# ORNSTEIN-UHLENBECK PINBALL AND THE POINCARÉ INEQUALITY IN A PUNCTURED DOMAIN.

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Abstract. In this paper we study the Poincaré constant for the Gaussian measure restricted to $D = \mathbb{R}^d - \mathcal{B}$ where $\mathcal{B}$ is the disjoint union of bounded open sets. We will mainly look at the case where the obstacles are Euclidean balls $B(x_i, r_i)$ with radii $r_i$, or hypercubes with vertices of length $2r_i$, and $d \geq 2$. This will explain the asymptotic behavior of a $d$-dimensional Ornstein-Uhlenbeck process in the presence of obstacles with elastic normal reflections (the Ornstein-Uhlenbeck pinball).

*Key words :* Poincaré inequalities, Lyapunov functions, hitting times, obstacles.

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## 1. Introduction.

In order to understand the goal of the present paper let us start with a well known question: how many non overlapping unit discs can be placed in a large square $S$? This problem of discs packing has a very long history including the following other question: is it possible to perform an algorithm yielding to a perfectly random configuration of $N$ such discs at a sufficiently quick rate (exponential for instance) ? This is one of the origin of the Metropolis algorithms as refereed in [DLM11].

The meaning of perfectly random is the following: the configuration space for the model is $S^N$, describing the location of the $N$ centers of the $N$ discs $B(x_i, 1)$, but under the constraints $d(x_i, \partial S) \geq 1$ and for all $i \neq j$, $|x_i - x_j| \geq 2$. The remaining domain $D$ is quite complicated, and randomness is described by the uniform measure on $D$.

The answer to the second question is positive, essentially thanks to compactness, but the exponent in the exponential rate of convergence is strongly connected with the Poincaré constant for the uniform measure on $D$ which is, at the present stage, far to be known (the only known upper bounds are disastrous).

One can of course ask the same questions replacing the square by the whole euclidean space, and the uniform measure by some natural probability measure, for instance the gaussian one. But this time even the finiteness of the Poincaré constant is no more clear. A very partial study ($N = 2, 3$) of this problem is done in [CFKR16].

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In all cases, the probability measure under study, and supported by the complicated state space $D$ is actually an invariant (even reversible) measure for some Markovian dynamics, one can study by itself, and which furnishes a possible algorithm. The boundary of $D$ becomes a reflecting boundary for the dynamics.

In this paper we intend to study the asymptotic behavior of a $d$-dimensional Ornstein-Uhlenbeck process in the presence of bounded obstacles with elastic normal reflections (looking like a random pinball). The choice of an Ornstein-Uhlenbeck (hence of an invariant measure of gaussian type) is made for simplicity as it captures already all the new difficulties of this setting, but a general gradient drift diffusion process (satisfying an ordinary Poincaré inequality) could be considered.

Of course for the packing problem in the whole space the obstacles are not bounded, but it seems interesting to look first at the present setting. Our model is also motivated by others considerations we shall give later.

All over the paper we assume that $d \geq 2$. We shall mainly consider the case where the obstacles are non overlapping euclidean balls or smoothed $l^\infty$ balls (hence smoothed hypercubes) of radius $r_i$ and centers $(x_i)_{1\leq i\leq N\leq +\infty}$, as overlapping obstacles could produce disconnected domains and thus non uniqueness of invariant measures (as well as no Poincaré inequality). We shall also look at different forms of obstacles when it can enlighten the discussion.

To be more precise, consider for $1 \leq N \leq +\infty$, $\mathcal{X} = (x_i)_{1\leq i\leq N\leq +\infty}$ a locally finite collection of points, and $(r_i)_{1\leq i\leq N\leq +\infty}$ a collection of non negative real numbers, satisfying

$$|x_i - x_j| > r_i + r_j \text{ for } i \neq j\,. \tag{1.1}$$

The Ornstein-Uhlenbeck pinball will be given by the following stochastic differential system with reflection

$$\begin{cases} dX_t = dW_t \,-\, \lambda\, X_t\, dt \,+\, \sum_i \, (X_t - x_i)\, dL_t^i\,, \\ L_t^i = \int_0^t \, \mathbb{1}_{|X_s - x_i| = r_i}\, dL_s^i. \end{cases} \tag{1.2}$$

Here $W$ is a standard Wiener process and we assume that $\mathbb{P}(|X_0 - x_i| \geq r_i \text{ for all } i) = 1$. $L^i$ is the local time description of the elastic and normal reflection of the process when it hits $B(x_i, r_i)$.

Existence and non explosion of the process, which is especially relevant for $N = +\infty$, will be discussed in Appendix A. The process lives in

$$\bar{D} = \mathbb{R}^d - \{x\,;\, |x - x_i| < r_i \;\text{ for some } i\}\,, \tag{1.3}$$

that is, we have removed a collection of non overlapping balls (or more generally non overlapping obstacles).

It is easily seen that the process admits an unique invariant (actually reversible) probability measure $\mu_{\lambda,\mathcal{X}}$, which is simply the Gaussian measure restricted to $D$, i.e.

$$\mu_{\lambda,\mathcal{X}}(dx) = Z_{\lambda,\mathcal{X}}^{-1}\, \mathbb{1}_D(x)\, e^{-\,\lambda\,|x|^2}\, dx\,, \tag{1.4}$$

where $Z_{\lambda,\mathcal{X}}$ is of course a normalizing constant. *Hence the process is positive recurrent.*

The question is to describe the rate of convergence for the distribution of the process at time $t$ to its equilibrium measure.

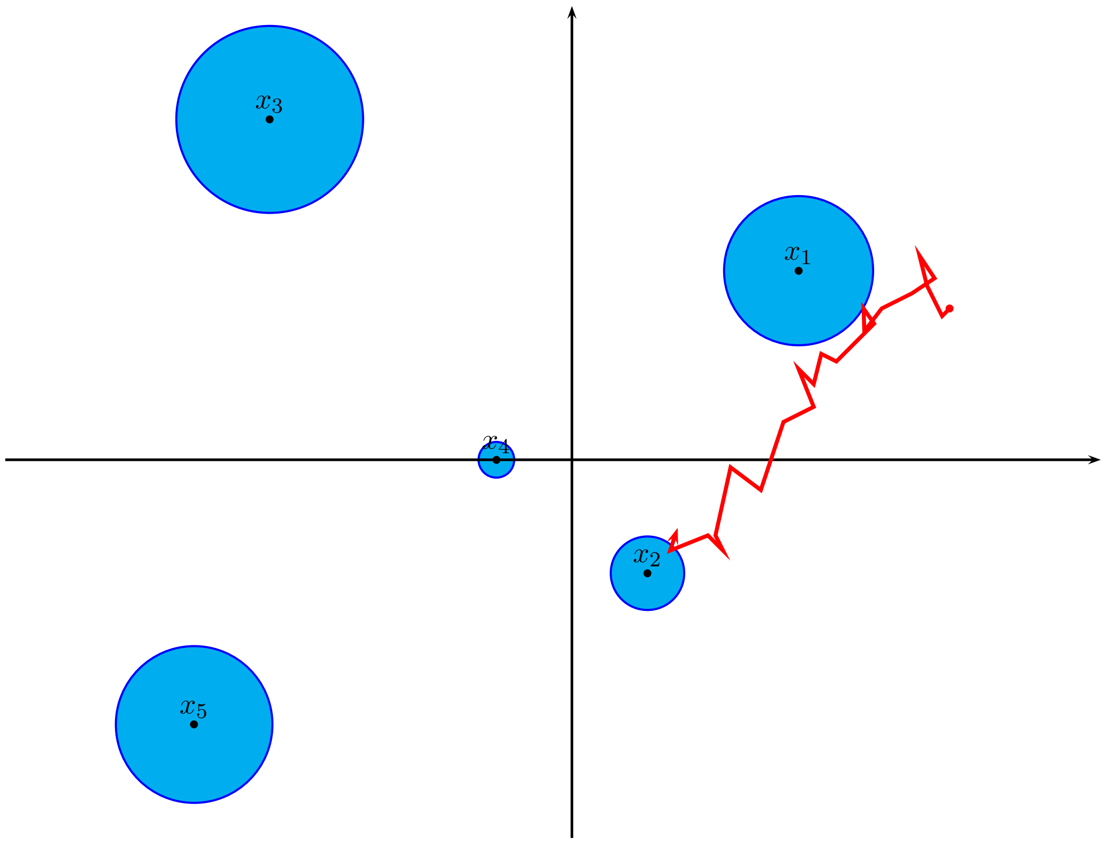


FIGURE 1. An Ornstein-Uhlenbeck particle in a billiard

To this end we shall look at the Poincaré constant of $\mu_{\lambda,\mathcal{X}}$ since it is well known that this Poincaré constant captures the exponential rate of convergence to equilibrium for symmetric processes (see e.g. [CGZ13] lemma 2.14 and [BCG08] theorem 2.1). Other functional inequalities (logarithmic Sobolev inequality, transportation inequality, ...) could be equally considered and the techniques developed here could also prove to be useful in these cases (for examples Lyapunov techniques have been introduced in the study of Super Poincaré inequalities in [CGWW09], including logarithmic Sobolev inequalities).

When the number of obstacles $N$ is finite, one can see, using Down, Meyn and Tweedie results [DMT95] and some regularity results for the process following [Cat86, Cat87], that the process is exponentially ergodic. It follows from [BCG08] theorem 2.1, that $\mu_{\lambda,\mathcal{X}}$ satisfies some Poincaré inequality, i.e. for all smooth $f$ (defined on the whole $\mathbb{R}^d$)

$$\mathrm{Var}_{\mu_{\lambda,\mathcal{X}}}(f) \leq C_P(\lambda,\mathcal{X}) \int |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} \,. \tag{1.5}$$

But the above method furnishes non explicit bounds for the Poincaré constant $C_P(\lambda,\mathcal{X})$.

Our first goal is thus to obtain reasonable and *explicit* upper and lower bounds for the Poincaré constant. Surprisingly enough (or not) the case of one hard obstacle already contains non trivial features.

Our second goal is to look at the case of *infinitely many* obstacles, for which the finiteness of the Poincaré constant is not even clear.

Part of the title of the paper is taken from a paper by Lieb et altri [LSY03] which is one of the very few papers dealing with Poincaré inequality in a sub-domain. Of course, one cannot get any general result due to the fact that one can always remove an, as small as we want, subset disconnecting the whole space; so that the remaining sub-domain cannot satisfy some Poincaré inequality. Hence doing this breaks the ergodicity of the process.
The method used in [LSY03] relies on the extension of functions defined in $D$ to the whole space. But the inequality they obtain, involves the energy of this extension (including the part inside $D^c$), so that it is not useful to get a quantitative rate of convergence for our process.

Our model can be used (or modified) as a model for crowds displacements (involving several particles in the obstacles environment). In particular the design of small obstacles that should kill the Poincaré constant is interesting.

Let us now describe the main results and main methods contained in the paper.

First, it is easily seen, thanks to homogeneity, that

$$C_P(\lambda, \mathcal{X}) = \frac{1}{\lambda}\, C_P(1, \sqrt{\lambda}\, \mathcal{X})\,. \tag{1.6}$$

where $\sqrt{\lambda}\,\mathcal{X}$ is the homotetic of $\mathcal{X}$, i.e. the collection of $B(\sqrt{\lambda}\, x_i, \sqrt{\lambda}\, r_i)$. Hence we have one degree of freedom in the use of all parameters. This homogeneity property will be used in the paper to improve some bounds.

The first section is peculiar. We look at a single spherical obstacle centered at the origin. We show that the Poincaré constant is given by

$$C_P(\lambda, B(0,r)) \, \approx \, \frac{1}{\lambda} + \frac{r^2}{d}$$

i.e. is up to some universal constant the sum of the Poincaré constant of the gaussian distribution $1/2\lambda$ and the one of the uniform measure on the sphere of radius $r$ i.e. $r^2/d$. For the process this reflects the fact that it hits a neighborhood of the origin with an exponential rate given by $\lambda$ but turns around the sphere with an exponential rate given by $r^2/d$. This is also in accordance with what is expected when $\lambda \to +\infty$ ($\mu_{\lambda,\mathcal{X}}$ is close to the uniform measure on the sphere) or $r \to 0$ where the obstacle disappears.
We also look at the usual perturbation method for Poincaré inequalities when the center is no more located at the origin (see Proposition 2.4 and Proposition 2.5) with results that are not entirely satisfactory. The result for the obstacle $B(0,r)$ can be used, through the decomposition of variance method, to obtain results for a general single ball $B(y,r)$. This is explained in the third Appendix.

The next two sections 3 and 4 are devoted to our main goals in the case of spherical obstacles: obtain explicit controls for the Poincaré constant in the presence of a single obstacle, extend it to a finite number of obstacles, prove that it is still finite in the case of an infinite number of obstacles.

In section 3 we develop a "local" Lyapunov method (in the spirit of [BBCG08]) around the obstacle. Under a restriction to small sizes, it is possible to give some explicit Lyapunov function. As in recent works ([BHW12, AKM12]) the difficulty is then to piece together the Lyapunov functions we may build near the obstacle and far from the obstacle and the origin. Let us describe the main results and methods.

