Some Remarks on Weighted Logarithmic Sobolev Inequality

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ABSTRACT. We give here a simple proof of weighted logarithmic Sobolev inequality, for example for Cauchy type measures, with optimal weight, sharpening results of Bobkov-Ledoux [12]. Some consequences are also discussed.

1. INTRODUCTION

In a recent paper, Bobkov and Ledoux [12, Theorem 3.4] proved that for a generalized Cauchy measure on $\mathbb{R}^n$, i.e.,

$$\mathrm{d}\nu_\beta(x) = \frac{1}{Z}(1+|x|^2)^{-\beta}\,\mathrm{d}x$$

for $\beta > n/2$, the following weighted logarithmic Sobolev inequality holds, provided $\beta \geq (n+1)/2$: for any smooth bounded f,

$$\mathrm{Ent}_{\nu_\beta}(f^2) = \nu_\beta\left(f^2\log\left(\frac{f^2}{\nu_\beta(f^2)}\right)\right) \leq \frac{1}{\beta-1}\int|\nabla f(x)|^2(1+|x|^2)^2\,\mathrm{d}\nu_\beta(x).$$

Simple test functions, however, indicate that the weight $(1+|x|^2)^2$ is not optimal: one may hope for a weight function behaving as $c(1+|x|^2)\log(e+|x|^2)$ at infinity, which will be a corollary of our main result.

It will thus be our purpose to prove inequalities of the type

$$\mathrm{Ent}_\mu(f^2) \leq c\int|\nabla f|^2\omega\,\mathrm{d}\mu$$

for some weight $\omega \geq 1$, and more generally weighted F-Sobolev inequalities with more general F's replacing the logarithm. It has to be noted that by Chen-Wang [20], every probability measure with positive density verifies a weighted logarithmic Sobolev inequality. One of the aims of the present paper is to propose an (optimal) estimation of this weight.

The (in a particular sense) case of weighted Poincaré inequalities is studied in [12] for Cauchy type measures and in [15] in more general situations. Consequences in terms of concentration of measure or isoperimetry are described in details in the latter reference.

It should also be interesting to look at weights that go to 0 at infinity (instead of weights bounded by 1 from below). Part of the results in [15] and in the present paper extend to this situation.

Our strategy will be the following:

(1) Consider a dual form of the weighted logarithmic Sobolev inequality (or more generally F-Sobolev inequality): the Super weighted Poincaré inequality.
(2) Use a Lyapunov condition to prove these Super weighted Poincaré inequalities.
(3) Show that these Super weighted Poincaré inequalities are equivalent to weighted F-Sobolev inequality (and in particular weighted logarithmic Sobolev inequality).

Let us then introduce the so-called Super weighted Poincaré inequality for a probability measure μ, in a simple context (namely, when the underlying carré du champ is in fact the square length of the gradient). It is inspired by the pioneering works on Super Poincaré inequality by Wang [42]. Given a weight ω larger than 1, we say that μ satisfies a Super weighted Poincaré inequality if for all f smooth and bounded, there exists a non-increasing function β_ω such that for all $s > 0$,

$$\int f^2 \, \mathrm{d}\mu \leq s \int |\nabla f|^2 \omega \, \mathrm{d}\mu + \beta_\omega(s)(\mu(|f|))^2. \tag{1.1}$$

When $\omega = 1$, this is the usual Super Poincaré inequality which describes properties of the measure stronger than the usual Poincaré inequality. If we add some additional weight ω (tending to infinity as $|x| \to \infty$, for example), we will be able to give an inequality adapted to measures "above" and "below" Poincaré and even be able to play between the weight and β.

Weighted Poincaré inequalities have been recently investigated by Bobkov-Ledoux [12] in particular for their interest in deviation inequalities, and by Cattiaux et al. [15], who showed their link with weak Poincaré inequalities and isoperimetric inequalities. They have been also intensively studied, in a converse form, in PDE theory to establish exponential convergence to equilibrium for fast diffusion equations (see [8, 22]). In parallel, Cattiaux et al. [18] have studied Super Poincaré inequalities using Lyapunov conditions (see also [2, 3]). We will combine here these two approaches to study these Super weighted Poincaré inequalities.

2. Results and Examples

2.1. A Lyapunov condition for Super weighted Poincaré inequality. Lyapunov conditions appeared a long time ago in relation to the problem of convergence to equilibrium for Markov processes; see [25, 37–39] and references therein. In addition, they have been used to study large and moderate deviations for empirical functionals of Markov processes (see Donsker-Varadhan [23, 24], Kontoyaniis-Meyn [35, 36], Wu [44], Guillin [31, 32], ...). Their use to provide functional inequalities has been very recently deeply investigated with some success: Lyapunov-Poincaré inequalities [3], Poincaré inequalities [2], transportation inequalities [19], Super Poincaré inequalities [18], weighted and weak Poincaré inequalities [15] (also see the recent survey [17]). We will take advantage of the approach of these last two papers to build our main results, but let us first describe our framework.

Let E be some Polish state space, μ be a probability measure, and P_t be a μ-symmetric diffusion semigroup with generator L on $L^2(E, \mu)$. The main assumption on L is that there exists some algebra $\mathcal{A}$ of bounded and uniformly continuous functions, containing constant functions, which is dense in the domain of L in the graph norm of $\mathcal{L}$ on $L^2(\mu)$. It enables us to define a "carré du champ" Γ, i.e., for $f, g \in \mathcal{A}$, $L(fg) = fLg + gLf + 2\Gamma(f, g)$. We will also assume that Γ is a derivation (in each component), i.e., $\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h)$, i.e., we are in the standard "diffusion" case in [1] and we refer to the Introduction of [13] for more details. For simplicity we set $\Gamma(f) = \Gamma(f, f)$. Also, since L generates a diffusion, we have the following chain rule formula: $\Gamma(\Psi(f), \Phi(g)) = \Psi'(f)\Phi'(g)\Gamma(f, g)$.

