\times\mathbb{R}^d)$, such that:

i) $\mathbb{P}_{u,x}(u_0=u,X_0=x)=1$,

ii) $u_t=u+t\quad\mathbb{P}_{u,x}$ a. s.,

iii) $\mathbb{P}_{u,x}$ is an extremal $(\frac{\partial}{\partial u}+L)$-diffusion with initial measure $\delta_{u,x}$.

Here, the path space is $\mathscr{C}_0^\infty([0,T],\mathbb{R}\times\mathbb{R}^d)$, and extremal means that $\mathbb{P}_{u,x}$ is an extremal solution of the martingale problem (3.5)ii), replacing A by L.

We emphasize that (3.7) is concerned with the (now homogeneous) time-space process. Actually, (3.5) should be written in this time-space context, replacing ν_0 by $\delta_0\otimes\nu_0$. We also define

$$\mathbb{P}_{\nu_0}=\int\mathbb{P}_{u,x}\;\delta_0^{(u)}\otimes\nu_0(dx).$$

Now define

$$M_s=X_s-X_0-\int_0^s b(u_v,X_v)dv$$

and as in (1.10) introduce

$$\text{(3.8)}\qquad\begin{cases} G_T &=\exp\big(\int_0^T\beta(t,X_t).dM_t-\frac{1}{2}\int_0^T|(\sigma\,\beta)(t,X_t)|^2\,dt\big)\,,\\[2ex] &\text{if } T\in\cup_n[0,T_n]\,,\quad T_n=\inf\{t\geq0,\ \int_0^t|(\sigma\,\beta)(s,X_s)|^2\,ds\geq n\}\,,\\[2ex] G_T &=\liminf G_{T_n}\,,\ \text{otherwise.}\end{cases}$$

Define the drift transformed

$$\mathbb{Q} = G_T \, \mathbb{P}_{\nu_0} \, . \tag{3.9}$$

As in section 1, we will also denote by $\mathbb{Q}$ the Föllmer measure associated with G_T on the space of explosive trajectories and we shall impose the (forward) FINITE ENERGY CONDITION

$$\int_0^T \int |\sigma \, \beta|^2(s,x) \, ds \, \nu_s(dx) < +\infty, \, . \tag{3.10}$$

In this case, we shall show that $\mathbb{Q}$ solves the stochastic quantization problem, under mild conditions on a and b.

Since we now know how to do in the symmetric case, we are tempted to adapt the approach of section 2. But we immediately have to face various difficulties :

Step.1. First we do no more assume that ν is given by a flow of probability densities. But even if so, i.e. assuming $\nu_t = \rho_t \, dx$, the cut-off of Step.1 in section 2 is badly behaved. Indeed, because of the time derivative, the cut-off does not satisfy anymore a Fokker-Planck equation. Furthermore, the relationship between β and log ρ is no more clear at all.

Step.2. Ito formula is no more available for log ρ unless we assume strong regularity assumptions. Recent extensions of Ito formula to $\mathscr{C}^1$ functions, due to Föllmer, Protter and Shiryaev or Russo and Vallois, are mainly available in one dimension, and $\mathscr{C}^1$ is too strong for our purpose.
Hence one can try to mollify ρ, by taking convolution with some smooth kernel. But here again, except in the flat case when Δ and convolution are commuting, the Fokker-Planck equation is lost.
In addition, Lyons-Zheng decomposition strongly used the reversibility of the underlying process $\mathbb{P}$, and it is hard to find here an analogue.

Step.3. Our definition of G_T is slightly changed, when comparing with the stationary case. Indeed the cut-off by the indicator ($T < \tau$ in (2.7)) disappeared, and stopping times are not the same as in section 2.

Hence, the strategy of section 2, is no more appropriate. However, in the flat case, and assuming some Hölder regularity on ρ, Zheng [72] succeeded in proving that $\mathbb{Q}$ solves the stochastic quantization problem. Also see the works by Nagasawa and his coauthors with similar regularity assumptions.

In the next subsection, we shall give the flavor of the strategy of proof we proposed in [17]. In the following one, we shall state precise results we then obtained.

3.1. **The stochastic strategy.**

Step.1. Of course we will have to use some approximations. To this end we consider a sequence β_k of measurable and bounded functions with compact support, that converges to β in $\mathbb{L}^2(\nu) \overset{\text{def}}{=} \mathbb{L}^2(d\nu_t \, dt)$, and consider the associated $\mathbb{Q}^k_{s,x}$ defined as in (3.9) just replacing β by β_k and $\mathbb{P}_{\nu_0}$ by $\mathbb{P}_{s,x}$ in (3.8). Of course the Q^k's are conservative, thanks e.g. to Novikov

criterion. We simply write $\mathbb{Q}^k$ for the one with initial measure ν_0.
The aim is to prove that for any good enough f ,

$$\mathbb{E}^{\mathbb{Q}^k}[f(X_t)] \leq \int f(x)\, d\nu_t + C \parallel \sigma(\beta - \beta_k) \parallel_{\mathbb{L}^2(\nu)} , \tag{3.11}$$

where C is independent of k.
To this end, introduces

$$f_k(s,x) = \mathbb{E}^{\mathbb{Q}^k_{s,x}}[f(X_{t-s})]$$

which satisfies, if f is good enough, the "heat" equation

$$(\frac{\partial}{\partial t} + L) f_k + (a\beta_k.\nabla f_k) = 0 \ , \text{ on } [0,t].$$

Applying the weak forward equation for the flow ν_t, we get

$$\int f_k(t,x)\, d\nu_t - \int f_k(0,x)\, d\nu_0 = \int_0^t \int [a(\beta - \beta_k).\nabla f_k](s,x)\, d\nu_s\, ds\, .$$

But $f_k(t,x) = f(x)$ and $\int f_k(0,x)\, d\nu_0 = \mathbb{E}^{\mathbb{Q}^k}[f(X_t)]$. Therefore we get

$$\mathbb{E}^{\mathbb{Q}^k}[f(X_t)] \leq \int f(x)\, d\nu_t + \parallel \sigma\nabla f_k \parallel_{\mathbb{L}^2(\nu)} \parallel \sigma(\beta - \beta_k) \parallel_{\mathbb{L}^2(\nu)} \ . \tag{3.12}$$

In order to obtain (3.11), it is thus enough to get some uniform bound for

$$\parallel \sigma\nabla f_k \parallel_{\mathbb{L}^2(\nu)} \ .$$

This bound is obtained by applying the weak forward equation to f_k^2, provided f is bounded. This step is an adaptation of Mikami's ideas in [52].