First we are able to find explicit Lyapunov functions in the neighborhood of the obstacles provided

$$\forall i\,,\, r_i < r\sqrt{\lambda} = \sqrt{(d-1)/2}\,-\,2^{-\frac{3}{4}}\,.$$

This implies some limitation for the dimension namely

$$d \geq 7\,.$$

If in addition the balls $B(x_i, r_i + b(\lambda)))$ are non-overlapping (here $b(\lambda)$ is some explicit constant), then one obtains an explicit upper bound for the Poincaré constant. This is explained in subsection 3.2 in particular in Proposition 3.10.
The remaining of section 3 is then dedicated to get rid of the dimension restriction still for small obstacles i.e. provided

$$\forall i\,,\, r_i \,\leq\, \frac{1}{2}\,\sqrt{(d-1)/2}\,.$$

In subsections 3.3 and 3.4 we show how to control the variance of functions compactly supported in the exterior of a large ball containing the origin. As a consequence we get in subsections 3.5 and 3.6 a general result for the Poincaré constant when there is only one obstacle, gathering all what was done in these subsections and the previous section.
Finally we prove the finiteness of the Poincaré constant for an infinite number of small obstacles uniformly disconnected, that is such the distance between two distinct obstacles is uniformly larger than some $\varepsilon > 0$ in Corollary 3.20. If we are not able to give a precise description of the Poincaré constant in general, we can give some provided all obstacles are far enough from the origin i.e. if

$$\forall i\,,\, |x_i|\sqrt{\lambda} > c\sqrt{d}$$

for some constant $c$ (see proposition 3.16 and the explanations at the beginning of subsection 3.7.)
We close section 3 by a subsection explaining what happens if we replace euclidean balls by hypercubes.

In section 4 we use the results in [CGZ13] in order to build new Lyapunov functions near the obstacles, this time without restriction on the radius. To this end, we study in details how the process avoids a spherical obstacle, using stochastic calculus. This allows us to build a new Lyapunov function near the obstacle, which is given by some exponential moment of the time needed to go around the obstacle. Useful results on the Laplace transform of exit times for some linear processes are recalled in the second Appendix. This new Lyapunov function is then used in subsection 4.2 to obtain an upper estimate for the Poincaré constant in a shell around a spherical obstacle. Together with the method in section 3, we can then show (see proposition 4.12) that provided

$$\forall i\,,\, |x_i| > r_i + m \quad \text{ and } \quad r_i > \frac{1}{2}$$

for some large enough $m$ the Poincaré constant is finite and obtain an upper bound for it. Finally we can extend the result in the case of infinitely many large obstacles. Hence sections 4 and 3 are complementary.

Gathering all this, we have the following key result: for a spherical obstacle located far from the origin, the Poincaré constant does not depend on the radius (contrary to what we conjectured in a previous version of this work). This result allows us to show the following general result in the case of an infinite number of spherical obstacles

**Theorem 1.7.** *Let $\mathcal{X} = (x_i)_{1\leq i<+\infty}$ a locally finite collection of distinct points, ordered such that $|x_i| \leq |x_{i+1}|$ for all $i$, and $\mathcal{R} = (r_i)_{1\leq i<+\infty}$ a collection of non-negative numbers. Assume that there exists $\varepsilon > 0$ with $|x_i - x_j| > r_i + r_j + \varepsilon$ for all $i \neq j$.*
*Then for any $\lambda > 0$, the measure $\mu_{\lambda,\mathcal{X}}$ defined in (1.4) has a finite Poincaré constant and the reflected Ornstein-Uhlenbeck process in $D$ (defined in (1.3)) is exponentially ergodic.*

Section 5 is devoted to obtain lower bounds. We show in particular that if we replace euclidean balls by hypercubes, the situation is drastically changed since each obstacle (in a particular configuration) gives some contribution $e^{cr^2}$ where $r$ denotes the length of an edge of the hypercube. In particular large obstacles far from the origin can make the Poincaré constant go to $\infty$. We give two approaches of this result: one using exit times for the stochastic process, the second one using isoperimetric ideas. The same isoperimetric ideas are used to give a lower bound for the Poincaré constant in the case of spherical obstacles. To conclude the section we show that replacing balls by some non convex small and far obstacles can kill the exponential ergodicity. This situation is analogous to the one obtained with "touching" spherical obstacles.

The conclusion is that, presumably for uniformly convex obstacles (with an uniform curvature bounded from below uniformly in the location of the obstacles too) a similar result as for spherical obstacles holds true and our method can be used. The only difficulty is to find the good Lyapunov functions. A lack of uniform convexity has some disastrous consequences on the Poincaré constant, even for small and far obstacles.

**Dedication** During the revision of the paper, we learned about the death of Marc Yor. Everybody knows what a tragedy it is for Probability theory. It turns out that some beautiful results of Marc Yor on exit times for general squared radial Ornstein-Uhlenbeck processes recalled in an Appendix, are crucial in the present paper.

## 2. Some results when $N = 1$.

2.1. **The case of one centered ball, i.e. $y = 0$.** Assume $N = 1$ and the obstacle is the euclidean ball $B(y, r)$ with $y = 0$. In this case $\mu_{\lambda,\mathcal{X}} = \nu^0_{\lambda,r}$ is the standard gaussian measure with variance $\frac{1}{2\lambda}$ restricted to $D = \mathbb{R}^d - B(0, r)$. More generally we will denote by $\nu^y_{\lambda,r}$ the gaussian measure with mean $y$ and variance $\frac{1}{2\lambda}$ restricted to $\mathbb{R}^d - B(0, r)$.
$\mu_{\lambda,\mathcal{X}}$ is spherically symmetric. Though it is not log-concave, its radial part, proportional to

$$\mathbb{1}_{\rho>r}\, \rho^{d-1}\, e^{-\lambda\, \rho^2}$$

is log concave in $\rho$ so that we may use the results in [Bob03], yielding

**Proposition 2.1.** *When $\mathcal{X} = B(0,r)$, the measure $\mu_{\lambda,\mathcal{X}}$ satisfies a Poincaré inequality* (1.5) *with*

$$\frac{1}{2}\left(\frac{1}{2\lambda}+\frac{r^2}{d}\right) \leq \max\left(\frac{1}{2\lambda}\,,\,\frac{r^2}{d}\right) \leq C_P(\lambda, B(0,r)) \leq \frac{1}{\lambda}+\frac{r^2}{d}\,.$$

*Proof.* For the upper bound, the only thing to do in view of [Bob03] is to estimate $\mathbb{E}(\xi^2)$ where $\xi$ is a random variable on $\mathbb{R}^+$ with density

$$\rho \mapsto A_\lambda^{-1}\,\mathbb{1}_{\rho>r}\,\rho^{d-1}\,e^{-\lambda\,\rho^2}\,. \tag{2.2}$$

But

$$A_\lambda = \int_r^{+\infty} \rho^{d-1}\,e^{-\lambda\,\rho^2}\,d\rho \,\geq\, r^{d-2}\int_r^{+\infty}\rho\,e^{-\lambda\,\rho^2}\,d\rho \,=\, \frac{r^{d-2}\,e^{-\lambda\,r^2}}{2\lambda}\,.$$

A simple integration by parts yields

$$\mathbb{E}(\xi^2) = \frac{d}{2\lambda} + \frac{r^d\,e^{-\lambda\,r^2}}{2\,\lambda\,A_\lambda} \,\leq\, \frac{d}{2\lambda}+r^2\,.$$

The main result in [Bob03] says that

$$C_P(\lambda, B(0,r)) \leq \frac{13}{d}\,\mathbb{E}(\xi^2)\,,$$

hence the result with a constant 13.

Instead of directly using Bobkov's result, one can look more carefully at its proof. The first part of this proof consists in establishing a bound for the Poincaré constant of the law given by (2.2). Here, again, we may apply Bakry-Emery criterion (which holds true on an interval), which furnishes $1/(2\lambda)$. The second step uses the Poincaré constant of the uniform measure on the unit sphere, i.e. $1/d$, times the previous bound for $\mathbb{E}(\xi^2)$. Finally these two bounds have to be summed up, yielding the result.

For the lower bound it is enough to consider the function $f(z) = \sum_{j=1}^d z_j$. Indeed, the energy of $f$ is equal to $d$. Furthermore on one hand

$$\text{Var}_{\mu_{\lambda,\mathcal{X}}}(f) = \frac{\int_r^{+\infty}\rho^{d+1}\,e^{-\lambda\,\rho^2}\,d\rho}{\int_r^{+\infty}\rho^{d-1}\,e^{-\lambda\,\rho^2}\,d\rho} \geq r^2\,,$$

while on the other hand, an integration by parts shows that

$$\text{Var}_{\mu_{\lambda,\mathcal{X}}}(f) = \frac{d}{2\lambda} + \frac{r^d\,e^{-\lambda r^2}}{2\lambda\,\int_r^{+\infty}\rho^{d-1}\,e^{-\lambda\,\rho^2}\,d\rho} \geq \frac{d}{2\lambda}$$

yielding the lower bound since the maximum is larger than the half sum. □

This result is satisfactory since we obtain the good order. Notice that when $r$ goes to 0 we recover (up to some universal constant) the gaussian Poincaré constant, and when $\lambda$ goes to $+\infty$ we recover (up to some universal constant) the Poincaré constant of the uniform measure on the sphere $rS^{d-1}$ which is the limiting measure of $\mu_{\lambda,\mathcal{X}}$. Also notice that the obstacle is really an obstacle since the Poincaré constant is larger than the gaussian one.

**Remark 2.3.** It is immediate that the same upper bound is true (with the same proof) for $\nu^0_{\lambda,r,R}(dx) = Z^{-1}_{\lambda,r,R}\,\mathbb{1}_{R>|x|>r}\,e^{-\lambda\,|x|^2}$ i.e. the gaussian measure restricted to a spherical shell $\{R > |x| > r\}$. For the lower bound some extra work is necessary. ◇

2.2. **A first estimate for a general $y$ using perturbation.** An intuitive idea to get estimates on the Poincaré constant relies on the Lyapunov function method developed in [BBCG08] which requires a local Poincaré inequality usually derived from Holley-Stroock perturbation's argument. To be more precise, let us introduce $\nu^y_{\lambda,r}$ which is the gaussian measure with mean $y \in \mathbb{R}^d$ restricted to $\mathbb{R}^d - B(0,r)$, and its natural generator

$$L_y = \frac{1}{2}\,\Delta - \lambda\,\langle x+y, \nabla\rangle\,.$$

If we consider the function $x \mapsto h(x) = |y+x|^2$ we see that

$$L_y h(x) \, = d - 2\lambda |x+y|^2 \leq \, - \lambda\, h(x) \quad \text{ if } \quad |x| \geq |y| + (d/\lambda)^{1/2}\,.$$

So we can use the method in [BBCG08]. Consider, for $\varepsilon > 0$, the ball

$$U = B\left(0, \left(|y| + (d/\lambda)^{1/2}\right) \vee (r+\varepsilon)\right)\,.$$

$h$ is a Lyapunov function satisfying

$$L_y h \leq -\lambda h + d\, \mathbb{1}_U\,.$$

Since $U^c$ does not intersect the obstacle $B(0,r)$, we may follow [CGZ13] and obtain that

$$C_P(\nu^y_{\lambda,r}) \, \leq \, \frac{4}{\lambda} + \left(\frac{4}{\lambda} + 2\right) \, C_P(\nu_{\lambda,r}, U+1)\,,$$

where $C_P(\nu_{\lambda,r}, U+1)$ is the Poincaré constant of the measure $\nu^y_{\lambda,r}$ restricted to the shell

$$S = \left\{ r < |x| < 1 + \left(\left(|y| + (d/\lambda)^{1/2}\right) \vee (r+\varepsilon)\right)\right\}\,.$$

Actually since $h$ may vanish, we first have to work with $h+\eta$ for some small $\eta$ (and small changes in the constants) and then let $\eta$ go to 0 for the dust to settle.

Now we apply Holley-Stroock perturbation argument. Indeed

$$\nu^y_{\lambda,r}(dx) = C(y,\lambda)\, e^{-2\lambda\,\langle x,y\rangle}\, \nu^0_{\lambda,r}(dx)$$

for some constant $C(y,\lambda)$. In restriction to the shell $S$, it is thus a logarithmically bounded perturbation of $\nu^0_{\lambda,r}$ with a logarithmic oscillation less than

$$4\lambda\,|y| \, \left(1 + \left(\left(|y| + (d/\lambda)^{1/2}\right) \vee (r+\varepsilon)\right)\right)$$

so that we have obtained

$$C_P(\lambda, B(y,r)) \leq \frac{4}{\lambda} + \left(2 + \frac{4}{\lambda}\right) \, \left(\frac{1}{\lambda} + \frac{r^2}{d}\right) \, e^{4\lambda\,|y|\,\left(1+\left(\left(|y|+(d/\lambda)^{1/2}\right)\vee(r+\varepsilon)\right)\right)}\,.$$

The previous bound is bad for small $\lambda' s$ but one can use the homogeneity property (1.6), and finally, letting $\varepsilon$ go to 0

**Proposition 2.4.** *For a general $y$, the measure $\mu_{\lambda,B(y,r)}$ satisfies a Poincaré inequality* (1.5) *with*

$$C_P(\lambda, B(y,r)) \leq \, \frac{2}{\lambda} \, \left(2 + 3\, \left(1 + \frac{r^2\,\lambda}{d}\right) \, e^{4\sqrt{\lambda}\,|y|\,\left(1+\left(|y|\sqrt{\lambda}+d^{1/2}\right)\vee r\sqrt{\lambda}\right)}\right)\,.$$

The previous result is not satisfactory for large values of $|y|$, $r$ or $\lambda$. In addition it is not possible to extend the method to more than one obstacle. Finally we have some extra dimension dependence when $y = 0$ due to the exponential term. Our aim will now be to improve this estimate.