In particular, if $E = \mathbb{R}^n$, $\mu(\mathrm{d}x) = p(x)\,\mathrm{d}x$ and $L = \Delta + \nabla \log p \cdot \nabla$, we may consider the algebra generated by C^∞ functions with compact support and the constant functions as the interesting subalgebra $\mathcal{A}$, and then $\Gamma(f, g) = \nabla f \cdot \nabla g$.

Now we define the notion of φ-Lyapunov function. Let $W \geq 1$ be a smooth enough function on E and φ be a C^1 positive increasing function defined on $\mathbb{R}^+$. We say that W is a φ-Lyapunov function if there are a family of increasing sets $(A_r)_{r\geq 0} \subset E$ such that $\bigcup_r A_r = E$ (we say that the family A_r is exhausting) and some $b \geq 0$ such that for some $r_0 > 0$,

$$LW \leq -\varphi(W) + b\mathbf{1}_{A_{r_0}}. \tag{2.1}$$

This latter condition is sometimes called a "drift condition", but we prefer to call it a Lyapunov condition. One has very different behavior depending on φ: if φ is linear, then a Poincaré inequality is valid, whereas when φ is super-linear (or more generally in the form $\varphi \times W$, where φ tends to infinity), we have stronger inequalities (Super Poincaré, ultracontractivity, ...), and finally if φ is sub-linear, we are in the regime of weak Poincaré inequalities. We will cover both weak and super Poincaré inequalities by playing with the weight function.

We are now in position to state our main theorem:

Theorem 2.1. *Assume that L satisfies the Lyapunov condition* (2.1) *and* μ *satisfies some local Super Poincaré inequality, i.e., there exists* β_{loc} *decreasing in* s *(for all* r*) such that for all* $s > 0$*,*

$$\int_{A_r} f^2 \, d\mu \le s \int \Gamma(f) \, d\mu + \beta_{\mathrm{loc}}(r,s) \Big(\int_{A_r} |f| \, d\mu \Big)^2 . \tag{2.2}$$

We also introduce some $\psi : [1,\infty[\to [1,\infty[$ *which is increasing and such that*

$$0 < (\varphi/\psi)'(W).$$

We finally assume that $G(r) := 1/(\inf_{A_r^c} \psi(W))$ *goes to* 0 *as* r *goes to infinity.*

Then μ *satisfies a Super weighted Poincaré inequality, i.e., for all* $s > 0$*,*

$$\int f^2 \, d\mu \le 2s \int \frac{\Gamma(f)}{(\varphi/\psi)'(W)} \, d\mu + \tilde\beta(s) \Big(\int |f| \, d\mu \Big)^2 , \tag{2.3}$$

where

$$\tilde\beta(s) = c_{r_0} \beta_{\mathrm{loc}}(G^{-1}(s), s/c_{r_0}),$$

$G^{-1}(s) = \inf\{t > 0 \mid G(t) > s\}$ *is the right inverse of G, and*

$$c_{r_0} = 1 + b \frac{\sup_{A_{r_0}} (\psi/\varphi)(W)}{\inf_{A_{r_0}^c} \psi(W)} .$$

Remark 2.2. In fact, it is of course sufficient to verify some local Super weighted Poincaré inequality, but as the weight is usually bounded on the subset A_r, they are equivalent (up to the constants involved). Even if we play with r, as the weight is supposed to be greater than 1 they are implied by the local Super Poincaré inequalities as used here.

Remark 2.3. In the particular case where $\Gamma(f,g) = \nabla f \cdot \nabla g$, one can take a more general Lyapunov condition; namely, $\varphi(W)$ may be replaced by $\varphi \times W$ for some functional φ and the same for ψ appearing in the theorem. The modifications are straightforward but give a hard-to-read result, and we leave then the details to those who need such a framework.

Remark 2.4. In practice, A_r will often be level sets of the Lyapunov function W or balls of radius r. The local Super Poincaré inequality will then be obtained by perturbation of the Super weighted Poincaré inequality on balls for the underlying (Lebesgue) measure.

Proof. Let us begin with quite easy estimates: for $r \geq r_0$,

$$\begin{aligned}\int f^2\,\mathrm{d}\mu &= \int_{A_r} f^2\,\mathrm{d}\mu + \int_{A_r^c} f^2\,\mathrm{d}\mu\\
&= \int_{A_r} f^2\,\mathrm{d}\mu + \int_{A_r^c} \frac{\psi(W)\varphi(W)}{\psi(W)\varphi(W)} f^2\,\mathrm{d}\mu\\
&\leq \int_{A_r} f^2\,\mathrm{d}\mu + \frac{1}{\inf_{A_r^c}\psi(W)}\int f^2\frac{\psi(W)}{\varphi(W)}\varphi(W)\,\mathrm{d}\mu\\
&\leq \int_{A_r} f^2\,\mathrm{d}\mu + b\frac{\sup_{A_{r_0}}((\psi/\varphi)(W))}{\inf_{A_r^c}\psi(W)}\int_{A_{r_0}} f^2\,\mathrm{d}\mu\\
&\quad + \frac{1}{\inf_{A_r^c}\psi(W)}\int\frac{-LW}{(\varphi/\psi)(W)} f^2\,\mathrm{d}\mu\\
&\leq \left(1 + b\frac{\sup_{A_{r_0}}((\psi/\varphi)(W))}{\inf_{A_r^c}\psi(W)}\right)\int_{A_r} f^2\,\mathrm{d}\mu\\
&\quad + \frac{1}{\inf_{A_r^c}\psi(W)}\int\frac{-LW}{(\varphi/\psi)(W)} f^2\,\mathrm{d}\mu.\end{aligned}$$