Step.2. We shall show that

$$\mathbb{E}^{\mathbb{Q}}[f(X_t)\,_{t<\tau}] \leq \int f(x)\, d\nu_t\, , \tag{3.13}$$

where $\tau = \sup_n T_n$. For nonnegative f this is obtained by using Fatou's lemma and taking the $\liminf$ in

$$\mathbb{E}^{\mathbb{Q}^k}[f(X_t)\,_{t<T_n}]$$

first in k, then in n.
Next (3.13) extends to any nonnegative measurable f. Hence we may apply it with

$$f = |\sigma\,\beta|^2(t,.)$$

for each t, and taking the average in t, one obtains

$$\mathbb{E}^{\mathbb{Q}}[\int_0^T |\sigma\,\beta|^2(t,X_t)\,_{t<\tau}\, dt] \leq \int\int |\sigma\,\beta|^2(t,x)\, d\nu_t\, dt < +\infty \tag{3.14}$$

thanks to the finite energy condition.
Now remember the calculation we have done in (1.17). Thanks to our assumptions on $\mathbb{P}$, a similar calculation can be done here. Thus one half the energy of the drift is equal to relative entropy. In particular thanks to (1.15) and (3.14), the variables $G_{T\wedge T_n}$ are uniformly bounded in the Orlicz space $x\log x$, hence uniformly integrable, and

$$\mathbb{E}^{\mathbb{P}_{\nu_0}}[G_{T\wedge T_n}] = 1\, ,$$

for all n thanks to Novikov criterion. It follows that

$$\mathbb{E}^{\mathbb{P}_{\nu_0}}\left[G_{T\wedge\tau}\right] = 1$$

thanks to the continuity of G. It remains to show that

$$\mathbb{Q}(\tau < T) = 0\,.$$

To this end remark that

$$\mathbb{Q}(\tau < T) = \mathbb{Q}(\tau < T < \xi) = \mathbb{E}^{\mathbb{P}_{\nu_0}}\left[G_{T\wedge\tau}\ {}_{\tau<T}\right], \tag{3.15}$$

and the result will be shown if

$$G_{T\wedge\tau} = 0 \text{ on } \{\tau < T\} \quad , \mathbb{P}_{\nu_0} \text{ a.s.} \tag{3.16}$$

(3.16) is satisfied provided no SUDDEN DEATH occurs, i.e. if

$$\mathbb{P}_{\nu_0}(\tau = T_n \text{ for some } n) = 0\,.$$

In general, no sudden death follows from the estimates we have obtained (see the correction in [17]).
Hence $\mathbb{Q}$ is a probability measure such that $\mathbb{Q}(\tau \leq t) = 0$ for all t, and an A-diffusion.
The strategy in step.2. is partly inspired by [72].

Step.3. It remains to show that $\mathbb{Q}_t = \nu_t$ for all t. But recall (3.13) and $\mathbb{Q}(\tau \leq t) = 0$. It follows that $\mathbb{Q}_t \leq \nu_t$ hence equal since they are both probability measures.
Hence $\mathbb{Q}$ solves the stochastic quantization problem.

3.2. **Some existence results.**

The previous strategy can be used in several contexts. Some technical points have to be checked, and this implies to make some hypotheses on a and b. Here are some of these results.

Theorem 3.17 (see[17], Theorems 4.28 and 4.42)**.** *Assume that σ and b are locally Hölder continuous. Let ν be a solution of the $\mathscr{C}_0^\infty$-weak forward equation (3.4) such that the finite energy condition (3.10) holds. Assume furthermore that*
either
i) σ and b are $\mathscr{C}^{1,2,\alpha}$, for some $\alpha > 0$;
or
ii) a is uniformly elliptic.
Then, the measure $\mathbb{Q}$ defined by (3.9) solves the stochastic quantization problem. In addition $\mathbb{Q}$ is markovian and

$$H(\mathbb{Q}, \mathbb{P}_{\nu_0}) = \frac{1}{2}\int_0^T \int (\beta.a\beta)(s,x)\, ds\, \nu_s(dx) < +\infty.$$

Theorem 3.17 deals with a general flow of marginals. In [17] Theorem 4.48, another type of result is obtained with weaker assumptions on σ and b, but stronger on ν. Notice that, in case i) no ellipticity is required, contrary to the approach using time dependent Dirichlet forms.

In [60], Quastel and Varadhan studied the stochastic quantization problem, for divergence form operators on the torus, under weaker assumptions on a and b. Their result is slightly improved in [22]. Here is the result

Theorem 3.18. *Assume the state space is the torus $\mathbb{T}^d$ and that $L = \frac{1}{2}\nabla .a\nabla$ for some a such that $\sigma \in H^1(dt\otimes dx)$ on $[0,T]\times\mathbb{T}^d$. Let $\nu = \rho\, dx$ be a solution of the $\mathscr{C}^\infty$-weak forward equation (3.4) such that the finite energy condition (3.10) holds. Assume furthermore that ρ is bounded and satisfies $\sigma\nabla\rho \in \mathbb{L}^2(dt\otimes dx)$. Then there exists a L-diffusion $\mathbb{P}$ with initial law $\rho_0\, dx$, and $\mathbb{Q}$ defined by (3.9) solves the stochastic quantization problem.*

The proof uses integration by parts and mollifiers as in the symmetric case. One difficulty is that (3.7) is no more satisfied. That is why we first have to build $\mathbb{P}$.

3.3. About uniqueness.

We shall now discuss uniqueness. A nice consequence of the Markovian framework (3.7) is the following uniqueness result:

Theorem 3.19. *In Theorems 3.17, if we assume that $\mathbb{P}_{u,x}$ is the unique solution to the martingale problem $\mathscr{M}(\frac{\partial}{\partial t}+L,\mathscr{C}^{1,2},\delta_{u,x})$ for every (u,x), then $\mathbb{Q}$ is the unique A-diffusion such that*

$$\mathbb{Q}[\int_0^T(\beta.a\beta)(s,X_s)ds<+\infty]=1.$$

In particular, $\mathbb{Q}$ is the unique A-diffusion such that $\mathbb{Q}_t=\nu_t$ for every t.

The proof is based on [44] chapter 13. Similar statements are contained in [70]. As an immediate consequence we get :

Corollary 3.20. *Under the hypotheses of Theorem 3.17, $\mathbb{Q}$ is an extremal solution to $\mathscr{M}(\frac{\partial}{\partial t}+A,\mathscr{C}^{1,2},\nu_0)$.*

A similar statement can be shown under the hypotheses of Theorem 3.18.

An easy consequence is uniqueness for the weak forward equation. Indeed it immediately follows

Theorem 3.21. *Assume that the hypotheses of Theorem 3.17 are fulfilled. Then:*

i) *ν is the unique solution of the weak forward equation such that*

$$\int_0^T\int\beta.a\beta\,(s,x)\,ds\,\nu_s(dx)<+\infty,\ \text{starting from } \nu_0;$$

ii) *if $\nu_0^{'}\ll\nu_0$, then there exists a solution of the weak forward equation starting from $\nu_0^{'}$;*

iii) *if $\frac{d\nu_0^{'}}{d\nu_0}$ is bounded, the previous solution satisfies*

$$\int_0^T\int\beta.a\beta\,(s,x)\,ds\,\nu_s^{'}(dx)<+\infty,$$

and is the unique solution (starting from $\nu_0^{'}$) satisfying this condition.