Another possible way, in order to evaluate the Poincaré constant, is to write, for

$$g = f - \frac{\int f(x)\, e^{-\lambda \langle x,y\rangle}\, \nu^0_{\lambda,r}(dx)}{\int e^{-\lambda \langle x,y\rangle}\, \nu^0_{\lambda,r}(dx)}\,, \ \text{ so that } \int g(x)\, e^{-\lambda \langle x,y\rangle}\, \nu^0_{\lambda,r}(dx) = 0$$

$$\begin{aligned} \mathrm{Var}_{\nu^y_{\lambda,r}}(f) &\leq \int g^2\, d\nu^y_{\lambda,r} = C(\lambda,y,r) \int \left(g\, e^{-\lambda\langle x,y\rangle}\right)^2 d\nu^0_{\lambda,r} \\ &\leq C(\lambda,y,r)\, C_P(\lambda, B(0,r)) \int \left|\nabla\left(g\, e^{-\lambda\langle x,y\rangle}\right)\right|^2 d\nu^0_{\lambda,r} \\ &\leq 2\, C_P(\lambda, B(0,r)) \left(\int |\nabla g|^2\, d\nu^y_{\lambda,r} + \lambda^2\, |y|^2 \int g^2\, d\nu^y_{\lambda,r}\right). \end{aligned}$$

It follows first that, provided $2\, C_P(\lambda, B(0,r))\, \lambda^2\, |y|^2 \leq \frac{1}{2}$,

$$\int g^2\, d\nu^y_{\lambda,r} \leq 4\, C_P(\lambda, B(0,r)) \int |\nabla g|^2\, d\nu^y_{\lambda,r}\,,$$

and finally

**Proposition 2.5.** *If* $4\,\lambda\,|y|^2\,\left(1 + \frac{r^2\,\lambda}{d}\right) \leq 1$, *the measure* $\mu_{\lambda,\mathcal{X}}$ *where* $\mathcal{X} = B(y,r)$ *satisfies a Poincaré inequality* (1.5) *with*

$$C_P(\lambda, B(y,r)) \leq 4\,\left(\frac{1}{\lambda} + \frac{r^2}{d}\right).$$

One can note that under the condition $4\,\lambda\,|y|^2\,\left(1 + \frac{r^2\,\lambda}{d}\right) \leq 1$, Proposition 2.4 and Proposition 2.5 yield, up to some dimension dependent constant, similar bounds. Of course the first proposition is more general.

## 3. Using Lyapunov functions.

In what we did previously we have used Lyapunov functions vanishing in a neighborhood of the obstacle(s). Indeed a Lyapunov function (generally) has to belong to the domain of the generator, in particular its normal derivative (generally) has to vanish on the boundary of the obstacle. Since it seems that a squared distance is a good candidate it is natural to look at the geodesic distance in the punctured domain $D$ (see [ABB87] and also [Har94] for small time estimates of the density in this situation). Unless differentiability problems (the distance is not everywhere $C^2$) it seems that this distance does not yield the appropriate estimate (calculations being tedious).
Instead of trying to get a "global" Lyapunov function, we shall build "locally" such functions.

In this section we consider the case $1 \leq N \leq +\infty$ i.e. we may consider as well an infinite number of obstacles.

To be more precise, consider an open neighborhood (in $D$) $U$ of the obstacles and some smooth function $\chi$ supported in $D$ such that $\mathbb{I}_{U^c} \leq \chi \leq 1$ (in particular $\chi$ vanishes on the boundary of the obstacles). Let $f$ be a smooth (compactly supported) function and $m$ be such that $\int \chi\,(f-m)\,d\mu_{\lambda,\mathcal{X}} = 0$. Then

$$\begin{aligned} \mathrm{Var}_{\mu_{\lambda,\mathcal{X}}}(f) &\leq \int_D (f-m)^2 d\mu_{\lambda,\mathcal{X}} = \int_U (f-m)^2 d\mu_{\lambda,\mathcal{X}} + \int_{U^c} (f-m)^2 d\mu_{\lambda,\mathcal{X}} \\ &\leq \int_U (f-m)^2 d\mu_{\lambda,\mathcal{X}} + \int_{\mathbb{R}^d} \chi^2\,(f-m)^2 d\mu_{\lambda,\mathcal{X}} \\ &\leq \int_U (f-m)^2\,d\mu_{\lambda,\mathcal{X}} + \frac{1}{2\lambda}\int_{\mathbb{R}^d} |\nabla(\chi\,(f-m))|^2\,d\mu_{\lambda,\mathcal{X}} \\ &\leq \int_U (f-m)^2\,d\mu_{\lambda,\mathcal{X}} + \frac{1}{\lambda}\int_D \left(|\nabla\chi|^2\,(f-m)^2 \,+\chi^2\,|\nabla f|^2\right)\,d\mu_{\lambda,\mathcal{X}}\,, \end{aligned}$$

where we have used that $\mu_{\lambda,\mathcal{X}}$ is simply the gaussian measure on the support of $\chi$, introducing the Poincaré constant of the gaussian $1/2\lambda$. It follows

$$\begin{aligned} \mathrm{Var}_{\mu_{\lambda,\mathcal{X}}}(f) &\leq \int_D (f-m)^2 d\mu_{\lambda,\mathcal{X}} \\ &\leq \left(1+\frac{\parallel \nabla\chi \parallel_\infty^2}{\lambda}\right)\int_U (f-m)^2\,d\mu_{\lambda,\mathcal{X}} + \frac{1}{\lambda}\int_D |\nabla f|^2\,d\mu_{\lambda,\mathcal{X}}\,. \end{aligned} \tag{3.1}$$

We thus see that what we have to do is to get some bound for $\int_U (f-m)^2\,d\mu_{\lambda,\mathcal{X}}$ in terms of the energy $\int_D |\nabla f|^2\,d\mu_{\lambda,\mathcal{X}}$ for any smooth $f$ which is exactly what is done by finding a "local" Lyapunov function.

### 3.1. **Two useful lemmas on Lyapunov function method.**

We may now present two particularly useful lemmas concerning Lyapunov function method and localization. Let us begin by the following remark: in the previous derivation assume that for some $p>1$ and some constant $C$,

$$\int_U (f-m)^2 d\mu_{\lambda,\mathcal{X}} \leq \frac{\lambda}{p\,\parallel \nabla\chi \parallel_\infty^2}\int_{\mathbb{R}^d} \chi^2\,(f-m)^2 d\mu_{\lambda,\mathcal{X}} \,+\, C\int_D |\nabla f|^2\,d\mu_{\lambda,\mathcal{X}}. \tag{3.2}$$

Then, using the Poincaré inequality for the gaussian measure, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \chi^2\,(f-m)^2 d\mu_{\lambda,\mathcal{X}} &\leq \frac{1}{\lambda}\int_{\mathbb{R}^d} \left(|\nabla\chi|^2\,(f-m)^2 \,+\chi^2\,|\nabla f|^2\right)\,d\mu_{\lambda,\mathcal{X}} \\ &\leq \frac{\parallel \nabla\chi \parallel_\infty^2}{\lambda}\int_U (f-m)^2\,d\mu_{\lambda,\mathcal{X}} + \frac{1}{\lambda}\int_D |\nabla f|^2\,d\mu_{\lambda,\mathcal{X}} \\ &\leq \frac{1}{p}\int_{\mathbb{R}^d} \chi^2\,(f-m)^2 d\mu_{\lambda,\mathcal{X}} + \frac{1}{\lambda}\left(1+C\parallel \nabla\chi \parallel_\infty^2\right)\int_D |\nabla f|^2\,d\mu_{\lambda,\mathcal{X}} \end{aligned}$$

so that

$$\int_{\mathbb{R}^d} \chi^2\,(f-m)^2 d\mu_{\lambda,\mathcal{X}} \leq \frac{p}{(p-1)\lambda}\left(1+C\parallel \nabla\chi \parallel_\infty^2\right)\int_D |\nabla f|^2\,d\mu_{\lambda,\mathcal{X}}$$

and using (3.2)

$$\int_U (f-m)^2 d\mu_{\lambda,\mathcal{X}} \leq \left(C + \frac{1}{(p-1)\,\parallel \nabla\chi \parallel_\infty^2}\left(1+C\parallel \nabla\chi \parallel_\infty^2\right)\right)\int_D |\nabla f|^2\,d\mu_{\lambda,\mathcal{X}}$$

and finally

**Lemma 3.3.** *If* (3.2) *holds for some smooth* $\chi$ *supported in* $D$ *and such that* $\mathbb{1}_{U^c} \leq \chi \leq 1$, *then*

$$Var_{\mu_{\lambda,\mathcal{X}}}(f) \leq \frac{1}{p-1} \left( Cp + \frac{1}{\parallel \nabla\chi \parallel_\infty^2} + \frac{p(1 + C \parallel \nabla\chi \parallel_\infty^2)}{\lambda} \right) \int_D |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} \, .$$

From now on we assume that $\partial D$ is smooth enough and we denote by $n$ the normalized inward (pointing into $D$) normal vector field on $\partial D$.

Now recall the basic lemma used in [BBCG08, CGZ13] we state here in a slightly more general context (actually this lemma is more or less contained in [CGZ13] Remark 3.3)

**Lemma 3.4.** *Let* $f$ *be a smooth function with compact support in* $\bar{D}$ *and* $W$ *a positive smooth function. Denote by* $\mu^S_{\lambda,\mathcal{X}}$ *the trace (surface measure) on* $\partial D$ *of* $\mu_{\lambda,\mathcal{X}}$. *Then the following holds*

$$\int_D \frac{-LW}{W} \, f^2 \, d\mu_{\lambda,\mathcal{X}} \, \leq \, \frac{1}{2} \int_D |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} \, + \, \frac{1}{2} \int_{\partial D} \frac{\partial W}{\partial n} \, \frac{f^2}{W} \, d\mu^S_{\lambda,\mathcal{X}} \, .$$

*Proof.* We recall the proof for the sake of completeness. Using the first Green formula we have (recall that $n$ is pointing inward)

$$\begin{aligned} \int_D \frac{-2\, LW}{W} \, f^2 \, d\mu_{\lambda,\mathcal{X}} &= \int_D \left\langle \nabla \left( \frac{f^2}{W} \right) \, , \, \nabla W \right\rangle \, d\mu_{\lambda,\mathcal{X}} + \int_{\partial D} \frac{\partial W}{\partial n} \, \frac{f^2}{W} \, d\mu^S_{\lambda,\mathcal{X}} \\ &= 2 \int_D \frac{f}{W} \, \langle \nabla f, \nabla W \rangle \, d\mu_{\lambda,\mathcal{X}} \, - \int_D \frac{f^2}{W^2} \, |\nabla W|^2 \, d\mu_{\lambda,\mathcal{X}} + \int_{\partial D} \frac{\partial W}{\partial n} \, \frac{f^2}{W} \, d\mu^S_{\lambda,\mathcal{X}} \\ &= - \int_D \left| \frac{f}{W} \, \nabla W - \nabla f \right|^2 \, d\mu_{\lambda,/XX} \, + \int_D |\nabla f|^2 \, d\mu_{\lambda,r} + \int_{\partial D} \frac{\partial W}{\partial n} \, \frac{f^2}{W} \, d\mu^S_{\lambda,\mathcal{X}} \, . \end{aligned}$$

□

3.2. **Localizing around the obstacles.**

From now on for simplicity we will assume that $D^c = \cup_i B(x_i, r_i)$ where the $B$'s are non overlapping euclidean balls. We shall indicate at the end how the results extend to others situations, in particular to smoothed hypercubes.

We will construct first Lyapunov functions near the obstacles. Hence we will build open neighborhoods $U_i$ for each ball, and will assume that the $U_i$'s are non overlapping sets too.

Not to introduce immediately too much notations, we shall write things for one ball denoted by $B(y, r)$. Let $h > 0$ and assume that one can find a Lyapunov function $W$ such that $LW \leq -\theta W$ for $|x - y| \leq r + 2h$ and $\partial W/\partial n \leq 0$ on $|x - y| = r$. Choose some smooth function $\psi$ such that $\mathbb{1}_{\{|x-y| \leq r+2h\}} \geq \psi \geq \mathbb{1}_{\{|x-y| \leq r+h\}}$ and, for some $\varepsilon > 0$,

$$\parallel \nabla\psi \parallel_\infty \leq (1 + \varepsilon)/h \, .$$

Applying Lemma 3.4 to $\psi\, f$ we obtain thanks to (3.8)

$$\begin{aligned}
\int_{r<|x-y|<r+h} f^2\, d\mu_{\lambda,\mathcal{X}} &\leq \int_{r<|x-y|<r+2h} (\psi\, f)^2\, d\mu_{\lambda,\mathcal{X}} \\
&\leq \frac{1}{\theta} \int_{r<|x-y|<r+2h} \frac{-LW}{W}\, (\psi\, f)^2\, d\mu_{\lambda,\mathcal{X}} \\
&\leq \frac{1}{\theta} \int_{r<|x-y|<r+2h} |\nabla f|^2\, d\mu_{\lambda,\mathcal{X}} \\
&\quad + \frac{1}{\theta} \left(\frac{1+\varepsilon}{h}\right)^2 \int_{r+h<|x-y|<r+2h} f^2\, d\mu_{\lambda,\mathcal{X}}\,.
\end{aligned}$$

Me may of course let $\varepsilon$ go to 0.

Next choose $U = \cup_i B(x_i, r_i+h_i)$, $\mathbb{1}_D \geq \chi \geq \mathbb{1}_{U^c}$ and assume that the balls $B(x_i, r_i+2h_i)$ are non overlapping. Assume that one can find Lyapunov functions $W_i$ such that $LW_i \leq -\theta_i W_i$ for $|x-x_i| \leq r_i + 2h_i$ and $\partial W_i/\partial n \leq 0$ on $|x-x_i| = r_i$. Let $h = \min h_i$, $\theta = \min \theta_i$. Using a similar argument as before we may assume that actually $\parallel \nabla \chi \parallel_\infty = \frac{1}{h}$.