Applying Lemma 2.5 below to the second term, and the local Super Poincaré inequality and the fact that $(\varphi/\psi)'(W) > 0$ to the first, we get

$$\begin{aligned}\int f^2\,\mathrm{d}\mu \leq &\left(s\left(1 + b\frac{\sup_{A_{r_0}}(\psi/\varphi)(W)}{\inf_{A_r^c}\psi(W)}\right) + \frac{1}{\inf_{A_r^c}\psi(W)}\right)\int\frac{\Gamma(f)}{(\varphi/\psi)'(W)}\,\mathrm{d}\mu\\
&+ \beta_{\mathrm{loc}}(r,s)\left(1 + b\frac{\sup_{A_{r_0}}(\psi/\varphi)(W)}{\inf_{A_r^c}\psi(W)}\right)\left(\int|f|\,\mathrm{d}\mu\right)^2.\end{aligned}$$

Recall now

$$c_{r_0} = 1 + b\frac{\sup_{A_{r_0}}(\psi/\varphi)(W)}{\inf_{A_{r_0}^c}\psi(W)}$$

and $\tilde{s} = sc_{r_0}$, so that, since A_r^c is decreasing in r, the last inequality furnishes

$$\int f^2\,\mathrm{d}\mu \leq (\tilde{s} + G(r))\int\frac{\Gamma(f)}{(\varphi/\psi)'(W)}\,\mathrm{d}\mu + \beta_{\mathrm{loc}}(r,\tilde{s}/c_{r_0})c_{r_0}\left(\int|f|\,\mathrm{d}\mu\right)^2.$$

Choose now $r = G^{-1}(\tilde{s})$ to conclude. □

One crucial element of the proof above was the following lemma, borrowed from [15], whose proof is reproduced here for completeness (it also shows the necessity for L to be a diffusion).

Lemma 2.5. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^1 increasing function. Then, for any $f \in \mathcal{A}$ and any positive $h \in D(\mathcal{E})$,*

$$\int\frac{-Lh}{\psi(h)} f^2\,\mathrm{d}\mu \leq \int\frac{\Gamma(f)}{\psi'(h)}\,\mathrm{d}\mu.$$

Proof. Since L is μ-symmetric, using that Γ is a derivation and the chain rule formula, we have

$$\int \frac{-Lh}{\psi(h)} f^2 \, d\mu = \int \Gamma\left(h, \frac{f^2}{\psi(h)}\right) d\mu = \int \left(\frac{2f\Gamma(f,h)}{\psi(h)} - \frac{f^2\psi'(h)\Gamma(h)}{\psi^2(h)}\right) d\mu.$$

Since ψ is increasing and according to Cauchy-Schwarz inequality, we get

$$\begin{aligned}\frac{f\Gamma(f,h)}{\psi(h)} &\leq \frac{f\sqrt{\Gamma(f)\Gamma(h)}}{\psi(h)} = \frac{\sqrt{\Gamma(f)}}{\sqrt{\psi'(h)}} \cdot \frac{f\sqrt{\psi'(h)\Gamma(h)}}{\psi(h)} \\ &\leq \frac{1}{2}\frac{\Gamma(f)}{\psi'(h)} + \frac{1}{2}\frac{f^2\psi'(h)\Gamma(h)}{\psi^2(h)}.\end{aligned}$$

The result follows. □

2.2. Equivalence with weighted F-Sobolev inequality. Let F be a continuous function, such that $\sup_{0<r<1} |rF(r)| < \infty$, $F(1) = 0$ and $\lim_{x\to+\infty} F(x) = +\infty$. We will say that the probability measure μ satisfies a defective weighted F-Sobolev inequality, with constants C_1 and C_2, and weight ω, if for all smooth and bounded f with $\mu(f^2) = 1$,

$$\int f^2 F(f^2) \, d\mu \leq C_1 \int \Gamma(f)\omega \, d\mu + C_2.$$

Notice that, modifying if necessary the constant C_2, we may replace F by F_+. This inequality will be called tight, or simply a weighted F-Sobolev inequality, if $C_2 = 0$.

When $\omega = 1$, it is known that if μ satisfies a defective F-Sobolev inequality and a Poincaré inequality, and with some (slight) additional assumptions on F, then μ satisfies a (tight) F-Sobolev inequality. The case $F = \log$ is known as the Rothaus lemma, and the previous general result is obtained in [4, Lemma 9 and Theorem 10].

The reader will easily check that the proofs in [4] extend to the weighted case, i.e., a weighted Poincaré inequality (with weight ω) and a weighted defective F-Sobolev inequality (with the same ω) imply a tight weighted F-Sobolev inequality, under the same assumptions as in [4, Lemma 9]. These assumptions are satisfied when $F(x) = \log_+(x)$ (see Remark 15 in [4]). We thus have that a weighted log-Sobolev inequality implies a weighted $\log_+$-Sobolev inequality, and together with a weighted Poincaré inequality implies a tight weighted $\log_+$-Sobolev inequality, hence a tight weighted log-Sobolev inequality.

We shall use this line of reasoning in various situations below, without mentioning it explicitly.

Now let us make a simple remark: if in the Super weighted Poincaré inequality we assume moreover that β_ω tends to a constant smaller than 1 as $s \to \infty$

(which is a quite weak hypothesis), the Super weighted Poincaré inequality implies a weighted Poincaré inequality. Indeed, applying (1.1) with $f = g - \mu(g)$, we get

$$(1 - \beta_\omega(s)) \operatorname{Var}_\mu(g) \le s \int \Gamma(g)\omega \, \mathrm{d}\mu,$$

thanks to Cauchy-Schwarz inequality, and the result follows if we take a large enough s for the left-hand side to be positive.

The next proposition is adapted from the works of Wang [42] and Theorems 3.3.1 and 3.3.3 in [43].

Proposition 2.6.

(1) *If μ satisfies a defective weighted F-Sobolev inequality with constants C_1, C_2, then there exist c_1, c_2 such that for all smooth bounded functions f and for all $s > 0$,*

$$\int f^2 \, \mathrm{d}\mu \le s \int \Gamma(f)\omega \, \mathrm{d}\mu + c_1 F^{-1}(c_2(1 + 1/s))\mu(|f|)^2,$$

where $F^{-1}(s) = \inf\{r \ge 0 \mid F(r) \ge s\}$.