Remark 3.22. Let us say a few words about the problem of SUDDEN DEATH, which can enlighten the reader on our choices. Föllmer's measure theory deals with (right) continuous supermartingales. This explains our choice of G in (3.9), instead of using cut-off. No occurrence of sudden death means that both choices are the same. In earlier papers, many authors preferred cut-off, but then have to manage discontinuities (for example Nagasawa did so), implying more difficulties in the proofs. L. Wu has studied in more details such kind of problem.

Remark 3.23. The above strategy can be extended to the case of bounded domains. This is done in [39], which extends [56].

4. **Time reversal and applications.**

In the previous section, we have built a solution of the stochastic quantization problem just assuming the forward energy condition. In [35] and [36], Föllmer suggested that duality is automatically built in, as a consequence of the invariance of relative entropy under time reversal. We shall explain this point now. Furthermore, the duality equation that we shall prove has many other nice consequences : a priori regularity for the flow of marginals, properties of invariant measures, non attainability of the nodal set.

Denote by R the time reversal operator on Ω, i.e.

$$R(X)\,:\,(t\mapsto X_{T-t}\overset{\text{def}}{=}\overline{X}_t). \tag{4.1}$$

Generally, we shall use a bar for every notation concerning the time reversed process. For instance, $\overline{\mathbb{P}}$ will be the $\mathbb{P}$ law of $\overline{X}$. The main idea of [35] and [36] is that relative entropy is preserved under time reversal, i.e.

$$H(\mathbb{Q},\mathbb{P})=H(\overline{\mathbb{Q}},\overline{\mathbb{P}}). \tag{4.2}$$

Hence, if $\overline{\mathbb{P}}$ is good enough, Girsanov transformation theory furnishes a backward drift $\overline{\beta}$ of finite energy. The first point is to describe $\overline{\mathbb{P}}$.

Time reversal results for non singular diffusions are well known. We shall mainly use the ones of Hausmann-Pardoux ([42]) and Millet-Nualart-Sanz ([53]). The following is Theorem 2.3 in [53] (see also Theorem 2.1 in [42]).

Theorem 4.3. *Assume that σ and b are globally Lipschitz in space, uniformly in time. If, in addition:*

i) $\forall\, t>0,\quad \mathbb{P}_t=\mu_t(dx)=p_t(x)dx;$

ii) $div(a(t,x)\,p_t(x))\in\mathbb{L}^1_{loc}(dt\times dx)$ *where* $div(ap)$ *is the vector field*

$$(\sum_j \partial_j(a_{ij}p))_{i=1,\dots,d},$$

then, $\overline{\mathbb{P}}$ *is on* $\overline{\Omega} = \mathscr{C}^0([0,T[,\mathbb{R}^d)$ *a* $\overline{L}$*-diffusion, with*

$$\overline{L}(t,x) = \frac{1}{2}\sum_{ij} \overline{a}_{ij}(t,x)\,\partial_i\partial_j \,+\, \sum_i \overline{b}_i(t,x)\,\partial_i$$

where $\overline{a}(t,x) = a(T-t,x)$,
and $\overline{b}(t,x) = -b(T-t,x) + \frac{1}{p_{T-t}(x)}\, div(a(T-t,x)\,p_{T-t}(x))\ {}_{p_{T-t}(x)\neq 0}$.

The global Lipschitz condition can be relaxed into a local one with some extra (intricate) hypotheses (see [53], Section 3).
Of course, it is useful to know some conditions for (4.3) i) and ii) to hold. These conditions depend on what is assumed for μ_0. Without any assumption some ellipticity or hypoellipticity is required.

Proposition 4.4. *Assume that one of the following conditions holds:*

i) *σ and b are $\mathscr{C}^{0,2}$, with bounded derivatives of first and second order, and a is uniformly elliptic;*

ii) *$\sigma_1,...,\sigma_d$ are $\mathscr{C}_b^{\beta,\infty}$,and $\mathscr{L}ie\ (\sigma_1,...,\sigma_d)(0,x)$ is uniformly full on $supp\,(\mu_0)$,*

then, (4.3) i) and ii) hold.

Case i) is contained in [53], and case ii) in [20].
Once μ_0 is assumed to be absolutely continuous, much weaker conditions are allowed.

Proposition 4.5. *In addition to the Lipschitz regularity, assume that $\mu_0 = p_0(x)\,dx$ where p_0 belongs to some weighted $\mathbb{L}^2$ space. If one of the following conditions holds:*

i) *$div(a(t,x)\,p_0(x)) \in \mathbb{L}^1_{loc}(dt\times dx)$ and μ_0 is stationary;*

ii) *σ and b are $\mathscr{C}^{\alpha,2}$, with bounded derivatives up to order 2;*

iii) *a is uniformly elliptic;*

then, (4.3) i) and ii) hold.

Case i) is clear. Cases ii) and iii) are contained in [42]. Actually these authors relax the regularity on b in case ii) (which can also be obtained by using the diffeomorphism property of the associated stochastic flow, see e.g. [45]).

We now turn to the singular diffusion.
Let $\mathbb{Q}$ be defined as in (3.9), and assume that $\mathbb{Q}$ solves the stochastic quantization problem. Then, we know that:

$$H(\mathbb{Q},\mathbb{P}_{\mu_0}) = H(\nu_0,\mu_0) + \frac{1}{2}\int_0^T \int (\beta.a\beta)(s,X_s)\,ds\,\nu_s(dx), \tag{4.6}$$

which is finite, thanks to the finite energy condition (3.10) provided that $H(\nu_0,\mu_0) < +\infty$. Since relative entropy is preserved under time reversal, we thus have:

$$H(\overline{\mathbb{Q}},\overline{\mathbb{P}}_{\mu_0}) = H(\mathbb{Q},\mathbb{P}_{\mu_0}) < +\infty,$$

so that $\overline{\mathbb{Q}} \ll \overline{\mathbb{P}}_{\mu_0}$. It follows from [44] (12.17) that $\overline{\mathbb{Q}}$ is a $\overline{A}$-diffusion, with

$$\overline{A} = \overline{L} + (\overline{a}\overline{\beta})$$

for some given function $\overline{\beta}(s, X_s)$.

One difficulty is now the following : if $\overline{\mathbb{P}}_{\mu_0}$ is not an <u>extremal</u> $\overline{L}$-diffusion, we cannot get an explicit expression for $\frac{d\overline{\mathbb{Q}}}{d\overline{\mathbb{P}}_{\mu_0}}$. Fortunately, one can again control the energy of the backward drift thanks to the following :

Lemma 4.7. *Assume $H(\nu_0, \mu_0) < +\infty$. Then*

$$H(\nu_T, \mu_T) + \frac{1}{2}\int_0^T \mathbb{E}^{\overline{\mathbb{Q}}}[\overline{\beta}_s.\overline{a}(s, X_s)\,\overline{\beta}_s]\,ds \le H(\overline{\mathbb{Q}}, \overline{\mathbb{P}}_{\mu_0}) < +\infty.$$

In particular $\overline{\beta}$ satisfies the finite energy condition. Equality holds if $\overline{\mathbb{P}}_{\mu_0}$ is an extremal $\overline{L}$-diffusion.