The previous inequality applied to $f-m$ in each ball yields

$$\int_U (f-m)^2\, d\mu_{\lambda,\mathcal{X}} \leq \frac{1}{\theta} \int_D |\nabla f|^2\, d\mu_{\lambda,\mathcal{X}} + \frac{1}{\theta\, h^2} \int_{\mathbb{R}^d} \chi\, (f-m)^2\, d\mu_{\lambda,\mathcal{X}} \tag{3.5}$$

i.e. (3.2) is satisfied with

$$C = \frac{1}{\theta} \quad \text{and} \quad p = \lambda\, \theta\, h^4\,, \tag{3.6}$$

provided the latter is larger than 1.

We may thus apply lemma 3.3 and obtain

**Lemma 3.7.** *Let $h > 0$ and $\theta > 0$. Assume that for $h_i \geq h$ the balls $B(x_i, r_i + 2h_i)$ are non overlapping. Assume in addition that one can find Lyapunov functions $W_i$ such that $LW_i \leq -\theta_i W_i$ for $|x-x_i| \leq r_i + 2h_i$, $\partial W_i/\partial n \leq 0$ on $|x-x_i| = r_i$, $\theta_i \geq \theta$.*
*Then, provided $\lambda\, \theta\, h^4 > 1$,*

$$Var_{\mu_{\lambda,\mathcal{X}}}(f) \leq \frac{h^2\,(2 + (\theta+\lambda)\, h^2)}{\lambda\, \theta\, h^4\, -\, 1} \int_D |\nabla f|^2\, d\mu_{\lambda,\mathcal{X}}\,.$$

Hence all we have to do is to find a "good" Lyapunov function.

For the moment, $U$ will be an open ball centered at $y$. Without loss of generality (if necessary) we may assume that $y = (a, 0)$ for some $a \in \mathbb{R}^+$, 0 being the null vector of $\mathbb{R}^{d-1}$. The (non normalized) normal vector field at the boundary of $B(y,r)$, pointing inward $D$, is thus $x - y = (x^1 - a, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$.

We shall exhibit some Lyapunov function $W_y$ near the obstacle. For $|\bar{x}| \leq r + 2h$ define

$$W_y(x^1, \bar{x}) = (r + 2h + \varepsilon)^2 - |\bar{x}|^2\,.$$

Then $\nabla W_y(x^1, \bar{x}) = (0, -2\bar{x})$ and

$$\frac{\partial W_y}{\partial n}(x^1, \bar{x}) = -\,\frac{2|\bar{x}|^2}{|x-y|}\, \leq 0\,. \tag{3.8}$$

Now $LW_y = -(d-1) + 2\lambda|\bar{x}|^2$ so that $LW_y \leq -2\lambda W_y$ provided

$$d-1 \geq 2\lambda(r+2h+\varepsilon)^2. \tag{3.9}$$

As before we may let $\varepsilon$ go to 0 so that we obtain (3.5) with $\theta = 2\lambda$ and $p = 2\lambda^2 h^4 > 1$. Choosing $h = b/\sqrt{\lambda}$, with $p = 2b^4 > 1$, we see that we must have $d \geq 7$ and $r\sqrt{\lambda} \leq \sqrt{(d-1)/2} - 2b$. Finally we have shown

**Proposition 3.10.** *Let $b > 0$ and $r > 0$ be such that $2b^4 > 1$ and $r\sqrt{\lambda} \leq \sqrt{(d-1)/2} - 2b$, so that $d \geq 7$.*
*Let $D^c = \cup_i B(x_i, r_i)$ where $r_i \leq r$ for all $i$. Assume that the balls $B(x_i, r_i + 2b/\sqrt{\lambda})$ are non overlapping.*
*Then the measure $\mu_{\lambda,\mathcal{X}}$ satisfies a Poincaré inequality (1.5) with*

$$C_P(\lambda, \mathcal{X}) \leq \frac{1}{\lambda}\,\frac{b^2(3b^2+2)}{2b^4-1}.$$

The dimension dependence clearly indicates that, even for small $r$'s, we presumably did not find the good Lyapunov function. However for large dimensions we see that small enough obstacles do not alterate the finiteness of the Poincaré constant.
Also notice that if we define $\beta = \frac{2b\sqrt{2}}{\sqrt{d-1}}$ the condition on $r$ reads

$$r\sqrt{\lambda} \leq (1-\beta)\sqrt{(d-1)/2} \quad \text{for some } \beta \text{ such that} \quad 1 > \beta > \frac{2^{5/4}}{\sqrt{d-1}}. \tag{3.11}$$

In the next three subsections we shall adapt the previous method in order to cover all dimensions but for far enough obstacles.

### 3.3. Localizing away from the obstacles and the origin.

Consider now $W(x) = |x|^2$ so that for $1 > \eta > 0$,

$$LW(x) = d - 2\lambda W(x) \leq -2\lambda(1-\eta)W(x) \quad \text{for } |x| \geq \sqrt{\frac{d}{2\lambda\eta}}.$$

We will obtain some Dirichlet-Poincaré bound, i.e. we look at functions $g$ which are smooth and compactly supported in $|x| \geq \sqrt{\frac{d}{2\lambda\eta}}$ (hence vanish on the boundary of this large ball). But we also have to assume that no obstacle intersects the boundary of this region of the space. Hence we have to replace the sphere $\{|x| = \sqrt{d/2\lambda\eta}\}$ by some smooth hypersurface $S$ such that $S \subset D$ and $\sqrt{d/2\lambda\eta} \leq d(0,S) \leq c\sqrt{d/2\lambda\eta}$ for some $c > 1$ and for all $x_i \in \mathcal{X}$, $B(x_i, r_i + 3h_i) \cap S = \emptyset$. We also assume that the balls $B(x_i, r_i + 3h_i)$ are non overlapping.

It will be clear in what follows that such an $S$ does exist, but for the moment the existence of $S$ is an assumption. The whole space $D$ in thus divided in two connected components $D_0$ containing 0 and $D_\infty$ such that $S$ is the boundary of both.

We consider now the $x_i \in \mathcal{X}$ such that $B(x_i, r_i + 3h_i) \subset D_\infty$, in particular $|x_i|$ is large enough. We denote by $\mathcal{X}_\infty$ this set.

Let $g$ be compactly supported in $D_\infty$. For all $1 \leq \varepsilon \leq 2$ we apply lemma 3.4 in

$$D_\varepsilon = D_\infty \cap_{x_i \in \mathcal{X}_\infty} \{|x - x_i| \geq r_i + \varepsilon h_i\},$$

i.e.
$$\int_{D_\varepsilon} \frac{-LW}{W}\, g^2\, d\mu_{\lambda,\mathcal{X}} \;\leq\; \frac{1}{2}\int_{D_\varepsilon} |\nabla g|^2\, d\mu_{\lambda,\mathcal{X}} \,+\, \frac{1}{2}\int_{\partial D_\varepsilon} \frac{\partial W}{\partial n}\, \frac{g^2}{W}\, d\mu^\varepsilon_{\lambda,\mathcal{X}}\,,$$
where $\mu^\varepsilon_{\lambda,\mathcal{X}}$ denotes the trace of $\mu_{\lambda,\mathcal{X}}$ on the boundary $\partial D_\varepsilon$.
It yields for all $\varepsilon$ as before
$$\begin{aligned}\int_{D_\varepsilon} g^2\, d\mu_{\lambda,\mathcal{X}} &\leq \int_{D_\varepsilon} g^2\, d\mu_{\lambda,\mathcal{X}}\\ &\leq \frac{1}{2\lambda\,(1-\eta)}\int_{D_\varepsilon} \frac{-LW}{W}\, g^2\, d\mu_{\lambda,\mathcal{X}}\\ &\leq \frac{1}{4\lambda\,(1-\eta)}\int_{D_1} |\nabla g|^2\, d\mu_{\lambda,\mathcal{X}} \,+\\ &\quad + \frac{1}{4\lambda\,(1-\eta)}\int_{\partial D_\varepsilon} \frac{\partial W}{\partial n}\, \frac{g^2}{W}\, d\mu^\varepsilon_{\lambda,\mathcal{X}}\,.\end{aligned}$$
Remark that $(1/W)\,|\frac{\partial W}{\partial n}|(x) \leq 2/|x|$ so that we obtain
$$\int_{D_2} g^2\, d\mu_{\lambda,\mathcal{X}} \leq$$
$$\leq \frac{1}{4\lambda\,(1-\eta)}\left(\int_{D_1} |\nabla g|^2\, d\mu_{\lambda,\mathcal{X}} + \sum_{x_i\in\mathcal{X}_\infty} \frac{2}{(|x_i|-r_i-2h_i)}\int_{|x-x_i|=r_i+\varepsilon h_i} g^2\, d\mu^\varepsilon_{\lambda,\mathcal{X}}\right).$$
Integrating the previous inequality with respect to $\varepsilon$ for $1\leq\varepsilon\leq 2$ we obtain

**Lemma 3.12.** *With the notations of this subsection, let $g$ be a smooth function compactly supported in $D_\infty$, then*
$$\int_{D} g^2\, d\mu_{\lambda,\mathcal{X}} \leq$$
$$\leq \frac{1}{4\lambda\,(1-\eta)}\left(\int_{D_1} |\nabla g|^2\, d\mu_{\lambda,\mathcal{X}} + \sum_{x_i\in\mathcal{X}_\infty} \frac{2}{h_i\,(|x_i|-r_i-2h_i)}\int_{r_i+h_i\leq|x-x_i|\leq r_i+2\,h_i} g^2\, d\mu_{\lambda,\mathcal{X}}\right).$$

## 3.4. **Localizing away from the origin for the far enough obstacles.**

Now we shall put together the previous two localization procedures.
Remark that, during the proof of lemma 3.7 (more precisely with an immediate modification), we have shown the following : provided we can find a Lyapunov function in the neighborhood $|x-y|\leq r+3h$ of the obstacle $|x-y|\geq r$,
$$\int_{r<|x-y|<r+2h} f^2\, d\mu_{\lambda,\mathcal{X}} \leq \frac{1}{\theta}\int_{r<|x-y|<r+3h} |\nabla f|^2\, d\mu_{\lambda,\mathcal{X}} + \frac{1}{\theta\, h^2}\int_{r+2h<|x-y|<r+3h} f^2\, d\mu_{\lambda,\mathcal{X}}\,,$$
so that using the Lyapunov function $W_y$ in subsection 3.2 (yielding $\theta = 2\,\lambda$) we have, provided $d-1\geq 2\lambda\,(r+3h)^2$,
$$\int_{r<|x-y|<r+2h} f^2\, d\mu_{\lambda,\mathcal{X}} \leq \frac{1}{2\lambda}\int_{r<|x-y|<r+3h} |\nabla f|^2\, d\mu_{\lambda,\mathcal{X}} + \frac{1}{2\lambda\, h^2}\int_{r+2h<|x-y|<r+3h} f^2\, d\mu_{\lambda,\mathcal{X}}\,. \tag{3.13}$$

Hence we have to assume that, at least for the far enough obstacles, $d-1 \geq 2\lambda\,(r_i+3h_i)^2$. At the same time, lemma 3.12 shows that we have to choose $h_i$ as large as possible. So in the sequel we choose

$$\lambda = 1 \quad , \quad b<1 \quad , \quad h_i = h = \frac{b}{3}\,\sqrt{(d-1)/2} \quad , \eta = \frac{1}{2}\,.$$

In order to fulfill the conditions in the previous subsection, we have to assume that for all far enough $x_i$, (i.e. all $x_i$ such that $|x_i| > c(\sqrt{d}+\sqrt{(d-1)/2})$ for some $c \geq 1$)

$$r_i \leq (1-b)\,\sqrt{(d-1)/2}\,.$$

We thus make the following assumption

**Assumption 3.14.** *Ordering the $x_i$'s such that $|x_i| \leq |x_{i+1}|$ for all $i$, we assume that there exists some $0 \leq n < +\infty$ such that $r_i \leq (1-b)\,\sqrt{(d-1)/2}$ for some $b<1$ and all $i \geq n$. In addition we assume that for $i \geq n$ the balls $B(x_i, r_i+3h)$ are non overlapping.*

Consider now the smallest $c \geq \frac{1}{h^3\,\sqrt{d}}$ (this value will be explained below) such that the open ball $B_d = B(0, c\sqrt{d})$ contains all the $B(x_i, r_i+1)$ for $i<n$. $B_d$ can contain or intersect only a finite number of balls $B(x_i, r_i+h)$ for $i \geq n$. If such a ball is included in $B_d$ there is nothing to do. If such a ball intersects $B_d$ but is not contained in $B_d$ we may smoothly deform the boundary of $B_d$ in order to push $B(x_i, r_i+h)$ in the interior of the modified domain. We can do so for all balls intersecting the boundary and in addition in a such a way that all others $B(x_i, r_i+3h)$ are still in the exterior of the modified domain. The boundary of this deformation of $B_d$ is denoted by $S$ and it is easily seen that with this construction we are in the situation of the previous subsection.
From now on we use the notation $D_0$, $D_\infty$ and $D_\varepsilon$ introduced therein.