(2) *If μ satisfies a Super weighted Poincaré inequality*

$$\int f^2 \, \mathrm{d}\mu \le s \int \Gamma(f)\omega \, \mathrm{d}\mu + \beta_\omega(s)\mu(|f|)^2,$$

then μ satisfies a defective weighted F-Sobolev inequality with

$$F(r) = \frac{c_1(\varepsilon)}{r} \int_0^r \xi(\varepsilon t) \, \mathrm{d}t - c_2(\varepsilon)$$

for all $0 < \varepsilon < 1$, where $c_1(\varepsilon)$ and $c_2(\varepsilon)$ are some constants, and

$$\xi(t) = \sup_{r>0} \left(\frac{1}{r} - \frac{\beta_\omega(r)}{rt} \right),$$

where $\beta_\omega^{-1}(t) = \inf\{r \ge 0 \mid \beta_\omega(r) \le t\}$.

Using this result, one sees that if a Super weighted (with weight ω) Poincaré inequality is valid with $\beta_\omega(s) = s^{-N} e^{c(1+1/s)}$, then a ($\omega$) weighted logarithmic Sobolev inequality is valid. In the preceding subsection we have presented conditions to verify Super weighted Poincaré inequalities; we only have now to illustrate them through examples. It will be the purpose of the next subsection.

2.3. Examples. We consider here the $\mathbb{R}^n$ situation with $\mathrm{d}\mu(x) = p(x)\,\mathrm{d}x$ and $L = \Delta + \nabla \log p \cdot \nabla$, where p is smooth enough and positive, and $\cdot$ is the Euclidean inner product. Recall the following elementary lemma, whose proof can be found in [2]. This lemma will be helpful in dealing with κ-concave measures.

Lemma 2.7. *If V is convex and $\int e^{-V(x)}\,\mathrm{d}x < +\infty$, then*

(1) *for all x, $x\cdot\nabla V(x) \geq V(x) - V(0)$;*
(2) *there exist $\delta > 0$ and $R > 0$ such that for $|x| \geq R$, $V(x) - V(0) \geq \delta|x|$.*

Another helpful result is the following result concerning the validity of a Super Poincaré inequality for Lebesgue measures on balls: for all $r > 0$, denote by $B(0,r)$ the Euclidean ball in $\mathbb{R}^n$. Then there exists c_n such that, for all smooth f and all $s > 0$,

$$\int_{B(0,r)} f^2\,\mathrm{d}x \leq s\int_{B(0,r)} |\nabla f|^2\,\mathrm{d}x + c_n(1+s^{-n/2})\Big(\int_{B(0,r)} |f|\,\mathrm{d}x\Big)^2. \tag{2.4}$$

Such an inequality will be particularly efficient when dealing with radial type measures, as a perturbation argument to get the local Super Poincaré inequality will be easy to do.

Indeed, we immediately obtain

(2.5)

$$\int_{B(0,r)} f^2\,\mathrm{d}\mu \leq s\int_{B(0,r)} |\nabla f|^2\,\mathrm{d}\mu + c_n\left(1+\left(\frac{s\inf_{B(0,r)} p}{\sup_{B(0,r)} p}\right)^{-n/2}\right)\left(\frac{\sup_{B(0,r)} p}{\inf^2_{B(0,r)} p}\right)\left(\int_{B(0,r)} |f|\,\mathrm{d}\mu\right)^2.$$

For more general types of measures, it is not so difficult to get local inequalities for level sets of the potential; see [18, Proposition 3.6].

2.3.1. Cauchy type measures. Let $\mathrm{d}\mu(x) = (V(x))^{-(n+\alpha)}\,\mathrm{d}x$ for some positive convex function V and some $\alpha > 0$. Let us begin by establishing a Lyapunov condition:

Lemma 2.8. *Let $L = \Delta - (n+\alpha)(\nabla V/V)\nabla$ with V convex and $\alpha > 0$. Then, there exist $k \in (2, \alpha+2)$, b, $R > 0$, and a function $W \geq 1$ such that*

$$LW \leq -\varphi(W) + b\mathbf{1}_{B(0,R)}$$

with $\varphi(u) = cu^{(k-2)/k}$ for some constant $c > 0$. Furthermore, one can choose $W(x) = |x|^k$ for x large.

Proof. Let $L = \Delta - (n+\alpha)(\nabla V/V)\nabla$ and choose $W \geq 1$ smooth, satisfying $W(x) = |x|^k$ for $|x|$ large enough and $k > 2$ that will be chosen later. For $|x|$ large enough we have

$$LW(x) = k(W(x))^{(k-2)/k}\left(n+k-2-\frac{(n+\alpha)x\cdot\nabla V(x)}{V(x)}\right).$$

Using (1) in Lemma 2.7 (since $V^{-(n+\alpha)}$ is integrable, e^{-V} is also integrable), we have

$$n + k - 2 - \frac{(n+\alpha)x \cdot \nabla V(x)}{V(x)} \leq k - 2 - \alpha + (n+\alpha)\frac{V(0)}{V(x)}.$$

Using (2) in Lemma 2.7, we observe that we can choose $|x|$ large enough for $V(0)/V(x)$ to be less than ε, say, $|x| > R_\varepsilon$. It remains for us to choose $k > 2$ and $\varepsilon > 0$ such that

$$k + n\varepsilon - 2 - \alpha(1-\varepsilon) \leq -\gamma$$

for some $\gamma > 0$. We have shown that, for $|x| > R_\varepsilon$,

$$LW \leq -k\gamma\varphi(W),$$

with $\varphi(u) = u^{(k-2)/k}$ (which is increasing since $k > 2$). A compactness argument achieves the proof. □