The proof is an easy application of the variational definition of relative entropy in 1.15. Since $\mathbb{Q} \ll \mathbb{P}_{\mu_0}$, $\nu_t \ll \mu_t$, and with assumption (4.3) i), we have:

$$\text{for } t \in]0, T],\ \nu_t(dx) = \rho_t(x)dx = \gamma_t(x)\,p_t(x)dx. \tag{4.8}$$

As in [35], we can describe the relationship between β, $\overline{\beta}$ and ρ.

Proposition 4.9. *Assume that $H(\mathbb{Q}, \mathbb{P}_{\mu_0})$ is finite, and that we are in one of the situations of Proposition 4.4 or 4.5. Then, for dt almost every $t \in]0, T]$, every f and φ in $\mathscr{C}_0^\infty(\mathbb{R}^d)$, we have:*

$$-\mathbb{E}^{\mathbb{Q}}[(\nabla\varphi\,.\,a\,\nabla f)(t, X_t)]$$

$$= \mathbb{E}^{\mathbb{Q}}[f(X_t)\,((L_t\varphi + \overline{L}_{T-t}\varphi)(t, X_t) + \nabla\varphi(X_t).a(t, X_t)(\beta(t, X_t) + \overline{\beta}(T-t, X_t)))],$$

provided that, for every $k \in \mathbb{N}$ and every $\varepsilon \in]0, T]$,

$$\int_\varepsilon^T \int_{|x|\le k} |\frac{div(ap)}{p}|(t, x)\,dt\,\nu_t(dx) < +\infty.$$

The proof is a straightforward copy of what is done in [35], using Ito formula in both directions of the time (remember the Nelson's forward and backward derivatives in section 1), and the finite energy condition to control $\mathbb{L}^2$ norms. Let us say at this point, that Picard [58] has obtained prior time reversal results in a similar but different framework. For the case of reflected diffusions in bounded domains see [13] and [57].

Applying 4.9 to a function $\varphi \in \mathscr{C}_0^\infty$ such that $\varphi(x) = e.x$ on the support of f, where e is a fixed element of $\mathbb{R}^d$, we thus get the following integration by parts formula:

Corollary 4.10. *Under the hypotheses of Proposition 4.9, for every $e \in \mathbb{R}^d$, we have:*

$$-\mathbb{E}^{\mathbb{Q}}[e.a(t, X_t)\nabla f(X_t)]$$

$$= \mathbb{E}^{\mathbb{Q}}[f(X_t)\,\{e.a(t, X_t)(\beta(t, X_t) + \overline{\beta}(T-t, X_t))\}] + \mathbb{E}^{\mathbb{Q}}[f(X_t)\,\{e.\frac{div(ap)}{p}(t, X_t)\}].$$

Corollary 4.10 is what we call the DUALITY EQUATION. Its statement is similar to the classical regular case (i.e 4.3) but cannot be deduced from 4.3 due to the lack of regularity of β.

Instead of giving proofs (see [22] and [21]), let us explain the relationship and differences between the duality equation and the Lyons-Zheng decomposition in the symmetric case.

We already mentioned that the strength of Lyons-Zheng decomposition is that it makes disappear the annoying zero energy (second order) terms in Ito decomposition. In the symmetric case, we get a pathwise version. In the non stationary case, the duality equation plays a similar role. Here again, second order terms disappear. But contrary to the symmetric case, we do no more have a pathwise equation but an averaged equation.

Now,remembering that

$$\mathbb{Q}_t = \rho_t \, dx$$

for all t, Corollary 4.10 can be rewritten in terms of ρ. As usual

$$\sigma \nabla \rho \stackrel{\text{def}}{=} \nabla(\rho \, \sigma) - \rho \nabla \sigma$$

in the sense of Schwartz distributions $\mathscr{D}'$, whenever the right hand side makes sense.

Lemma 4.11. *Under the hypotheses of Corollary 4.10, for all $t \in [0,T]$, there exists $\eta_t \in \mathbb{L}^\infty_{loc}$ such that*

$$\sigma \nabla \rho_t = \rho_t \sigma \{ \frac{\nabla p_t}{p_t} + (\beta(t,.) + \overline{\beta}(T-t,.))\} + \rho_t \eta_t \quad in \ \mathscr{D}' \, .$$

The above statement can look strange. Indeed if we replace σ by a, 4.11 is an immediate consequence of the duality equation. When a is uniformly elliptic, one should think that a similar statement holds without a just dividing by a^{-1}. This feeling is not true for two main reasons : first, β only satisfies $\sigma \, \beta \in \mathbb{L}^2(\rho \, dx)$ (finite energy condition) and β alone is not necessarily in $\mathscr{D}'$; second products in $\mathscr{D}'$ have to be used with caution. Do not think that the first argument can be bypassed by looking at the (time-space) support of ρ : this support does not need to be open.
So, what allows to "divide" by σ is the finite energy condition, and the proof of 4.11 uses regularization. $\eta = 0$ is the elliptic case, but does not need to vanish when σ is degenerate.

We can now state

Theorem 4.12. *Assume that the hypotheses of Theorem 3.17 are fulfilled. Assume in addition that σ and its first order derivatives are locally bounded, that $H(\nu_0, \mu_0) < +\infty$, and that:*
i) either one of the hypotheses of Proposition 4.4 or Proposition 4.5 is satisfied;
ii) or $\mu_0 = p_0 dx$ is a reversible Probability measure of the Markov process $(\mathbb{P}_x)_{x \in \mathbb{R}^d}$.
Assume in addition that

$$(\sigma \frac{\nabla p_t}{p_t}) \in \mathbb{L}^2_{loc}(d\nu_t \, dt).$$

Then, $d\nu_t(x) = \rho_t(x)\,dx$ *for every* $t \in]0,T]$*, and*

$$\sigma \nabla \rho_t \in \mathbb{L}^1_{loc}(dx) \quad and \quad \int_\varepsilon^T \int_K \frac{|\sigma \nabla \rho_t|^2}{\rho_t}\,dt\,dx < +\infty,$$

for any compact subset K *of* $\mathbb{R}^d$ *and any* $\varepsilon > 0$*. Furthermore when* ρ *is locally bounded, then* $\sigma \nabla \rho_t \in \mathbb{L}^2_{loc}(dx)$*.*
In cases ii) or i) with the hypotheses of Proposition 4.5, one may take $\varepsilon = 0$*.*

A similar statement can be shown (but is less interesting) when we replace 3.17 by 3.18 (see [22]).