Now for a smooth function $g$ with compact support included in $D_\infty$, we denote

$$A = \int_{D_\infty - D_2} g^2\, d\mu_{\lambda,\mathcal{X}}\,,$$

$$B = \int_{D_2} g^2\, d\mu_{\lambda,\mathcal{X}}\,,$$

and

$$C = \int_{D_\infty} |\nabla g|^2\, d\mu_{\lambda,\mathcal{X}}\,.$$

According to (3.13) and to lemma 3.12, we obtain

$$A \leq \frac{1}{2}\,\left(C + \frac{1}{h^2}\,B\right) \quad \text{ and } \quad B \leq \frac{1}{2}\,\left(C + \frac{2}{h\,c\sqrt{d}}\,A\right)\,.$$

Hence,

$$A \,\leq\, \frac{1}{2}\,\left(1+\frac{1}{2\,h^2}\right)\,C \,+\, \frac{1}{2\,h^3\,c\,\sqrt{d}}\,A\,,$$

and thanks to our choice of $c$ we get finally

$$A \,\leq\, \left(1+\frac{1}{2\,h^2}\right)\,C \quad , \quad B \leq\, \left(1+h^2\right)\,C\,.$$

This yields

**Lemma 3.15.** *Let $0 < b < 1$ and $h = \frac{b}{3}\sqrt{(d-1)/2}$. Assume that $\lambda = 1$, and Assumption 3.14 is satisfied. Then, for all smooth function $g$, compactly supported in $D_\infty$ (which depends on $b$), it holds*

$$\int g^2 \, d\mu_{\lambda,\mathcal{X}} \, \leq \, K \int |\nabla g|^2 \, d\mu_{\lambda,\mathcal{X}} \, ,$$

*with*

$$K = 2 + \frac{1}{2h^2} + h^2 \, .$$

### 3.5. Localizing around the origin for a far enough single obstacle.

Assume that $N = 1$ and that the single obstacle is far enough, i.e. $n = 0$ in Assumption 3.14. Notice that in this situation $D_\infty$ is simply the exterior of a large ball $B(0, c\sqrt{d})$. To get some bound for the Poincaré constant, it remains now to follow the method in [BBCG08, CGZ13]. Let $f$ be a smooth function with compact support. Assume that we are in the situation of lemma 3.15 (in particular $\lambda = 1$).

Recall that $\mu_{\lambda,\mathcal{X}}$ restricted to the ball $\{|x| \leq c\sqrt{d}\}$ is just the gaussian measure restricted to the ball (since this ball does not intersect the obstacle), hence satisfies a Poincaré inequality with a constant less than $\frac{1}{2}$. If

$$m = \int_{|x| \leq c\sqrt{d}} f \, d\mu_{\lambda,\mathcal{X}} / \mu_{\lambda,\mathcal{X}}(|x| \leq c\sqrt{d}) \, ,$$

we have

$$\mathrm{Var}_{\mu_{\lambda,\mathcal{X}}}(f) \leq \int_D (f-m)^2 d\mu_{\lambda,\mathcal{X}}$$

so that it is enough to control the second moment of $\bar{f} = f - m$.

We write

$$\bar{f} = \chi \, \bar{f} + (1-\chi) \, \bar{f} = \chi \bar{f} + g$$

where $\chi$ is 1-Lipschitz and such that $\mathbb{1}_{|x| \leq c\sqrt{d}-1} \leq \chi \leq \mathbb{1}_{|x| \leq c\sqrt{d}}$. $g$ is thus compactly supported in $|x| \geq c\sqrt{d}$ so that we may apply what precedes. In particular

$$\begin{aligned}
\int_D \bar{f}^2 \, d\mu_{\lambda,\mathcal{X}} \;&\leq\; 2 \int_{|x| \leq c\sqrt{d}} \bar{f}^2 \, d\mu_{\lambda,\mathcal{X}} + 2 \int_D g^2 \, d\mu_{\lambda,\mathcal{X}} \\
&\leq\; \int_{|x| \leq c\sqrt{d}} |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} + 2K \int_D |\nabla g|^2 \, d\mu_{\lambda,\mathcal{X}} \\
&\leq\; \int_{|x| \leq c\sqrt{d}} |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} + 4K \int_{x \in D, |x| \geq c\sqrt{d}-1} |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} + \\
&\qquad\quad + 4K \int_{c\sqrt{d} \geq |x| \geq c\sqrt{d}-1} \bar{f}^2 \, d\mu_{\lambda,\mathcal{X}} \\
&\leq\; (1+2K) \int_{|x| \leq c\sqrt{d}} |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} + 4K \int_{x \in D, |x| \geq c\sqrt{d}-1} |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} \\
&\leq\; (1+6K) \int_D |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} \, .
\end{aligned}$$

We have thus proved, using (1.6)

**Proposition 3.16.** *Assume that $N=1$, and that for some $0<b<1$, we have $r\sqrt{\lambda} \leq (1-b)\sqrt{(d-1)/2}$, $h=\frac{b}{3}\sqrt{(d-1)/2}$ and $|y|\sqrt{\lambda} > c\sqrt{d}$ for $c > 1/(h^3\sqrt{d})$. Then the measure $\mu_{\lambda,\mathcal{X}}$ satisfies a Poincaré inequality* (1.5) *with*

$$C_P(\lambda, B(y,r)) \leq \frac{1}{\lambda}\,(1+6K)\ ,$$

*with $K = 2 + h^2 + \frac{1}{2h^2}$.*

The main interest of the previous proposition is that it shows that for a single far enough small obstacle the Poincaré constant does not depend on the location of the obstacle. We also have tried to trace a little bit the constants to show that we obtain some tractable explicit upper bound, the final step being to optimize in $b$ (left to the reader).

### 3.6. **A general result for a single obstacle with small radius.**

We can gather together all the previous results in the case $N=1$. For the sake of simplicity the next theorem is not optimal, but readable.

**Theorem 3.17.** *There exists some universal constant $\kappa$ such that if*

$$r\sqrt{\lambda} \leq \frac{1}{2}\sqrt{(d-1)/2}\,,$$

*the measure $\mu_{\lambda,\mathcal{X}}$ where $\mathcal{X}=\{y\}$ is a singleton, satisfies a Poincaré inequality* (1.5) *with*

$$C_P(\lambda, B(y,r)) \leq \frac{\kappa}{\lambda}\,.$$

*Proof.* If $d$ is big enough ($d \geq 33$) we may use Proposition 3.10. If $d \leq 33$ and $|y|\sqrt{\lambda}$ large, we may apply Proposition 3.16 with $b = 1/(2\sqrt{d-1})$. Finally, if $d \leq 33$ and $|y|\sqrt{\lambda}$ is small we may use Proposition 2.4. □

**Remark 3.18.** In comparison with Proposition 3.10, we have spent a rather formidable energy in order to cover the small dimension situation. But the alternate method we have developed for large $|y|$ will be useful in other contexts, in particular for an infinite number of obstacles.
It is also worth noticing that we have used Proposition 2.4 that cannot be extended to more than one obstacle. ♢

### 3.7. **The case of infinitely many obstacles.**

Now consider the case with more than obstacle. If we look at the localization procedure in subsection 3.5 we see that a key point is to get the value (or a bound) for the Poincaré constant in a neighborhood of the origin. If all obstacle are far enough we can mimic what is done in subsection 3.5. But in general, the $n$ introduced in Assumption 3.14 is not equal to 0, so that we have to look at the Poincaré constant in $D_0$. Since this set is compact and

with a smooth boundary, the finiteness of the Poincaré constant is ensured, for instance by the Down-Meyn-Tweedie theory as we indicate in the introduction.
Unfortunately it is very hard to get some explicit upper bound of this constant depending on all points $x_i$ in $\mathcal{X}$ such that the obstacles $B(x_i, r_i)$ are subsets of $D_0$. Exactly the same problem occurs in [DLM11] where the value of the Poincaré constant (or the spectral gap) for the parameter $\varepsilon$ (using the notations therein) is shown to be quadratic in $\varepsilon$, but with an unknown constant pre-factor.

We can nevertheless mimic what we did in subsection 3.5 replacing the value 1/2 by the unknown Poincaré constant in $D_0$. This yields

**Theorem 3.19.** *For any $1 \leq N \leq +\infty$ (in particular $N = +\infty$), under Assumption 3.14, $\mu_{\lambda,\mathcal{X}}$ satisfies a Poincaré inequality with constant $C_P(\lambda, \mathcal{X}) = \frac{\kappa}{\lambda} < +\infty$ where $\kappa$ depends on $n$, $d$ and the structure of the (finite) number of the obstacles that are close to the origin.*
*More precisely, with the notations of Proposition 3.16, $\kappa \leq 4K + (2+4K)C_P(n)$ where $C_P(n)$ denotes the Poincaré constant in $D_0$.*

**Corollary 3.20.** *Ordering the $x_i$'s such that $|x_i| \leq |x_{i+1}|$ for all $i$, assume that there exists some $0 \leq n < +\infty$ such that $r_i \leq (1-b)\sqrt{(d-1)/2}$ for some $b < 1$ and all $i \geq n$, and that in addition there exists $\varepsilon > 0$ such that for all pair $i \neq j$, $dist(B(x_i, r_i), B(x_j, r_j)) \geq \varepsilon$.*
*Then $C_P(\lambda, \mathcal{X}) < +\infty$.*

*Proof.* Take $b' = \left(\varepsilon/2\sqrt{(d-1)/2}\right) \wedge b$. The condition on the radii $r_i$ is still satisfied replacing $b$ by $b'$ while $h = \frac{b'}{3}\sqrt{(d-1)/2}$ satisfies $6h \leq \varepsilon$. Hence the balls with radii $r_i + 3h$ are non overlapping and we may apply the previous Theorem. □

### 3.8. **Others obstacles like hypercubes.**

Replacing euclidean balls by others geometries of obstacles requires first to find a Lyapunov function in the neighborhood of each obstacle as in subsection 3.2. We will not discuss this in details here, but only consider the case where we replace the euclidean ball $B(x_i, r_i)$ by some hypercube, in a nice position.
Namely we consider the $x$'s such that $x = x_i + (z\,x_i + y_i)$ where $y_i$ belongs to the hyperplane orthogonal to $x_i$ intersected with the $d-1$ $l^\infty$ ball of radius $r_i\sqrt{d}$ and $z \in [-r_i\sqrt{d}, r_i\sqrt{d}]$. In other words we consider hypercubes in $d$ dimensions such that, first the line connecting the origin to the center of mass $x_i$ of the hypercube is orthogonal to some face of the latter, second the hypercube is included in the euclidean ball $B(x_i, r_i)$.
In this situation the function $W_{x_i}$ introduced in subsection 3.2 (replacing $y$ by $x_i$) is still a Lyapunov function with a non-positive normal derivative on the boundary of the hypercube. The reader who is afraid by the singularities of the boundary can "smooth the corners".
The results in subsections 3.6 and 3.7 easily extend, but this time with $r_i \leq b$ for some constant $b$ independent of the dimension. Of course we have to assume that all the obstacles are in the nice position described above.

## 4. General spherical obstacles using stochastic calculus.

As we have seen, provided we are able to find a good Lyapunov function near the obstacles, we are able to control (even if not explicitly) the Poincaré constant in $D$. The choice we made in the previous section implies a limitation for the radius of the obstacles. What we shall do now is to find a new Lyapunov function near the obstacles. This Lyapunov function will be built by trying to understand how fast the process goes around the obstacles.

Indeed recall the following results on the exponential moments of hitting times (see e.g.[CGZ13]).

**Proposition 4.1.** *Let $U$ be a bounded connected subset with smooth boundary of $D$ and $T_U$ denotes the hitting time of $U$.*

- *Assume that for some $\theta > 0$ and all $x \in D$, $\mathbb{E}_x\left(e^{\theta T_U}\right) < +\infty$. Define $W(x) = \mathbb{E}_x\left(e^{\theta T_U}\right)$. Then $W$ belongs to the domain of the generator $L$ of the reflected Ornstein -Uhlenbeck process (in particular $\partial W/\partial n = 0$ on $\partial D$), and satisfies $LW \leq -\theta W$ outside of $U$.*
- *For all $x \in D$,*

$$\mathbb{E}_x\left(e^{\theta T_U}\right) < +\infty \quad \text{for all } \theta < \theta(U), \text{ with } \theta(U) = \frac{\mu_{\lambda,\mathcal{X}}(U)}{16\, C_P(\lambda, \mathcal{X})}\,.$$

Actually, [CGZ13] only dealt with diffusion processes, without reflection. But the proof of this Proposition lies on three facts which are still true here: the symmetry of $\mu_{\lambda,\mathcal{X}}$, the existence of a density for the law at time $t > 0$ of the process starting at any $x$, the results of Proposition 1.4 and Remark 1.6 in [CG08] which hold true for general Markov processes with a square gradient operator.

Hence provided we can control exponential moments of hitting times, we can build (non explicit) Lyapunov functions.

The discussion below is done for a single obstacle $B(y, r)$. We shall conclude at the end of the section for more than one obstacle.