Consider now the case studied in [12] of the (generalized) Cauchy measure:

$$p(x) = Z_\beta^{-1}(1+|x|^2)^{-\beta}, \quad \beta > n/2.$$

Lemma 2.8 gives us a Lyapunov condition. Using (2.5), we get local Super Poincaré inequalities

$$\int_{B(0,R)} f^2 \, \mathrm{d}\mu \leq s \int_{B(0,R)} |\nabla f|^2 \, \mathrm{d}\mu + c_n(1 + s^{-n/2}(1+R^2)^{\beta n/2})(1+R^2)^{2\beta} Z_\beta \Big(\int_{B(0,R)} |f| \, \mathrm{d}\mu \Big)^2.$$

Choose now $\psi(v) = \log(v)$ for large v (and ψ smooth). Theorem 2.1 together with Proposition 2.6 thus furnishes (up to local modifications, e.g., for large $|x|$'s)

$$\varphi(u) = u^{k-2/k}, \quad \psi(u) = \log(u), \quad W(x) = |x|^k, \quad (\psi(W))(x) = k\log|x|,$$

hence

$$G(r) = \frac{1}{k\log r}, \quad G^{-1}(s) = e^{1/ks},$$

so that

$$\Big(\frac{\varphi}{\psi}\Big)'(u) \sim \frac{c}{u^{2/k}\log u}, \quad \omega(x) \sim \Big(\frac{1}{(\varphi/\psi)'(W)}\Big)(x) \sim c|x|^2 \log|x|,$$

and finally, for small s,

$$\beta_\omega(s) \sim s^{-n/2} e^{c/s}.$$

We have thus obtained the following result.

Corollary 2.9. *Cauchy measures $\mu(\mathrm{d}x) = Z_\beta^{-1}(1+|x|^2)^{-\beta}$ for $\beta > n/2$ verify the following weighted logarithmic Sobolev inequality: there exists $C = C(\beta, n)$ such that for all smooth bounded functions f,*

$$\mathrm{Ent}_\mu(f^2) \le C \int |\nabla f(x)|^2 (1+|x|^2) \log(e+|x|^2)\,\mathrm{d}\mu(x).$$

We then obtain the correct order of magnitude of the weight in this inequality, compared to [12, Theorem 3.4]. However, it has to be noted that we are losing the pretty expression of the constant in front of the weighted energy. Note that in dimension 1, Barthe-Zhang [7] obtained the same weight.

2.3.2. Exponential measure. We will look at the exponential measure

$$\nu(\mathrm{d}x) = Z_n^{-1} e^{-|x|}\,\mathrm{d}x.$$

It is well known that the exponential measure satisfies a Poincaré inequality. It is also easy to see that considering $W(x) = e^{a|x|}$ for $|x| \ge R$, we get, if $a < 1$, for R large enough,

$$LW(x) = a\left(\frac{n-1}{|x|} + a - 1\right) W(x) \le -\lambda W + b\mathbf{1}_{B(0,R)}$$

and thus the Lyapunov condition.

Using (2.5) with the choice $\psi(v) = \log(v)$ for large v (and ψ smooth), we get the following result.

Corollary 2.10. *The exponential measure ν satisfies the following weighted logarithmic Sobolev inequality: there exists $C = C(\beta, n)$ such that for all smooth bounded functions f,*

$$\mathrm{Ent}_\mu(f^2) \le C \int |\nabla f(x)|^2 (1+|x|)\,\mathrm{d}\mu(x).$$

As a comparison, let us recall a result of Bobkov-Ledoux [11, Equation (1.6)] which states that for the one-sided exponential $\tilde{\nu}$ (in dimension one),

$$\mathrm{Ent}_{\tilde{\nu}}(f^2) \le 4 \int x (f'(x))^2\,\mathrm{d}\tilde{\nu}.$$

We then recover in any dimension their result directly (they can only use tensorization to get the n-dimensional version of this inequality) and may extend it to other potentials.

Remark 2.11. Actually, the proof above covers a very large class of measures satisfying a Poincaré inequality—namely, measures $\mu(\mathrm{d}x) = e^{-V}\,\mathrm{d}x$ such that $V \to +\infty$ as $|x| \to +\infty$ and satisfying the following condition:

$$\text{there exists } 0 < a < 1 \text{ such that } \liminf_{|x|\to+\infty} (a|\nabla V|^2 - \Delta V) = B > 0.$$

Indeed, in this case we have $\varphi(u) = \lambda u$ (for some $\lambda > 0$) and $W = e^{AV}$ for some well-chosen positive constant A.

Choosing again $\psi(u) = \log u$ for large u's, we obtain the weight $\omega(x) = |x|$ for large $|x|$'s. If we assume in addition that there exists some constant $c > 0$ such that, for all R and all x such that $|x| = R$,

$$c \sup_{|y|=R} V(y) \leq V(x) \leq \frac{1}{c} \inf_{|y|\geq R} V(y),$$

it is not difficult to see that $G^{-1}(s) \sim (\bar{V})^{-1}(1/s)$, where $\bar{V}(R) = \inf_{|y|\geq R} V(y)$ is increasing. Using (2.5) again, we obtain that $\beta_\omega(s) \sim \exp(C/s)$ and hence the same weighted logarithmic Sobolev inequality as in the previous corollary.

We do not know whether this is true for any measure satisfying the Poincaré inequality. Indeed, we know that there exists some Lyapunov function W yielding a linear φ, but we do not know in full generality how to compare W and the potential V, so we cannot give an explicit formula for β_ω.

3. Properties and Applications

3.1. Concentration of measure. We will present here two different approaches to get concentration inequalities. The first one, due to Bobkov-Ledoux [12], uses controls on the growth of moments. As we obtain optimal weight by our approach, we will compare in some examples what the implications of these better controls are. The other one is based on the derivation of a suitable transportation cost information inequality following the approach of [9], based on the Hamilton-Jacobi equation.