If the statement of the above Theorem is precise, it is a little bit intricate. First, one can suppress some *loc* subscripts both in the hypotheses and the conclusions. Next one can give a condensed "rough" statement : if the hypotheses of Theorem 4.12 are fulfilled then, roughly speaking

$$\sigma \nabla \sqrt{\rho} \text{ belongs to } \mathbb{L}^1([0,T], \mathbb{L}^2)\,, \tag{4.13}$$

or, if one prefers

$$\sigma \nabla \log \rho \text{ belongs to } \mathbb{L}^2(d\nu_t\,dt)\,. \tag{4.14}$$

Of course (4.13) is clearly the analogue of our assumption on ψ in the stationary case.

Finally, since we have a pair of processes in duality $(\mathbb{Q}, \overline{\mathbb{Q}})$, one can obtain a TIME DEPENDENT LYONS-ZHENG DECOMPOSITION for smooth f, namely

(4.15)

$$\begin{aligned} f(t,X_t) - f(0,X_0) &= \frac{1}{2}\,(M^f_t + \overline{M}^f_{T-t} - \overline{M}^f_T) \\ &+ \frac{1}{2}\int_0^t (a(\beta - \overline{\beta}))\,.\nabla\, f(u,X_u)\,du \\ &+ \int_0^t (\partial_u + (b - \frac{1}{2}\frac{div(ap_u)}{p_u})\,.\nabla)\, f(u,X_u)\,du \end{aligned}$$

where M (resp. $\overline{M}$) is a $\mathbb{Q}$ (resp. $\overline{\mathbb{Q}}$) martingale with the ad hoc brackets. Also notice that we have made the abuse of notation $\overline{\beta} = \overline{\beta}(T-t,.)$.
According to (4.14), one can try to apply (4.15) with $f = \log\,\rho$. Of course it is immediately seen that the only annoying term will be the time derivative ∂_u. Recalling the strategy we used in the stationary case, such a study will yield indications on the problem of attainability of the nodes.

5. **Back to Schrödinger equation.**

In this section we assume that we have solved the stochastic quantization problem of section 3, and that we may apply the results of the previous section on time reversal. That is, the flow ν_t is given (without reference to anything else) and we assume enough regularity on the coefficients. Of course, the finite energy condition is assumed to hold.

Let us continue the analysis we did at the end of the preceding section.
Taking the expectation in (4.15), one obtains that ρ satisfies the so called current equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\,\nabla.\,(\rho\; a(-\beta+\overline{\beta})) - \nabla.\,(\rho\;(b - \frac{1}{2}\frac{div(ap_u)}{p_u})) \tag{5.1}$$

in $\mathscr{D}'$. One cannot divide (5.1) by ρ, but one can expect that the chain rule furnishes (formally)

$$\begin{aligned}\frac{\partial (log\,\rho)}{\partial t} &= \frac{1}{2}\,\nabla.\,(a(-\beta+\overline{\beta})) + \frac{1}{2}\,\frac{\sigma\nabla\rho}{\rho}\,.\,\sigma(-\beta+\overline{\beta}) \\ &-\nabla\,.\,(b-\frac{1}{2}\frac{div(ap_u)}{p_u}) - \frac{\nabla\rho}{\rho}\,.\,(b-\frac{1}{2}\frac{div(ap_u)}{p_u}).\end{aligned} \tag{5.2}$$

Hence one can control ∂_t log ρ provided one controls the right hand side in (5.2). The second and the third term are well behaved. The fourth one give some trouble since no σ is in, but forget about this and focus and the worse term, the first one

$$\nabla.\,(a(-\beta+\overline{\beta}))\,.$$

Recall that, in order to obtain the time dependent Lyons-Zheng decomposition, we have taken the average of the forward and the time reversed Ito formulas. If we take instead, the difference, we obtain

$$\begin{aligned}\int_0^t [\sum_{ij} a_{ij}\,\partial_i\partial_j + a(\beta+\overline{\beta})\,.\nabla + \frac{div(ap_u)}{p_u}\,.\nabla]\,f\,(u,X_u)\,du \\ = M_t^f - \overline{M}_T^f + \overline{M}_{T-t}^f.\end{aligned} \tag{5.3}$$

(5.3) allows to control terms like

$$\nabla.\,(a\,\nabla\,f)\,.$$

Hence, if β is a GRADIENT one can expect to get nice controls. Remember that in section 1, β was a gradient, but now it is not necessarily so.

At this point let us introduce a LEAST ACTION PRINCIPLE, i.e. MINIMIZATION OF ENTROPY.

Indeed if ρ satisfies the Fokker-Planck equation for some β of finite energy, it still satisfies the Fokker-Planck equation for $B=\beta+B^{\perp}$ where $B^{\perp}$ is any vector field of finite energy in $\mathbb{L}^2(d\nu_t\,dt)$ such that

$$\int_0^T \int (B^{\perp}.\,a\nabla f)(s,x)\,\rho(s,x)\,ds\,dx = 0,$$

for all smooth f.
Among all possible β's, there is one which minimizes the energy, namely β_{min}, which is the projection of β onto the $\mathbb{L}^2(d\nu_t\, dt)$ closure of the gradient of smooth functions, that is

(5.4) there exists a sequence of smooth functions h_n such that

$$\lim_{n\to+\infty} \int_0^T \int |\sigma\,(\beta_{min} - \nabla h_n)|^{\,2}(s,x)\,\rho(s,x)\,ds\,dx = 0.$$

The associated $\mathbb{Q}_{min}$ then minimizes relative entropy. We refer to [17] and [19] for details.

In the flat smooth case of [55], one can deduce that $\beta_{min} = \nabla\, h$ for some $\mathbb{L}^2_{loc}$ function h. This result is known in Analysis as de Rham's theorem, and can be obtained by using e.g. Poincaré inequality (other proofs using some lemmata of Peetre and Tartar are well known). A similar result is not yet known for the weighted Sobolev spaces we are using. Hence we will have to still work with the sequence h_n and use a limiting procedure. However in the rest of this section (which is an outline) we will write

$$\beta_{min} = \nabla\, h.$$

Note that we can use a similar argument to show that

$$\overline{\beta}_{min} = \nabla\, \overline{h}\,,$$

which also follows from the duality equation (thanks to a priori regularity).
Contrary to what Nelson does in [55], we shall not work with $\mathbb{Q}_{min}$, but only use β_{min}. The current equation for the log (5.2) then becomes

$$\frac{\partial(log\,\rho)}{\partial t} = \frac{1}{2}\,\nabla\,.\,(a(-\nabla h + \nabla\overline{h})) + \frac{1}{2}\,\frac{\sigma\nabla\rho}{\rho}\,.\,\sigma(-\nabla h + \nabla\overline{h})$$
$$-\nabla\,.\,(b - \frac{1}{2}\frac{div(ap_u)}{p_u}) - \frac{\nabla\rho}{\rho}\,.\,(b - \frac{1}{2}\frac{div(ap_u)}{p_u}). \tag{5.5}$$