### 4.1. **The rate of rotation.**

To understand how fast the process goes around the obstacle, we introduce a new stochastic process $Y_t$ which is just the reflected Ornstein-Uhlenbeck process in the shell $S = \{r \leq |x - y| \leq r + q\}$ for some positive $q$, i.e

$$\begin{cases} dY_t = dW_t \, - \, \lambda\, Y_t\, dt \, + \, (Y_t - y)\, dL_t\,, \\ L_t = \int_0^t \left(\mathbb{1}_{|Y_s - y| = r} - \mathbb{1}_{|Y_s - y| = r+q}\right) dL_s. \end{cases} \tag{4.2}$$

Next as usual, we assume that $y = (a, 0)$ and write the generic point of the euclidean space as $x = (x^1, \bar{x})$. Again $n$ denotes the normal vector field $(x^1 - a, \bar{x})$ (pointing either inward or outward), so that, for any nice function $g$, Ito formula yields

$$g(Y_t) = g(Y_0) + \int_0^t \nabla g(Y_s).dW_s + \int_0^t Lg(Y_s)ds + r\int_0^t \frac{\partial g}{\partial n}(Y_s)\, dL_s\,.$$

Finally we shall look at the process

$$Z_t = \arccos\left(\frac{Y_t^1 - a}{\sqrt{|\bar{Y}_t|^2 + (Y_t^1 - a)^2}}\right) \, = \, \varphi(Y_t)\,. \tag{4.3}$$

We can calculate

$$\nabla\varphi(x) = \left(\frac{-|\bar{x}|}{(x^1-a)^2+|\bar{x}|^2}\,,\;\frac{(x^1-a)\,\bar{x}}{|\bar{x}|\,((x^1-a)^2+|\bar{x}|^2)}\right) \quad \text{so that} \quad \frac{\partial\varphi}{\partial n}\,(x) = 0\,.$$

Consider

$$M = \{-r-q \leq x^1 - a \leq -r\,,\; \bar{x} = 0\}\,.$$

If $Y_0 \notin M$, i.e. $Z_0 \neq \pi$, we may apply Ito-Tanaka formula up to time $T_M$ (the first time $Y_.$ hits $M$) yielding for $t < T_M$,

$$\begin{aligned} Z_t^2 &= Z_0^2 + \int_0^t 2Z_s\,\langle \nabla\varphi(Y_s), dW_s\rangle + \int_0^t |\nabla\varphi(Y_s)|^2\,ds \qquad (4.4)\\ &\quad + \int_0^t \frac{Z_s(2\lambda\,a\,|\bar{Y}_s| + (d-2)\,(Y_s^1-a))}{|\bar{Y}_s|^2 + (Y_s^1-a)^2}\,ds\\ &= Z_0^2 + \int_0^t \frac{2Z_s}{\left(|\bar{Y}_s|^2+(Y_s^1-a)^2\right)^{1/2}}\,dB_s + \int_0^t \frac{1+Z_s(2\lambda\,a\,|\bar{Y}_s| + (d-2)\,(Y_s^1-a))}{|\bar{Y}_s|^2+(Y_s^1-a)^2}\,ds \end{aligned}$$

where $B_.$ is a new standard Brownian motion. We have considered $Z^2$ instead of $Z$ to kill the local time at 0 of $Z_.$ (since $t < T_M$ the local time of $Z_.$ at $\pi$ does not appear too).

Introduce the subset

$$K = \{x^1 - a < 0\,,\; |\bar{x}| \leq \eta < r\} \cap S\,.$$

Since $M \subset K$ we know that $T_K \leq T_M$ so that (4.4) holds for $t \leq T_K$. We want to estimate $T_K$ by comparing $Z_t$ with a simpler diffusion process for which estimates are easy to obtain (since they are known).

Set

$$A(t) = \int_0^t \frac{1}{\left(|\bar{Y}_s|^2+(Y_s^1-a)^2\right)}\,ds\,,$$

and $A^{-1}(t)$ the inverse of $A(.)$. Notice that $(t/(r+q)^2) \leq A(t) \leq (t/r^2)$ so that $r^2 t \leq A^{-1}(t) \leq (r+q)^2 t$.

Define the time changed process $\tilde{Y}_t = Y_{A^{-1}(t)} = (\tilde{Y}_t^1, \tilde{Y}_t^2)$ and $U_t = Z^2_{A^{-1}(t)}$. Then for $t < A(T_M)$, $U_.$ satisfies

$$U_t = Z_0^2 + \int_0^t 2\,\sqrt{U_s}\,d\tilde{B}_s + \int_0^t \left(1+\sqrt{U_s}\,(2\lambda\,a|\tilde{Y}_s^2| + (d-2)\,(\tilde{Y}_s^1-a))\right)ds\,, \qquad (4.5)$$

for some new Brownian motion $\tilde{B}_.$. In order to compare $U_.$ with some CIR process (see Appendix B) we have to bound the drift term from below.

Remark that for a point $\tilde{y} \in K^c$,

$$|\tilde{y}^2| = \sqrt{(|\tilde{y}^2|)^2 + (\tilde{y}^1-a)^2}\,\sin(\sqrt{u}) \geq r\,\sin(\sqrt{u}) \geq \frac{\eta}{\pi}\,\sqrt{u}\,.$$

Hence looking separately at the case $\tilde{y}^1 - a > 0$ and $\tilde{y}^1 - a \leq 0$ it follows that the drift term satisfies

$$1 + \sqrt{U_s}\,(2\lambda\,a|\tilde{Y}_s^2| + (d-2)\,(\tilde{Y}_s^1-a)) \;\geq\; 1 + \left(\frac{2\lambda\,a\,\eta - (d-2)(r+q)}{\pi}\right)\,U_s\,.$$

Hence up to time $T_K$, using standard comparison results for one dimensional diffusions, we know that $U_t \geq V_t$ where

$$dV_t = 2\sqrt{V_t}\, d\tilde{B}_t + (1 + 2\beta V_t)dt$$

i.e. $V_.$ is a generalized squared radial Ornstein-Uhlenbeck process, with $\beta = \frac{\lambda\, a\, \eta - (d-2)(r+q)}{\pi}$ and $\delta = 1$ provided $\beta \geq 0$.

It follows that $A(T_K)$ is smaller than the first hitting time of $\pi$ by $V_.$. According to (B.4), we thus have

$$\mathbb{E}_x(e^{\theta T_K}) < +\infty \quad \text{for all } x \in S \text{ provided} \quad \theta < \frac{\beta}{(r+q)^2}\,.$$

It is thus tempting to define $W(x) = \mathbb{E}_x\left(e^{\theta\, T_K}\right)$, which satisfies $LW = -\theta W$ in $S - K$. This is not yet enough but will be useful.

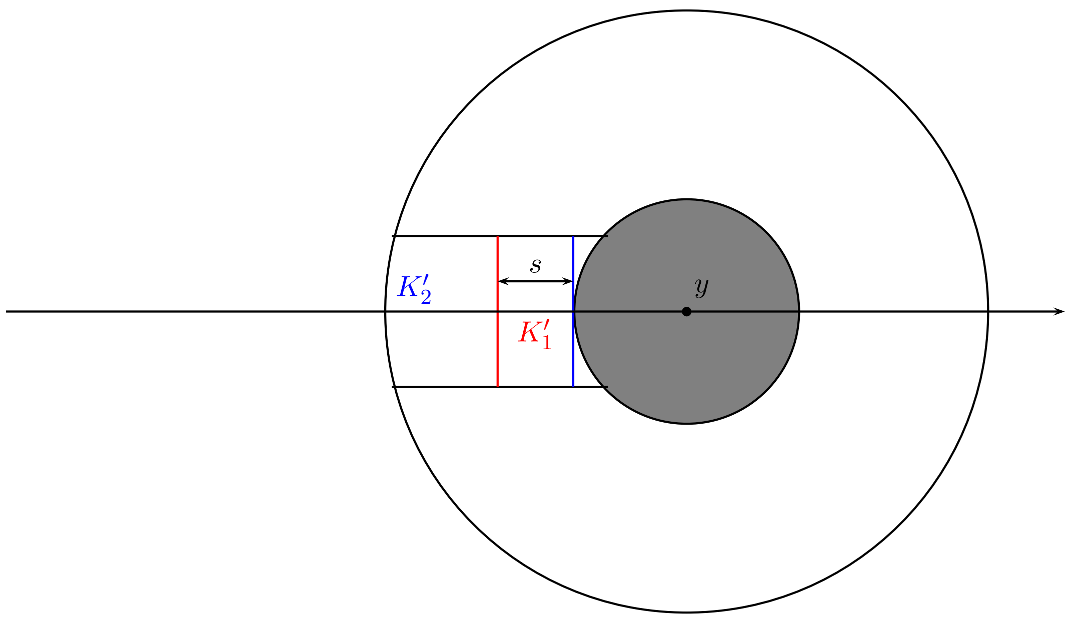


FIGURE 2. Rotation around the obstacle.

4.2. **The Poincaré inequality in the shell $S$.**

Using what precedes we shall prove the following first result

**Proposition 4.6.** *Let $\eta$, $s$, $q$ be positive numbers such that $\eta + s < r$, $s < q$ and*

$$\beta = \frac{\lambda\, a\, \eta - (d-2)(r+q)}{\pi} > 0\,.$$

*Assume that $a > r + s + \frac{1}{\sqrt{\lambda}}$.*

*Then, the (non normalized) restriction of $\mu_{\lambda,r}$ to the shell $S=\{r\leq|x-y|\leq r+q\}$ satisfies a Poincaré inequality*

$$\int_S f^2\,d\mu_{\lambda,r}\leq C_P(\lambda,S)\int_S|\nabla f|^2\,d\mu_{\lambda,r}+\frac{1}{\mu_{\lambda,r}(S)}\left(\int_S f\,d\mu_{\lambda,r}\right)^2$$

*where*

$$C_P(\lambda,S)\leq\frac{2(r+q)^2}{\beta}+\frac{1}{\lambda}\left(1+\frac{2(r+q)^2}{\beta\,s^2}\right)\left(\frac{2}{\lambda s^2}+\frac{5}{2}\right).$$

*Proof.* We shall use the results in the previous subsection. Define $W(x)=\mathbb{E}_x\left(e^{\theta T_K}\right)$ for $x\in S$. Then $W$ belongs to the domain of the generator of $Y_.$ (in particular the normal derivative on the shell's boundary vanishes) and satisfies $LW=-\theta W$ in $S-K$.
Consider now

$$K'=\left\{x^1-a<0\,,\,|\bar{x}|\leq\eta+s<r\right\}\cap S\,.$$

Then as before, using [CGZ13] formula (2.14) (in the present framework of our reflected Ornstein-Uhlenbeck process $Y_.$), we have

$$C_P(\lambda,S)\leq\frac{2}{\theta}+\left(\frac{2}{\theta\,s^2}+1\right)C_P(K')\,. \tag{4.7}$$

It remains to get some bound for $C_P(K')$.

Again we divide $K'$ in two overlapping parts:

$$K'_1=\left\{-r-q<-r-s<x^1-a<0\,,\,|\bar{x}|\leq\eta+s<r\right\}\cap S$$

and

$$K'_2=\left\{x^1-a<-r\,,\,|\bar{x}|\leq\eta+s<r\right\}\cap S\,.$$

Note that $K'_2$ is convex. Hence the restriction of the gaussian measure to $K'_2$ satisfies a Poincaré inequality with constant $1/2\lambda$.
As before it is then sufficient to build some Lyapunov function in $K'_1$. This time we choose $W(x)=(x^1)^2$. Note that, on one hand, the normal derivative of $W$ on $|\bar{x}|=\eta+s$ is equal to 0, while on the other hand, the (non normalized) inward normal derivative of $W$ on $|x-y|=r$ is equal to $2(x^1-a)x^1$. The latter is thus negative provided $x^1>0$, hence in particular if $a>r+s$.
In addition,

$$LW(x)=1-2\lambda(x^1)^2\leq-\lambda(x^1)^2\quad\text{ in }K'_1 \tag{4.8}$$

as soon as $a>r+s+(1/\sqrt{\lambda})$. Thus, as before we obtain

$$C_P(K')\leq\frac{1}{\lambda}\left(\frac{2}{\lambda s^2}+\frac{5}{2}\right)\,.$$

□

Now let $a'>r'+s'+1$, $y'=(a',0)$, $\eta'+s'<r'$. Define $(a,r,s,q,\eta)=\frac{1}{\sqrt{\lambda}}\,(a',b',c',q',\eta')$ so that $a>r+s+\frac{1}{\sqrt{\lambda}}$. Define $S'=\{r'\leq|x-y'|\leq r'+q'\}$. The homogeneity property (1.6) is still available in our situation yielding

$$C_P(1,S')=\lambda C_P(\lambda,S)$$

for all $\lambda$. Hence

$$C_P(1,S') \;\leq\; \frac{2\pi(r'+q')^2}{a'\eta' - (d-2)\frac{r'+q'}{\sqrt{\lambda}}} + \left(1+\frac{2(r'+q')^2}{\lambda\beta'\,(s')^2}\right)\left(\frac{2}{(s')^2}+\frac{5}{2}\right)$$

where the meaning of $\beta'$ is clear. Now we may let $\lambda$ go to infinity and obtain

**Proposition 4.9.** *Let $s<q$, and assume that $a>r+s+1$. Let $0<\eta<r-s$. Then, the (non normalized) restriction of $\mu_{1,r}$ to the shell $S=\{r\leq |x-y|\leq r+q\}$ satisfies a Poincaré inequality*

$$\int_S f^2\,d\mu_{1,r} \leq C_P(1,S)\int_S |\nabla f|^2\,d\mu_{1,r} + \frac{1}{\mu_{1,r}(S)}\left(\int_S f\,d\mu_{1,r}\right)^2$$

*where*

$$C_P(1,S) \leq \frac{2\pi(r+q)^2}{a\,\eta} + \left(\frac{2}{s^2}+\frac{5}{2}\right).$$

### 4.3. A new estimate for an obstacle which is not too close to the origin.

We may use Proposition 4.9 to build a new Lyapunov function near the obstacle when $\lambda = 1$.

In the situation of the proposition consider $T_{S/2}$ the hitting time of the "half" shell $S'=\{r+(q/2)\leq |x-y|\leq r+q\}$. Then according to proposition 4.1 we may define $W(x)=\mathbb{E}_x\left(e^{\theta T_{S/2}}\right)$ which satisfies $LW=-\theta W$ for $x\in S-S'$ and $\partial W/\partial n = 0$ on $|x-y|=r$, provided

$$\theta < \frac{1}{8C_P(1,S)}\,\frac{\mu_{1,r}(S')}{\mu_{1,r}(S)}\,. \tag{4.10}$$

Now we can first apply lemma 3.7 with $2h=q/2$, provided $\theta h^4>1$.