3.1.1. Growth of moments and deviation inequality. We briefly recall here the main results concerning concentration inequality obtained by Bobkov-Ledoux [12, Theorem 4.1, Corollary 4.2] and present their main result.

Theorem 3.1 **(Bobkov-Ledoux [12]).** *Assume that the following weighted logarithmic Sobolev inequality is satisfied:*

$$\mathrm{Ent}_\mu(f^2) \leq 2 \int |\nabla f|^2 \omega \, \mathrm{d}\mu.$$

Assume also that ω has a finite moment of order $p \geq 2$; then for any μ-centered 1-Lipshitz function f, one has

$$\|f\|_p \leq \sqrt{p-1} \|\omega\|_p.$$

It implies that if $\|\omega\|_p \leq C$,

$$\mu(|f| \geq t) \leq \begin{cases} 2e^{-t^2/2c^2e} & \text{if } 0 \leq t \leq C\sqrt{ep}, \\ 2e^{-t/Ce} & \text{if } C\sqrt{ep} \leq t \leq Cep, \\ 2(Cp/t)^p & \text{if } Cep \leq t. \end{cases} \tag{3.1}$$

Remark that the weight obtained by Bobkov-Ledoux for Cauchy measures ν_β is $\omega = (\beta-1)^{-1}(1+|x|^2)^2$, whereas ours is $\omega = C(1+|x|^2)\log(1+|x|^2)$, which thus allows integration for $L^p(\mu)$ for a larger p. In addition, Corollary 2.9 is obtained for $\beta > n/2$ instead of $\beta \geq (n+1)/2$. Thus our result furnishes in principle a larger strip of Gaussian concentration. However, the evaluation of C is quite bad here (due mainly to the local inequality). It thus raises the question of the optimal constant with our weight. In dimension 1, one may use the generalized Hardy inequality.

3.1.2. Transportation inequality. We give here quickly another way to derive concentration inequality, based on transportation inequality, as derived from logarithmic Sobolev type inequality by Bobkov-Gentil-Ledoux [9] using the Hamilton-Jacobi equation valid in a general Riemannian setting (see, for example, results on the Hamilton-Jacobi equation in this context in [21, 26]); see also [40] for a proof based on PDE and optimal transport, or [16] for a refined argument. Let d_ω be the new Riemannian distance associated to ω, i.e., C_{xy} is the set of all absolutely continuous paths $\gamma : [0,1] \to \mathbb{R}^d$ such that $\gamma(0) = x$ and $\gamma(1) = y$ and

$$d_\omega(x,y) := \inf_{\gamma \in C_{x,y}} \int_0^1 \sqrt{\omega(\gamma(s))^{-1}|\gamma'(s)|^2}\,\mathrm{d}s.$$

Theorem 3.2. *Suppose that μ satisfies a weighted logarithmic Sobolev inequality with weight 2ω, i.e., for all nice f*

$$\mathrm{Ent}_\mu(f^2) \leq 2\int |\nabla f|^2 \omega \,\mathrm{d}\mu.$$

Then μ satisfies the following weighted Transportation-Information inequality (ωT_2): for all probability measures ν with $\mathrm{d}\nu = f\,\mathrm{d}\mu$,

$$W_{2,\omega}^2(\nu,\mu) := \inf_{X\sim\nu,\, Y\sim\mu} \mathbb{E}(d_\omega^2(X,Y)) \leq \mathrm{Ent}_\mu(f), \tag{3.2}$$

and thus for every μ-centered function with $|f(x) - f(y)| \leq d_\omega(x,y)$, for all $r > 0$,

$$\mu(|f| \geq r) \leq 2e^{-r^2/2}.$$

3.2. Entropic convergence.

3.2.1. The natural diffusion associated to the weighted energy. As is well known, the logarithmic Sobolev inequality is equivalent to the exponential decay in $\mathbb{L}\log\mathbb{L}$ of the diffusion semi-group associated to the Dirichlet form present in the inequality. We then get that a weighted logarithmic Sobolev inequality for the measure $\mathrm{d}\mu = e^{-V(x)}\,\mathrm{d}x$

$$\mathrm{Ent}_\mu(f^2) \leq 2\int |\nabla f|^2 \omega \,\mathrm{d}\mu$$

implies that the semi-group (P_t^ω) with generator

$$L^\omega = \omega\Delta + (\nabla\omega - \omega\nabla V)\cdot\nabla$$

satisfies

$$\mathrm{Ent}_\mu(P_t^\omega f) \le e^{-t/2}\,\mathrm{Ent}_\mu(f).$$

As this semi-group is reversible with respect to μ, it is certainly possible to use the results of [18], via also Lyapunov conditions, to get this convergence, but it is far easier to get a Lyapunov condition on the generator L than on L^ω. Note that it may also be useful when one desires to sample from μ via a Langevin tempered diffusions type algorithm (see [25]): we provide here an easy way to find a diffusion coefficient leading to an exponential entropic convergence. It has to be noted that the approach is quite different than in Hwang et al. [34] or Franke et al. [27], where they add a divergence-free drift to accelerate the diffusion. Moreover, they are limited to cases where the initial measure μ satisfies a Poincaré inequality. One may also get deviation inequality for integral functional of this Markov process, once remarked that assuming weighted logarithmic Sobolev inequality implies a transportation cost (ωT_2) inequality, then we are using once again the weighted logarithmic Sobolev inequality: for all probability measures ν with $\mathrm{d}\nu = f\,\mathrm{d}\mu$,

$$W_{2,\omega}^2(\nu,\mu) \le \frac{1}{2}\int \frac{|\nabla f|^2}{f}\,\omega\,\mathrm{d}\mu,$$

which implies, by [33], that for all μ-centered functions f with $\|f\|_{Lip(\omega)} \le 1$ and for $(X_t^\omega)_{t\ge0}$, the Markov process with generator L^ω: for all positive r

$$\mathbb{P}_\nu\left(\frac{1}{t}\int_0^t f(X_s^\omega)\,\mathrm{d}s \ge r\right) \le e^{-r^2/2},$$

which may be useful in Monte-Carlo simulations.