Now use (5.3) with $f = -h + \overline{h}$. This yields

$$\int_0^t [\sum_{ij} a_{ij}\,\partial_i\partial_j + a(\beta+\overline{\beta})\,.\nabla + \frac{div(ap_u)}{p_u}\,.\nabla]\,(-h+\overline{h})\,(u,X_u)\,du$$
$$= M_t^{h-\overline{h}} - \overline{M}_T^{h-\overline{h}} + \overline{M}_{T-t}^{h-\overline{h}}. \tag{5.6}$$

Combining (4.15), (5.5), (5.6) and the duality equation, we finally obtain,

(5.7)

$$\begin{aligned}\log\rho(t,X_t) - \log\rho(0,X_0) \quad &= \frac{1}{2}\,(M_t^{\log\rho} + \overline{M}_{T-t}^{\log\rho} - \overline{M}_T^{\log\rho})\\ &+ \frac{1}{2}(M_t^{h-\overline{h}} - \overline{M}_T^{h-\overline{h}} + \overline{M}_{T-t}^{h-\overline{h}})\\ &+ \frac{1}{2}\int_0^t a(\beta-\overline{\beta}))\,.\nabla\,\log\,\rho\,(u,X_u)\,du\\ &- \int_0^t \nabla\,.\,(b - \frac{1}{2}\,\frac{div(ap_u)}{p_u})(u,X_u)\,du.\end{aligned}$$

Hence, formally, one can control the $\mathbb{Q}$ expectation of

$$\sup_{t\in[0,T]} |\log \rho(t,X_t) - \log \rho(0,X_0)|$$

by the $\mathbb{L}^2\,(d\nu_t\,dt)$ norm of $\sigma\,\nabla \log\,\rho$ and the relative entropy $H(\mathbb{Q},\mathbb{P})$.

Of course all the job is to give a rigorous meaning to all this derivation. This job is carried on in section 6 of [22]. We shall not give precise results here, because their formulation is quite intricate. Mainly, non attainability of the nodal set is shown when, either $\mathbb{P}_{\mu_0}$ is reversible or in the elliptic case.

Not to introduce disturbing technicalities let us come back to the classical flat case of Brownian motion i.e.

$$L_t = \frac{1}{2}\,\Delta.$$

Define

$$\theta = -h + \overline{h}\,.$$

Then the current equation for the *log* can be rewritten

$$\partial_t\, log\rho_t = \frac{1}{2}\,\Delta\,\theta_t + \frac{1}{2}\,\nabla log\rho_t\,.\,\nabla\theta_t\,. \tag{5.8}$$

Finally define the <u>wave function</u>

$$\psi_t = \rho_t^{\frac{1}{2}}\, e^{-\frac{1}{2}\,i\theta_t}. \tag{5.9}$$

An easy calculation shows that

$$i\,\partial_t\,\psi_t = -\,\frac{1}{2}\,\Delta\,\psi_t + V\,\psi_t, \tag{5.10}$$

where

$$2\,V(t,.) = \partial_t\theta_t - \frac{1}{4}\,|\nabla\theta_t|^2 + \frac{1}{4}\,|\nabla\, log\rho_t|^2 + \frac{1}{2}\,\Delta\, log\rho_t. \tag{5.11}$$

Due to the regularity results, V belongs to $H^{-1}((t,x)\,,\,\rho(t,x)>0)$. This is not satisfactory. Actually we would like that

$$V \in \mathbb{L}^1([0,T]\times H^{-1})$$

in order to $V\psi$ be well defined as an operator.

Of course we have one degree of freedom in the choice of the wave function. Indeed we may add to θ_t any function η which depends only on t. This will only modify V, adding $\partial_t\,\eta$. Choosing

$$\eta(t) = \int_{\rho_t>0} h(t,z)\,dz\,,$$

will minimize the H^{-1} norm of V and is thus the optimal choice.

Though the situation is not fully satisfactory, the derivation above indicates how one can build the potential V starting from the statistical observation of a particles system. Indeed recall the discussion in section 1. Relative entropy is the rate function for the large deviations of the empirical mean of the positions of Brownian particles, and $\mathbb{Q}_{min}$ is thus the most probable paths-law when one observes the flow of marginals ρ. Hence as we suggested as the end of section 1, not only the stochastic quantization problem, but also properties of the Schrödinger wave function are closely related to the Gibbs conditioning principle.

6. **The Large deviations approach.**

As we have seen in section 1 and in section 5, Schrödinger equation and stochastic quantization are closely related to some Gibbs conditioning principle, hence with LARGE DEVIATIONS. What is more surprising, is that stochastic quantization (i.e. an existence problem) can be directly solved by using large deviations results. This fact is very unusual, and was shown for the first time in [18].

But the interpretation in terms of a conditioning principle (assuming the stochastic quantization problem is solved) of the solution goes back to Fukushima and Takeda [41] in 1984, for the symmetric (stationary) case. Their result is not written for empirical measures but for the occupation measure (Donsker-Varadhan functional). The non stationary case for the occupation measure was further studied by Deuschel-Stroock [31], Roelly-Zessin [63] and Wu [69].

The empirical measure level (level 2 in large deviations vocabulary) appeared almost simultaneously in [36] and in papers by Aebi and Nagasawa (for all concerned with these works, as well as for previous Nagasawa's results on the stochastic quantization problem, we refer to Aebi's book [1]). Other references are available for Schrödinger bridges due to Wakolbinger, Dawson, Gorostiza ...

In this section, we shall first explain how the stochastic quantization problem is solved by a direct and simple Large Deviations argument. The results are contained in [18] for $\mathbb{R}^d$ diffusion processes, and in [19] for general Markov processes. We shall next derive the Gibbs conditioning principle, and try to give some refined versions of it.

6.1. **Stochastic quantization via Large Deviations.** We are still using the notations and assumptions of section 3, namely (3.7), but in addition we assume that the family $\mathbb{P}_{t,x}$ is Feller continuous, i.e. that the associated semi group maps $\mathscr{C}_b$ into $\mathscr{C}_b$. This is (mainly) satisfied when the hypotheses of 3.17 are fulfilled. The derivation below can be rigorously done for general Markov processes, just being cautious with "domains" (see [19]). As we did before, we shall only give the flavor. In particular domains problems and topological considerations (which are very important for Large Deviations) will be hidden.