It remains to choose all parameters. All conditions are satisfied for instance if

$$\frac{q^4}{4^4}\,\frac{1}{16C_P(1,S)}\,\frac{\mu_{1,r}(S')}{\mu_{1,r}(S)} > 1\,. \tag{4.11}$$

It is not too difficult to be convinced that the ratio of the two measures is uniformly (in $r$ and $y$) bounded from below, provided $a-r-q>1$ (1 can be replaced by any positive constant), i.e. provided the origin is far enough from $B(y,r+q)$. Indeed the measure restricted to $S$ is mainly concentrated near the point $(a-r-q,0)$ which belongs to both $S$ and $S'$.

Now look at the bound in Proposition 4.9. If $r$ is small (goes to 0), the bound for a given $a$ becomes very bad. Indeed, for $2/s^2$ to be nice, we have to choose $s$ bounded from below, so that $q$ is bounded from below too and since $\eta<r$ the term governed by $1/a\eta$ explodes.

Hence we may choose $r>(1-b)\sqrt{(d-1)/2}$ in order to cover the case which is not covered by Theorem 3.17, or simply $r>\frac{1}{2}$.

Now, we have clearly to choose $q$ as small as possible, but satisfying (4.11). To simplify choose $s=1$ so that $C_P(1,S)\leq c$ where $c$ is of order $5+(2\pi\,r(1+\frac{q}{r})^2/a)$. We see that for (4.11) to be satisfied we need $q$ to be greater than a constant of order at least 10. We have obtained, with $m=q+1$

**Proposition 4.12.** *One can find universal constants $m > 0$ and $C$ such that, provided $|y| > r + m$ and $r > \frac{1}{2}$, $C_P(1, B(y,r)) \leq C$.*

### 4.4. Finiteness of the Poincaré constant for an infinite number of spherical obstacles.

Of course we can use the previous construction of a Lyapunov function near the far enough obstacles together with the ideas of subsection 3.7 to cover the case of infinitely many obstacles. To this end, instead of using Lemma 3.7 we should also follow what we have done in subsections 3.3 and 3.4, i.e. replace $2\lambda$ ($= 2$ here) by $\theta$ defined above in (3.13). But we have to be accurate when using Lyapunov functions near the obstacles, that the enlargements we are using are non overlapping. In particular $q$ and $s$ have to be smaller than the half of the distance between obstacles.

**Theorem 4.13.** *Let $\mathcal{X} = (x_i)_{1\leq i<+\infty}$ a locally finite collection of distinct points, ordered such that $|x_i| \leq |x_{i+1}|$ for all $i$, and $\mathcal{R} = (r_i)_{1\leq i<+\infty}$ a collection of non-negative numbers. Assume that there exists $\varepsilon > 0$ with $|x_i - x_j| > r_i + r_j + \varepsilon$ for all $i \neq j$. Define $D = \mathbb{R}^d - \cup_i B(x_i, r_i)$ (for $d \geq 2$) where $B(y,r)$ denotes the euclidean ball with center $y$ and radius $r$.*
*Then for any $\lambda > 0$, the gaussian measure $\mu_{\lambda,\mathcal{X}}$ has a finite Poincaré constant and the reflected Ornstein-Uhlenbeck process in $D$ is exponentially ergodic.*

*Proof.* Since the conditions are still satisfied when dilating the space we may assume that $\lambda = 1$.
As for the proof of Corollary 3.20 we shall use the Lyapunov functions near the obstacles outside some large enough smooth subset containing the origin to be determined during the proof.
For small obstacles ($r_i \leq \frac{1}{2}\sqrt{d-1}$ for instance) we use the Lyapunov function in subsection 3.4. For the large obstacles we use the one introduced in the previous subsection. With the notations of subsection 3.4, and still with $h_i = h$, we obtain

$$A \leq \frac{1}{\theta}\left(C + \frac{1}{h^2}B\right) \quad , \quad B \leq \frac{1}{2}\left(C + \frac{2}{\alpha\, h}A\right) ,$$

where $\alpha = \min\{i \text{ large}\,;\, |x_i| - r_i - 2h\}$.
We have to choose $h, q, s$ of order $\varepsilon$ (up to well chosen constants), so that for $A$ and $B$ to be controlled by $C$ it is enough that $h^3\,\alpha \geq c(1 + (1/\varepsilon^2))$ for a large enough $c$.
But it is not difficult to see that $|x_i| - r_i \to +\infty$ as $i \to +\infty$, so that there exists a large enough constant $c > 0$ such that $|x_i| - r_i \geq c(1 + (1/\varepsilon^5))$ and we can conclude. □

## 5. Lower bounds for non spherical obstacles.

We obtained in the previous section that for far enough obstacles, the radii of the obstacles do not really increase the value of the Poincaré constant. Hence, roughly speaking, the only radius that really matters is the one of the obstacle containing the origin if such an obstacle exits (of course we did not prove the result in this so general form). But actually this property is strongly linked to the geometry of the obstacle, and we shall see below that replacing spherical obstacles by other geometries will drastically modify the result.

### 5.1. Lower bounds for Hypercubes via stochastic calculus.

Replace the ball $\{|x-y|<r\}$ with $y=(a,0)$ ($a \geq 0$) by an hypercube, $H_r = \{|x^1-a| < r\,,\, |x^j|<r \text{ for } j\geq 2\}$. As we already said, we may "smooth the corners" for the boundary to be smooth (replacing $r$ by $r+\varepsilon$), so that existence, uniqueness and properties of the reflected process are similar to those we have mentioned for the disc.

Consider the process $X_t$ starting from $x=(a+r,0)$. Denote by $S(r)=\min_{j\geq 2} S^j(r)$, where $S^j(r)$ is the exit time of $[-r,r]$ by the coordinate $X^j_.$. Up to time $S(r)$, the $X^j_.$'s ($j\geq 2$) are Ornstein-Uhlenbeck processes, starting at 0, $X^1_.$ is an Ornstein-Uhlenbeck process reflected on $a+r$, starting at $a+r$; and all are independent. Of course $S(r)=T_{U^c(r)}$ where the set $U(r)=\{x^1\geq a+r\,;\, \max_{j\geq 2}|x^j|\leq r\}$.
According to Proposition 4.1, if

$$\mathbb{E}_{(a+r,0)}\left(e^{\theta\, S(r)}\right)=+\infty \quad \text{then} \quad C_P(\lambda,\mathcal{X}) \geq \frac{\mu_{\lambda,\mathcal{X}}(U_r^c)}{16\,\theta}\,. \tag{5.1}$$

But according to (B.3) and to the independence of the coordinates of the process, this holds as soon as $\theta > \frac{(d-1)\lambda}{e^{\lambda r^2}-1}$. In particular since $\mu_{\lambda,\mathcal{X}}(U^c(r))\geq \frac{1}{2}$, we always have

**Theorem 5.2.** *Let $D=\mathbb{R}^d - H_r$ where $H_r$ is the hypercube described above. Then there exists an universal constant $C$ such that the Poincaré constant in $D$ satisfies $C_P(\lambda,r)\geq \frac{C\,e^{(\lambda r^2)}}{d\lambda}$.*

Recall that we have shown that for small enough obstacles ($r$ of order a dimension free constant) the Poincaré constant is bounded from above by some $\kappa/\lambda$.
What is interesting here is that the lower bound does not depend on the location of $y$. In particular consider the situation of Theorem 4.13 with an infinite number of hypercubes as obstacles, in the position described in subsection 3.8, i.e. the line joining the origin to the center of mass of each hypercube is orthogonal to some face of the latter. Of course for far enough obstacles the measure of $U_i^c(r_i)$ will still be larger than one half. So *if we allow the existence of a sequence of radii going to infinity the process is no more exponentially ergodic.*

### 5.2. An isoperimetric approach for hypercubes.

In this subsection, we present another approach for getting lower bounds. The easiest way to build functions allowing to see the lower bounds we have obtained in the previous subsection, is first to look at indicator of sets, hence isoperimetric bounds.
We define the Cheeger constant $C_C(\lambda,y,r)$ as the smallest constant such that for all subset $A\subset D$ with $\mu_{\lambda,\mathcal{X}}(A)\leq \frac{1}{2}$,

$$C_C(\lambda,y,r)\,\mu^S_{\lambda,\mathcal{X}}(\partial A)\geq\, \mu_{\lambda,\mathcal{X}}(A)\,. \tag{5.3}$$

Recall that $\mu^S_{\lambda,\mathcal{X}}(\partial A)$ denotes the surface measure of the boundary of $A$ in $D$ defined as

$$\liminf_{h\to 0}\,\frac{1}{h}\,\mu_{\lambda,\mathcal{X}}(A_h/A)$$

where $A_h$ denotes the euclidean enlargement of $A$ of size $h$. The important fact here is that $A$ is considered as a subset of $D$. In particular, if we denote by $\partial S_r$ the boundary of the square $S_r$ in the plane $\mathbb{R}^2$, $A \cap \partial S_r \subset D$ and so does not belong to the boundary of $A$ in $D$. The Cheeger constant is related to the $\mathbb{L}^1$ Poincaré inequality, and it is well known that

$$C_P \, \leq \, 4 \, C_C^2 \, , \tag{5.4}$$

while $C_P$ can be finite but $C_C$ infinite. Hence an upper bound for the Cheeger constant will provide us with an upper bound for the Poincaré constant while a lower bound can only be a hint.

5.2.1. *Squared obstacle.* For simplicity we shall first assume that $d = 2$, and use the notation in subsection 5.1. Consider for $a > 0$, the subset $A = \{x^1 \geq a + r \, , \, |x^2| \leq r\}$ with boundary $\partial A = \{x^1 \geq a + r \, , \, |x^2| = r\}$.
Recall the basic inequalities, for $0 < b < c \leq +\infty$,

$$\frac{b^2}{1+2b^2} \, \left(\frac{e^{-\,b^2}}{b} \, - \, \frac{e^{-\,c^2}}{c}\right) \leq \int_b^c \, e^{-\,u^2} \, du \leq \frac{1}{2b} \, \left(e^{-\,b^2} \, - \, e^{-\,c^2}\right) \, . \tag{5.5}$$

It follows, for $r\sqrt{\lambda}$ large enough (say larger than one)

$$\begin{aligned} \frac{\mu_{\lambda,\mathcal{X}}(A)}{\mu^S_{\lambda,\mathcal{X}}(\partial A)} \quad &= \quad \frac{\left(\int_{a+r}^{+\infty} e^{-\lambda z^2} \, dz\right) \, \left(\int_{-r}^{+r} e^{-\lambda u^2} \, du\right)}{2 \, e^{-\lambda r^2} \, \left(\int_{a+r}^{+\infty} e^{-\lambda z^2} \, dz\right)} \\ &\geq \quad \frac{1}{2\,\sqrt{\lambda}} \, e^{\lambda r^2} \, \left(1 \, - \, \frac{1}{r\sqrt{\lambda}} \, e^{-\lambda r^2}\right) \, , \end{aligned}$$

so that

$$C_C(\lambda, y, r) \geq \frac{1}{2\,\sqrt{\lambda}} \, e^{\lambda r^2} \, \left(1 \, - \, \frac{1}{r\sqrt{\lambda}} \, e^{-\lambda r^2}\right) \, . \tag{5.6}$$

Note that this lower bound is larger than the one obtained by combining Cheeger's inequality (5.4) and the lower bound for the Poincaré constant obtained in Theorem **??**, since this combination furnishes an explosion like $e^{\lambda r^2/2}$.

We strongly suspect, though we did not find a rigorous proof, that this set is "almost" the isoperimetric set, in other words that, up to some universal constant, the previous lower bound is also an upper bound for the Cheeger constant. In particular, we believe that this upper bound (hence the upper bound for the Poincaré constant) does not depend on $a$. Of course, since we know that the isoperimetric constant of the gaussian measure behaves like $1/\sqrt{\lambda}$, isoperimetric sets for the restriction of the gaussian measure to $D$ have some (usual) boundary part included in the boundary of the obstacle and our guess reduces to the following statement: if $r$ is large enough, for any subset $B \subset D$ with given gaussian measure, the standard gaussian measure of the part of the usual boundary of $B$ that does not intersect $\partial D$ is greater or equal to $C \, e^{-r^2}$ times the measure of the boundary intersecting $\partial D$.

5.2.2. *Hypercubes.* Of course, what we have just done immediately extends to $d$ dimensions, defining $A$ as $A = \{x^1 \geq a+r\,,\,|x^i| \leq r \text{ for all } 2 \leq i \leq d\}$ and furnishes exactly the same bound as (5.6) replacing 2 by $2(d-1)$, i.e. in dimension $d$

$$C_C(\lambda, y, r) \geq \frac{1}{2(d-1)\,\sqrt{\lambda}}\, e^{\lambda r^2}\, \left(1 - \frac{1}{r\sqrt{\lambda}}\, e^{-\lambda r^2}\right)\,. \tag{5.7}$$

In order to get a lower bound for the Poincaré constant, inspired by what precedes, we shall proceed as follows: denote by $A(u)$ the set

$$A(u) = \{x^1 \geq a+r\,,\,|x^i| \leq u \text{ for all } 2 \leq i \leq d\}\,,$$

and for $r > 1$, choose a Lipschitz function $f$ such that $\mathbb{1}_{A(r-1)} \leq f \leq \mathbb{1}_{A(r)}$, for instance $f(x) = (1 - d(x, A(r-1)))_+$.