3.2.2. Link with weak logarithmic Sobolev inequality. Two of the authors with I. Gentil introduced in [14] the weak logarithmic Sobolev inequalities, i.e., μ satisfies (WLSI) for some non-increasing function β if for all bounded smooth functions, for all $s>0$,

$$\mathrm{Ent}_\mu(f^2) \le \beta(s)\int |\nabla f|^2\,\mathrm{d}\mu + s\,\mathrm{Osc}(f)^2. \tag{3.3}$$

This is the weak counterpart of the classical Gross logarithmic Sobolev inequalities, as weak Poincaré inequalities of [41] were for the usual Poincaré inequalities. These weak logarithmic Sobolev inequalities are particularly useful for asserting the speed of convergence towards equilibrium (for the natural Markov process associated to μ) in entropy when dealing with particular initial measures (such as Dirac mass, not suitable to an L^2 analysis).

It was shown in [14] that weak logarithmic Sobolev inequalities are equivalent to some capacity/measure conditions. If in dimension one, these capacity/measure conditions can be translated into verifiable conditions; it is no more the case in larger dimensions and only a comparison, under some additional conditions, with Beckner inequalities (stronger than Poincaré) or weak Poincaré inequalities gave multidimensional examples. We will show here that weighted logarithmic Sobolev inequalities, together with some concentration estimates, enable us to obtain weak logarithmic Sobolev inequalities, so that Lyapunov type conditions plus concentration give a new set of conditions for weak logarithmic Sobolev inequalities.

Theorem 3.3. *Assume that* μ *satisfies the following weighted logarithmic Sobolev inequality:*

$$\mathrm{Ent}_\mu(f^2) \le \int \omega |\nabla f|^2 \, d\mu.$$

Then μ *satisfies a* (WLSI) *with function* $\beta(s) = g^{-1}(s)$, *where*

$$g(r) = \mu(B_r^c)\left[2\mathfrak{c} + \log\left(1 + \frac{e^2}{\mu(B_r^c)}\right)\right], \tag{3.4}$$

with $B_r = \{x \mid \omega \le r\}$ *and* $\mathfrak{c} > 0$ *explicit.*

Proof. Let us first recall the result of Theorem 2.2 of [14] (taking advantage of Remark 2.3), that is, a capacity measure condition for weak logarithmic Sobolev inequality.

To this end, let us recall the definition of the capacity of a given measurable set $A \subset \Omega$:

$$\mathrm{Cap}_\mu(A, \Omega) := \inf\left\{ \int |\nabla f|^2 \, d\mu \mid \mathbf{1}_A \le f \le \mathbf{1}_\Omega \right\},$$

where the infimum is taken over all Lipschitz functions. Finally, if A is such that $\mu(A) < \frac{1}{2}$, then

$$\mathrm{Cap}_\mu(A) := \inf\left\{ \mathrm{Cap}_\mu(A, \Omega) \mid A \subset \Omega,\ \mu(\Omega) \le \frac{1}{2} \right\}.$$

A sufficient condition for (3.3) to hold is then: for every A with $\mu(A) < \frac{1}{2}$,

$$\forall\, s > 0, \quad \frac{\mu(A) \log(1 + e^2/\mu(A))}{\beta(s)} \le \mathrm{Cap}_\mu(A). \tag{3.5}$$

We cannot use directly our weighted logarithmic Sobolev inequality with this notion of capacity, so we introduce the natural weighted capacity

$$\begin{aligned}
\overline{\mathrm{Cap}}_\mu(A,\Omega) &:= \inf\left\{\int |\nabla f|^2 \omega \,\mathrm{d}\mu \mid \mathbf{1}_A \le f \le \mathbf{1}_\Omega\right\},\\
\overline{\mathrm{Cap}}_\mu(A) &:= \inf\left\{\overline{\mathrm{Cap}}_\mu(A,\Omega) \mid A \subset \Omega,\ \mu(\Omega) \le \frac{1}{2}\right\}\\
&= \inf\left\{\int |\nabla f|^2 \omega \,\mathrm{d}\mu \mid f : \mathbb{R}^d \to [0,1],\ f\mathbf{1}_A = 1,\ \mu(f=0) \ge \frac{1}{2}\right\}.
\end{aligned}$$

Using Bobkov-Goetze's seminal work [10] or its refined version by Barthe-Roberto [6], the weighted logarithmic Sobolev inequality implies that for all A such that $\mu(A) < \frac{1}{2}$, there exists $\mathfrak{c}$ such that

$$\mu(A)\log\left(1+\frac{e^2}{\mu(A)}\right) \le \mathfrak{c}\overline{\mathrm{Cap}}_\mu(A).$$

Consider now the set $B_r = \{x \mid \omega \le r\}$; by a simple adaptation of the proof of Gozlan [30], we get that if $A \subset B_r$,

$$\overline{\mathrm{Cap}}_\mu(A) \le 2r\,\mathrm{Cap}_\mu(A) + 2\mu(B_r^c).$$

Remark now that the mapping $t \to t\log(1+e^2/t)$ is concave increasing for small values of t, so that for all A such that $\mu(A) \le \frac{1}{2}$,

$$\begin{aligned}
&\mu(A)\log\left(1+\frac{e^2}{\mu(A)}\right)\\
&\quad\le \mu(A\cap B_r)\log\left(1+\frac{e^2}{\mu(A\cap B_r)}\right) + \mu(A\cap B_r^c)\log\left(1+\frac{e^2}{\mu(A\cap B_r^c)}\right)\\
&\quad\le \mathfrak{c}\overline{\mathrm{Cap}}_\mu(A\cap B_r) + \mu(B_r^c)\log\left(1+\frac{e^2}{\mu(B_r^c)}\right)\\
&\quad\le 2\mathfrak{c}r\mathrm{Cap}_\mu(A) + \mu(B_r^c)\left[2\mathfrak{c}+\log\left(1+\frac{e^2}{\mu(B_r^c)}\right)\right].
\end{aligned}$$

Setting $s = \mu(B_r^c)[2\mathfrak{c} + \log(1+e^2/\mu(B_r^c))]$, we conclude the proof. □

If r is large enough, the concentration result of the previous section will give upper bounds for the second term of the left-hand side.