$t \mapsto \nu_t$ is thus a flow of probability measures satisfying the weak forward equation 3.4 for some β of finite energy. It is easy to see that this is equivalent to

$$\int f(T,x)\nu_T(dx) - \int f(0,x)\nu_0(dx) - \int_0^T \int (\partial_s + L)f(s,x)\, d\nu_s(x)\, ds \;= \int_0^T \int (\sigma\,\beta\,.\,\sigma\,\nabla f)(s,x)\, d\nu_s(x)\, ds\,, \tag{6.1}$$

for smooth f. But according to Riesz representation Theorem, (6.1) holds for some β (not necessarily an a priori given one) of finite energy if and only if the mapping

$$f \mapsto \int f(T,x)\nu_T(dx) - \int f(0,x)\nu_0(dx) - \int_0^T \int (\partial_s + L)f(s,x)\, d\nu_s(x)\, ds$$

is continuous when the space of smooth functions is equipped with the semi norm

$$\| \sigma \nabla f \|_{\mathbb{L}^2(d\nu_t\, dt)} \ .$$

This continuity property can be expressed in a VARIATIONAL form, namely

$$J_3(\nu) < +\infty \tag{6.2}$$

where

$$J_3(\nu) = \sup_f \tag{6.3}$$

$$\{\int f(T,x)\nu_T(dx) - \int f(0,x)\nu_0(dx) - \int_0^T \int (\partial_s + L)f(s,x)\, d\nu_s(x) ds - \frac{1}{2} \| \sigma \nabla f \|^2_{\mathbb{L}^2(d\nu_t\, dt)}\}.$$

Looking at J_3, we recognize a HAMILTON-JACOBI operator. This operator is connected to the moment generating function of the $\mathbb{P}$ process as follows.
For a continuous bounded function c define

$$g_c(t,x) = \mathbb{E}^{\mathbb{P}_{t,x}}[\exp \int_0^{T-t} c(t+s, X_s)\, ds]\,, \tag{6.4}$$

and

$$f_c(t,x) = \log\, g_c(t,x)\,. \tag{6.5}$$

Then f_c will satisfy the Hamilton-Jacobi equation

$$(\partial_s + L)f_c + \frac{1}{2}|\sigma \nabla f_c|^2 + c = 0\,. \tag{6.6}$$

Hence, provided

$$\sigma \nabla f_c \in \mathbb{L}^2(d\nu_t\, dt)$$

one has

$$\begin{aligned} &\int f_c(T,x)\nu_T(dx) - \int f_c(0,x)\nu_0(dx) - \int_0^T \int \{(\partial_s + L)f_c + \frac{1}{2}|\sigma \nabla f_c|^2\}(s,x)\, d\nu_s(x) ds \\ &= \int_0^T \int c(s,x)\, d\nu_s(x) ds - \int \log \mathbb{E}^{\mathbb{P}_x}[\exp \int_0^T c(s, X_s)\, ds]\; d\nu_0(x) \\ &\stackrel{\text{def}}{=} J_1(c, \nu)\,. \end{aligned} \tag{6.7}$$

It follows

$$J_3(\nu) \;\geq\; J_1(\nu) \stackrel{\text{def}}{=} \sup_{c \in \mathscr{C}_b} J_1(c, \nu)\,. \tag{6.8}$$

But now, J_1 looks like a CRAMER TRANSFORM (i.e. a Fenchel-Legendre conjugate function), and one can easily guess that it corresponds to some Large Deviations rate function. The only peculiar point is the “desintegration” with respect to ν_0.

Indeed consider an infinite collection of independent X_i with law $\mathbb{P}_{x_i}$ such that the empirical measure

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}$$

goes to ν_0 and introduce the EMPIRICAL PROCESS indexed by $\mathscr{C}_b$,

$$\overline{X}^n(f) = \frac{1}{n}\sum_{i=1}^{n}\big(\frac{1}{T}\int_0^T f(t, X_i(t))\,dt\big)\,.$$

An extension of Cramer's theorem, based on the general result by Dawson and Gärtner [29], shows that $\overline{X}^n$ satisfies a large deviations principle with speed n and rate function $J_1(\mu)$ if $\mu_0 = \nu_0$, $+\infty$ otherwise. The Feller assumption is required here.

We have chosen a presentation in terms of empirical process indexed by functions in order to exhibit the required duality. Of course $\overline{X}^n$ is nothing else than the empirical process $t \mapsto L_n^t$ we have introduced in (1.14). In particular, it is a (continuous) map of the empirical measure

$$L_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}$$

for which large deviations are governed by the RELATIVE ENTROPY thanks to Sanov's theorem. Hence using the CONTRACTION PRINCIPLE and goodness of the rate functions one obtains

$$J_1(\nu) = J_2(\nu) \overset{\text{def}}{=} \inf\{H(\mathbb{Q}, \mathbb{P}_{\nu_0})\,,\, \mathbb{Q}_t = \nu_t \text{ for all } t \in [0,T]\}\,. \tag{6.9}$$

But as we already discussed, if $J_2(\nu)$ is finite, then to the minimal $\mathbb{Q}_{min}$ one can associate some drift of finite energy β_{min}, and

$$J_3(\nu) = \frac{1}{2}\parallel \sigma\,\beta_{min} \parallel^2_{\mathbb{L}^2(d\nu_t\,dt)} = H(\mathbb{Q}_{min}, \mathbb{P}_{\nu_0}) = J_2(\nu)\,.$$

Thus

$$J_2(\nu) \geq J_3(\nu)$$

and accordingly, thanks to (6.8) and (6.9)

$$J_1(\nu) = J_2(\nu) = J_3(\nu)\,. \tag{6.10}$$

What does (6.10) say ? It says that if ν satisfies the weak forward equation for some drift β of finite energy, hence $J_3(\nu)$ is finite, then so does $J_2(\nu)$ and consequently one can solve the stochastic quantization problem for some (possibly different) β_{min}.
Actually, the fact that β_{min} is a function is not immediate since Girsanov theory only furnishes an adapted process. But taking appropriate conditional expectations, one can show that is is a function. See [17] for a (too) intricate proof, and [19] for a much more simple.
Of course ν is a solution of the Fokker-Planck equation for both β and β_{min} so that

$$\int_0^T\int \sigma\,(\beta - \beta_{min})\,.\,\sigma\,\nabla f\,d\nu_t\,dt = 0 \tag{6.11}$$

for all smooth f, i.e.

$$\beta - \beta_{min} \in Grad^{\perp}$$

where $Grad^{\perp}$ is the orthogonal set of the smooth functions for the semi norm $\| \sigma \nabla f \|_{\mathbb{L}^2(d\nu_t\, dt)}$. In order to achieve the stochastic quantization for a given β, one can thus use $\mathbb{Q}_{min}$ as a reference measure and build $\mathbb{Q}$ similarly to what has been done in section 3.1 step.2.

As a byproduct of the whole methodology, one obtains a very nice one to one correspondence between ENTROPY concepts for probability measures and ENERGY concepts for the drifts, that is, one can formulate entropy results in terms of some HILBERTIAN $\mathbb{L}^2$ norm.

(6.12) i) The minimal drift β_{min} belongs to the closure of smooth functions for the

$$\| \sigma \nabla f \|_{\mathbb{L}^2(d\nu_t\, dt)}$$

semi norm.