3.3. Modified logarithmic Sobolev inequalities. We will prove here that weighted logarithmic Sobolev inequalities imply modified logarithmic Sobolev inequalities (i.e., the energy is modified). These inequalities were initially introduced by Bobkov-Ledoux in [11], where they show that a Poincaré inequality implies a logarithmic Sobolev inequality for a particular class of functions

($|\nabla f/f| \leq c < C_{c_{SG}}$, where c_{SG} is the spectral gap constant). These results were later extended to measures between exponential and Gaussian by Gentil et al. [28, 29]. For recent results, giving nice conditions we will discuss later, see also [5].

Theorem 3.4. *Let H and H^* be a pair of dual convex Young functions, such that $H(|x|)/|x| \geq a > 0$ for large $|x|$ and $H^*(\varepsilon|x|) \leq b(\varepsilon)H^*(|x|)$ with $b(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Suppose now that the weighted logarithmic Sobolev inequality

$$\mathrm{Ent}_\mu(f^2) \leq \int |\nabla f|^2 \omega \, \mathrm{d}\mu$$

holds for some weight $\omega \geq 1$, that a Poincaré inequality holds and that for some $\alpha > 0$,

$$K := \int e^{\alpha H^y(\omega)} \, \mathrm{d}\mu < \infty. \tag{3.6}$$

Then the following modified logarithmic Sobolev inequality holds for sufficiently small ε and some constant C (explicit in the proof):

$$\mathrm{Ent}_\mu(f^2) \leq C \int \left(H\left(\varepsilon^{-1} \left|\frac{\nabla f}{f}\right|^2\right) f^2 + |\nabla f|^2 \right) \mathrm{d}\mu. \tag{3.7}$$

Proof. Actually, it is sufficient to get a defective modified logarithmic Sobolev inequality, since a Poincaré inequality allows us to tighten a defective inequality thanks to [5, Theorem 2.4]. We then have

$$\begin{aligned}
\mathrm{Ent}_\mu(f^2) &\leq \int |\nabla f|^2 \omega \, \mathrm{d}\mu = \int \varepsilon^{-1} \left|\frac{\nabla f}{f}\right|^2 \varepsilon \omega f^2 \, \mathrm{d}\mu \\
&\leq \int H\left(\varepsilon^{-1} \left|\frac{\nabla f}{f}\right|^2\right) f^2 \, \mathrm{d}\mu + \int H^*(\varepsilon\omega) f^2 \, \mathrm{d}\mu.
\end{aligned}$$

Choose now ε sufficiently small so that $b(\varepsilon) \leq \alpha/2$ so that

$$\begin{aligned}
\int H^*(\varepsilon\omega) f^2 \, \mathrm{d}\mu &\leq \frac{1}{2} \int \alpha H^*(\omega) f^2 \, \mathrm{d}\mu \\
&\leq \frac{1}{2} \int (\alpha H^*(\omega) - \log K) f^2 \, \mathrm{d}\mu + \frac{1}{2} \log K \int f^2 \, \mathrm{d}\mu \\
&\leq \frac{1}{2} \mathrm{Ent}_\mu(f^2) + \frac{1}{2} \log K \int f^2 \, \mathrm{d}\mu,
\end{aligned}$$

where we have used the variational formula for the entropy in the last line. Plugging the latter inequality in to the preceding one, we obtain the defective modified logarithmic Sobolev inequality

$$\mathrm{Ent}_\mu(f^2) \le 2\int H\left(\varepsilon^{-1}\left|\frac{\nabla f}{f}\right|^2\right) f^2\,\mathrm{d}\mu + \log(K)\int f^2\,\mathrm{d}\mu,$$

which ends the proof. □

One may then use the Lyapunov conditions used to derive a weighted logarithmic Sobolev inequality to get a generalization of Barthe-Kolesnikov [5, Theorems 5.27, 5.28].

Example. Consider the usual (for modified LSI) examples: $\mathrm{d}\mu = Z_\alpha e^{-|x|^\beta}$ for $1 < \alpha \le 2$ so that the Poincaré inequality is valid. Using the Lyapunov function $W(x) = e^{a|x|^\beta}$ for $a < 1$, one may easily derive the Lyapunov condition

$$LW \le -c|x|^{2(\beta-1)}W + b\mathbf{1}_{B(0,R)},$$

from which one deduces, using $\psi(w) = \log(w)$ and Theorem 2.1 (and Proposition 2.6),

$$\mathrm{Ent}_\mu(f^2) \le C\int |\nabla f|^2(1+|x|^{2-\beta})\,\mathrm{d}\mu.$$

Consider now the Young functions

$$H_\beta(x) = |x|^{\beta/(2(\beta-1))} \quad \text{and} \quad H_\beta^{\mathcal{Y}}(x) = c_\beta|x|^{\beta/(2-\beta)}$$

so that $H_\beta^{\mathcal{Y}}(\varepsilon\omega) = c_\beta\omega^{\beta/(2-\beta)}|x|^\beta$, which is easily seen to be integrable with respect to μ for ε sufficiently small. We then get

$$\mathrm{Ent}_\mu(f^2) \le C\int\left(\left|\frac{\nabla f}{f}\right|^{\beta/(\beta-1)} f^2 + |\nabla f|^2\right)\mathrm{d}\mu$$

for some constant C, which is a generalization in the multidimensional case of [28].

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