(6.12) ii) The set of all (markovian) probability measures with marginal laws ν_t satisfying $H(\mathbb{Q}, \mathbb{P}_{\nu_0}) < +\infty$, is in one to one correspondence with $Grad^{\perp}$, via

$$\beta = \beta_{min} + \beta^{\perp}$$

in particular stochastic quantization can be solved for all β of finite energy.

(6.12) iii) The Csiszar relation

$$H(\mathbb{Q}, \mathbb{P}_{\nu_0}) = H(\mathbb{Q}_{min}, \mathbb{P}_{\nu_0}) + H(\mathbb{Q}, \mathbb{Q}_{min})$$

holds, i.e. $\mathbb{Q}_{min}$ coincides with the Csiszar I-projection (see [27]).

This approach can be successfully used in very general contexts (see section 5 in [19]). The only restriction is the Feller property, and additional hypotheses have to be made for checking the condition

$$\sigma \nabla f_c \in \mathbb{L}^2(d\nu_t\, dt)\,.$$

In particular we recover most of the general symmetric case, part of the results for bounded domains and some results in infinite dimension.

6.2. **Gibbs conditioning principle.** Let us introduce

$$A_\nu = \{\, \mathbb{Q}\,,\, \mathbb{Q}_t = \nu_t \,\}\,,\; A_\nu^H = \{\, \mathbb{Q} \in A_\nu\,,\; H(\mathbb{Q},\mathbb{P}_\nu) < +\infty \,\}\,. \tag{6.13}$$

In section 1 we have seen that the conditional law of X_1 knowing that the empirical process $t \mapsto L_n^t$ is close to $t \mapsto \nu_t$ is asymptotically given by the minimizer of relative entropy. But to this end, one has first to choose some α neighborhood of $t \mapsto \nu_t$. Though it is natural to take some open ball of radius α, the behavior of the minimizing $\mathbb{Q}^\alpha$ when α goes to 0 is not clear.

Föllmer proposed an alternative blow up. Actually, imposing $\mathbb{Q}_t = \nu_t$ is equivalent to impose an infinite (countable) number of generalized moment conditions, i.e.

$$\mathbb{E}^{\mathbb{Q}}\,[F_j(X_{t_j})] = \int F_j(x)\, d\nu_{t_j}\,.$$

Defining $\mathbb{Q}^k$ as the probability measure that minimizes relative entropy under the first k moment constraints, one easily see that $\mathbb{Q}^k$ is a Cauchy sequence for the relative entropy pseudo-distance, hence in variation distance, thanks to Pinsker inequality. Thus it converges

in variation to $\mathbb{Q}_{min}$. Furthermore one can explicitly calculate $\mathbb{Q}^k$ by solving a k dimensional optimization problem, and $\mathbb{Q}^k$ is thus some Gibbs measure.
This idea was used by Aebi and Nagasawa in order to prove some version of the Gibbs conditioning principle. Their proof is for Schrödinger bridges but it can be adapted to Nelson processes.

Let T_k the set of dyadic numbers of level k in $[0, T]$, and choose some partition of $\mathbb{R}^d$ into 2^k measurable sets B_j^k in such a way that the partition at level $k+1$ is a refinement of the one at level k. Now define the 2^{-k} blow up of A_ν as

$$A_\nu^k = \{\mathbb{Q}\,,\, |\mathbb{Q}(X_t \in B_j^k) - \nu_t(B_j^k)| \le 2^{-k} \text{ for all } t \in T_k \text{ and all } j\}\,.$$

Then the following holds

Theorem 6.14. *The conditional law*

$$\mathbb{P}_n^k = \mathbb{P}_{\nu_0}^{\otimes n}(X_1 \in ./L_n \in A_\nu^k)$$

satisfies

$$\lim_k \lim_n \mathbb{P}_n^k = \mathbb{Q}_{min}\,.$$

The statement of 6.14, though very interesting for the statistical interpretation, is not yet fully satisfactory. Actually one should ask for a similar statement with $k = k(n)$ depending on n and also for exact bounds on errors. The strategy to get such results is quite clear : it requires exact bounds for the lower bound in Sanov theorem. We recently obtained these bounds in a work still in progress.
Also notice that the above approach using moment constraints instead of some open ball is certainly the good one from a practical point of view. Exact calculations and simulations can be done.

Let us indicate here that similar minimization of convex functionals under a finite or infinite number of linear constraints ($\nu \mapsto \int f(x)\, d\nu_t$ is linear) are studied for a long time in Optimization Theory and Convex Analysis. People like Rockafellar, Borwein, Lewis, Nussbaum ... have obtained relevant results. The papers by Léonard ([46] and [47]) contain results in this spirit which are particularly well adapted to our topic.

7. **Miscellaneous.**

7.1. **Conclusion.** In these lectures we have tried to show that stochastic modeling in quantum mechanics involves a great variety of ideas and exciting mathematical developments. As we said in the introduction, some new physical counterparts have been obtained during the last four years. If they are few, they are nevertheless an encouragement for the interested mathematicians. Nobody can tell today whether an approach or another is the good one or is condemned. But I really think that the relationship with statistical mechanics described in section 6 deserves further study. Anyhow, forgetting about Physics, substantial mathematical progresses have been made in various directions.
We shall briefly indicate now some connections with other areas and some open problems.

7.2. **Some connections with Statistical Mechanics and other Topics.** Connections with statistical mechanics are immediate in view of section 6. It turns out that the stochastic quantization problem is also important for other models. For example the work of Quastel and Varadhan [60] is a requirement for the study of the asymptotic behavior of a tagged particle in the exclusion process (see [59]). Time reversal using (local) relative entropy has been studied by Föllmer and Walkolbinger in [37] for interacting diffusion processes on a lattice (also see [54]). Very recently, Fradon and the author (see [15]) have used this approach to tackle an old problem, namely to show that all stationary measures of such particle systems are Gibbs.

When studying large deviations, one can replace empirical measures by weighted point measures. This strategy either called Maximum Entropy on the Mean (MEM) or weighted bootstrap has yield interesting results in the statistical resolution of some ill posed inverse problems (associated names are Dacunha-Castelle, Gamboa, Gassiat, Csiszar ...). It is used in [16] which as we already said, has something to do with bridges. It is also a well behaved approach for simulation.

7.3. **Open problems and recent developments.** Infinite dimensional state spaces like Hilbert spaces, $\mathbb{C}^*$ algebras, loop spaces are particularly relevant for quantum field theory. Some interesting results are now known for such state spaces in the symmetric case. Here again associated names are Albeverio, Röckner, Kondratiev and coauthors. Very few is known in the non stationary case, except some results due to Nelson and Carlen. Part of the methods and results of these lectures immediately extend to more general state spaces (it was mentioned in the course of the lectures). It is an open problem to extend all these results.

The "fine" study of the paths of Nelson's processes is also not well understood, as well as their behavior when the time goes to infinity. Such results should be interesting for scattering theory.

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