If $Z_\lambda$ denotes the (inverse normalizing) constant in front of the exponential density of the gaussian kernel (notice that $Z_\lambda$ goes to 0 as $\lambda$ goes to infinity), it holds

$$\begin{aligned}
\operatorname{Var}_{\mu_{\lambda,\mathcal{X}}}(f) &\geq \mu_{\lambda,\mathcal{X}}(A(r-1)) - (\mu_{\lambda,\mathcal{X}}(A(r)))^2 \\
&\geq Z_\lambda \int_{a+r}^{+\infty} e^{-\lambda u^2} du \left(\left(\int_{-r+1}^{r-1} e^{-\lambda u^2} du\right)^{d-1} - Z_\lambda \left(\int_{-r}^{r} e^{-\lambda u^2} du\right)^{2(d-1)} \int_{a+r}^{+\infty} e^{-\lambda u^2} du\right),
\end{aligned}$$

so that, there exists some universal constant $c$ such that, as soon as $r\sqrt{\lambda} > c$,

$$\operatorname{Var}_\mu(f) \geq \frac{Z_\lambda}{2} \int_{a+r}^{+\infty} e^{-\lambda u^2} du \left(\int_{-r+1}^{r-1} e^{-\lambda u^2} du\right)^{d-1}\,.$$

At the same time again if $r\sqrt{\lambda} > c$,

$$\int |\nabla f|^2\, d\mu_{\lambda,\mathcal{X}} \leq \int \left(\mathbb{1}_{A(r)} - \mathbb{1}_{A(r-1)}\right)\, d\mu_{\lambda,\mathcal{X}} \leq Z_\lambda \left(\int_{a+r}^{+\infty} e^{-\lambda u^2} du\right) \frac{e^{-\lambda(r-1)^2}}{(r-1)\lambda}\,(d-1) \left(\int_{-r}^{r} e^{-\lambda u^2} du\right)^{d-2}\,.$$

It follows, using homogeneity again, that

$$\begin{aligned}
C_P(\lambda, y, r) &\geq \frac{1}{2} \left(\frac{r\sqrt{\lambda} - 1}{(d-1)\lambda}\right) e^{(r\sqrt{\lambda}-1)^2}\, \frac{\left(\int_{-r\sqrt{\lambda}+1}^{r\sqrt{\lambda}-1} e^{-u^2} du\right)^{d-1}}{\left(\int_{-r\sqrt{\lambda}}^{r\sqrt{\lambda}} e^{-u^2} du\right)^{d-2}} \\
&\geq \left(\frac{r\sqrt{\lambda} - 1}{(d-1)\lambda}\right) e^{(r\sqrt{\lambda}-1)^2}\, \frac{1}{4\sqrt{\pi}} \left(1 - \frac{e^{-(r\sqrt{\lambda}-1)^2}}{r\sqrt{\lambda} - 1}\right)^{d-2}\,.
\end{aligned} \tag{5.8}$$

Notice that this lower bound is smaller (hence worse) than the one we obtained in Theorem 5.2, and also contain an extra dimension dependent term (the last one). But of course it is much easier to get.

Since 1 is arbitrary, we may replace $r\sqrt{\lambda} - 1$ by $r\sqrt{\lambda} - \varepsilon$ for any $0 \leq \varepsilon \leq 1$, the price to pay being some extra factor $\varepsilon^2$ in front of the lower bound for the Poincaré constant.

### 5.3. Back to spherical obstacles. Another lower bound.

It is tempting to develop the same approach in the case of spherical obstacles. First assume $\lambda = 1$.
Introduce for $0 \leq u \leq r$,

$$A(u) = \{x^1 \geq a \, , \, |\bar{x}| \leq u\} \cap D \, .$$

As before we consider, for $\varepsilon < u$, a function $\mathbb{1}_{A(u-\varepsilon)} \leq f \leq \mathbb{1}_{A(u)}$ which is $1/\varepsilon$-Lipschitz. Then

$$\mathrm{Var}_{\mu_{\lambda,\mathcal{X}}}(f) \geq \mu_{\lambda,\mathcal{X}}(A(u-\varepsilon)) - (\mu_{\lambda,\mathcal{X}}(A(u)))^2$$

and

$$\int |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} \leq (1/\varepsilon^2) \; (\mu_{\lambda,\mathcal{X}}(A(u)) - \mu_{\lambda,\mathcal{X}}(A(u-\varepsilon))) \; ,$$

with

$$\mu_{\lambda,\mathcal{X}}(A(u)) = Z_\lambda \, \sigma(S^{d-2}) \int_0^u \left( \int_{a+\sqrt{r^2-s^2}}^{+\infty} e^{-t^2} \, dt \right) s^{d-2} \, e^{-s^2} \, ds \, ,$$

and $\sigma(S^{d-2})$ is the Lebesgue measure of the unit sphere. It follows

$$\begin{aligned} (Z_\lambda)^{-1} \int |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} &\leq (\sigma(S^{d-2})/\varepsilon^2) \int_{u-\varepsilon}^u \left( \int_{a+\sqrt{r^2-s^2}}^{+\infty} e^{-t^2} \, dt \right) s^{d-2} \, e^{-s^2} \, ds \\ &\leq \frac{\sigma(S^{d-2})}{2\varepsilon^2} \int_{u-\varepsilon}^u \frac{s^{d-2}}{(a+\sqrt{r^2-s^2})} \, e^{-(a+\sqrt{r^2-s^2})^2} \, e^{-s^2} \, ds \\ &\leq \frac{\sigma(S^{d-2}) \, u^{d-2} \, e^{-(a^2+r^2)}}{2\varepsilon^2 \, (a+\sqrt{r^2-u^2})} \int_{u-\varepsilon}^u e^{-2a\sqrt{r^2-s^2}} \, ds \, . \end{aligned}$$

To get a precise upper bound for the final integral, we perform the change of variable $v = \sqrt{r^2-s^2}$ so that

$$\begin{aligned} \int_{u-\varepsilon}^u e^{-2a\sqrt{r^2-s^2}} \, ds &= \int_{\sqrt{r^2-u^2}}^{\sqrt{r^2-(u-\varepsilon)^2}} \frac{v}{\sqrt{r^2-v^2}} \, e^{-2av} \, dv \\ &\leq \frac{\sqrt{r^2-(u-\varepsilon)^2}}{2a(u-\varepsilon)} \left( e^{-2a\sqrt{r^2-u^2}} - e^{-2a\sqrt{r^2-(u-\varepsilon)^2}} \right) \\ &\leq \frac{\sqrt{r^2-(u-\varepsilon)^2}}{2a(u-\varepsilon)} \, e^{-2a\sqrt{r^2-(u-\varepsilon)^2}} \left( e^{\frac{2a\varepsilon(2u-\varepsilon)}{\sqrt{r^2-(u-\varepsilon)^2}+\sqrt{r^2-u^2}}} - 1 \right) . \end{aligned}$$

Again for $r \geq c$ for some large enough $c$, and $a+\sqrt{r^2-u^2} \geq 1$, for $u > 2\varepsilon$,

$$\begin{aligned}
\text{Var}_{\mu_{\lambda,\mathcal{X}}}(f) &\geq \frac{1}{2}\,\mu_{\lambda,\mathcal{X}}(A(u-\varepsilon))\\
&\geq \frac{1}{2}\,Z_\lambda\,\sigma(S^{d-2})\,e^{-(a^2+r^2)}\int_0^{u-\varepsilon}\frac{a+\sqrt{r^2-s^2}}{1+2(a+\sqrt{r^2-s^2})^2}\,s^{d-2}\,e^{-2a\sqrt{r^2-s^2}}\,ds\\
&\geq \frac{1}{2}\,Z_\lambda\,\sigma(S^{d-2})\,e^{-(a^2+r^2)}\int_{u-2\varepsilon}^{u-\varepsilon}\frac{a+\sqrt{r^2-s^2}}{1+2(a+\sqrt{r^2-s^2})^2}\,s^{d-2}\,e^{-2a\sqrt{r^2-s^2}}\,ds\\
&\geq \frac{1}{2}\,Z_\lambda\,\sigma(S^{d-2})\,e^{-(a^2+r^2)}\,\frac{a+\sqrt{r^2-(u-\varepsilon)^2}}{1+2(a+\sqrt{r^2-(u-2\varepsilon)^2})^2}\,(u-2\varepsilon)^{d-2}\\
&\qquad\qquad \int_{u-2\varepsilon}^{u-\varepsilon} e^{-2a\sqrt{r^2-s^2}}\,ds\,.
\end{aligned}$$

The last integral is bounded from below by

$$\int_{u-2\varepsilon}^{u-\varepsilon} e^{-2a\sqrt{r^2-s^2}}\,ds \;\geq\; \frac{\sqrt{r^2-(u-\varepsilon)^2}}{2a(u-\varepsilon)}\,e^{-2a\sqrt{r^2-(u-\varepsilon)^2}}\,\left(1-e^{\frac{-2a\varepsilon(2u-3\varepsilon)}{\sqrt{r^2-(u-\varepsilon)^2}+\sqrt{r^2-(u-2\varepsilon)^2}}}\right)$$

We thus deduce

$$C_P(1,B(y,r)) \geq \varepsilon^2\,\frac{(a+\sqrt{r^2-u^2})(a+\sqrt{r^2-(u-\varepsilon)^2})}{1+2(a+\sqrt{r^2-(u-2\varepsilon)^2})^2}\,\frac{(u-\varepsilon)^{d-2}}{u^{d-2}}\,H\,, \tag{5.9}$$

with

$$H=\frac{1-e^{\frac{-2a\varepsilon(2u-3\varepsilon)}{\sqrt{r^2-(u-\varepsilon)^2}+\sqrt{r^2-(u-2\varepsilon)^2}}}}{e^{\frac{2a\varepsilon(2u-\varepsilon)}{\sqrt{r^2-(u-\varepsilon)^2}+\sqrt{r^2-u^2}}}-1}\,.$$

For small $r$ (smaller than $c\sqrt{d-1}$ for some small enough $c$) it is not difficult to show that $C_P(1,B(y,r)) \geq c_d$, and presumably $c_d$ can be chosen independently of $d$, using again hitting times.

The bound (5.9) is not interesting if $a \gg r$, since in this case $H$ is very small, unless $\varepsilon$ is small enough (of order at most $r/a$), so that the lower bound we obtain goes to 0 with $r/a$. Hence we shall only look at the case where $a/r \leq C$. Since $2\varepsilon < u < r$, for $H$ to be bounded from below by some universal constant, we see that $au\varepsilon \leq cr$ for some small enough universal constant $c$, so that we have to choose $u$ and $\varepsilon$ of order $\sqrt{r/a}$. It is then not difficult to see that, combined with all what we have done before, this will furnish the following type of lower bound

**Proposition 5.10.** *There exists a constant $C_d$ such that for all $y$ and $r$,*

$$C_P(\lambda,B(y,r)) \geq \frac{C_d}{\lambda}\,\left(1+\frac{r}{|y|\vee 1}\right)\,.$$

Even for very large $r's$, the previous method furnishes a dimension dependent bound. Proposition 5.10 is interesting when the obstacle contains the origin, in which case we have a linear dependence in $r/|y|$. Of course, when $y=0$ we know that the lower bound growths as $r^2$. Also notice that for large $a$ the previous lower bound is similar to the upper bound we have obtained in the previous section.

5.4. **How to kill the Poincaré constant with far but small non convex obstacles.**

In a previous subsection we have seen that an infinity of appropriately oriented squared obstacle with "centers" and radii going to infinity furnishes an infinite Poincaré constant. We shall see here that if we break the convexity of the obstacle, even small obstacles at infinity can kill the Poincaré constant.

For simplicity, we will assume that $d = 2$, and we shall look at $\lambda = 1$ with a non-convex bounded obstacle, namely we consider

$$D^c = \{0 \leq y - x^1 \leq \alpha\,;\, |x^2| \leq \alpha\} \cup \{0 \leq x^1 - y \leq \alpha\,;\, \frac{\alpha}{2} \leq |x^2| \leq \alpha\}\,.$$

We simply denote by $\mu$ the gaussian measure restricted to $D$.

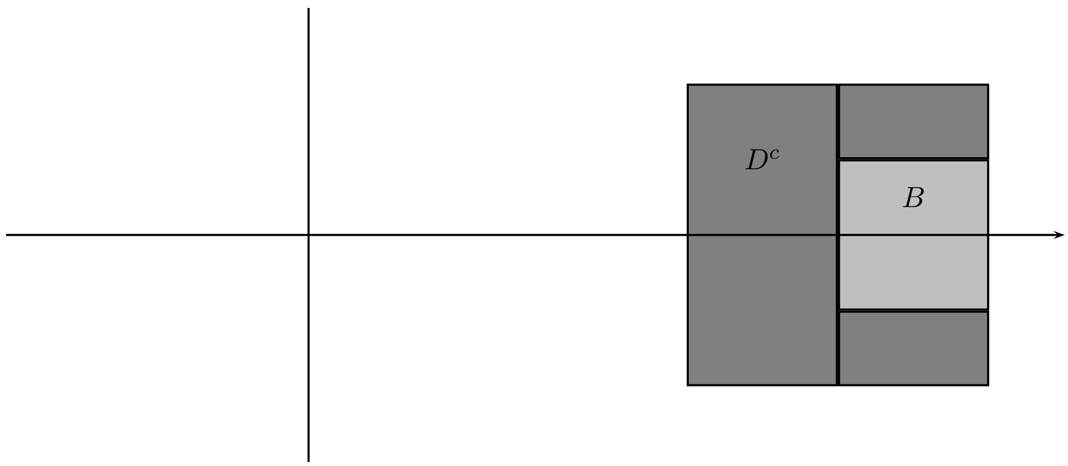


FIGURE 3. A non convex obstacle $D^c$ in gray. The trap $B$ in lightgray

As in the previous subsections we shall introduce some $2/\alpha$-Lipschitz function $f$ such that $\mathbb{1}_A \leq f \leq \mathbb{1}_B$ with $A = \{0 \leq x^1 - y \leq \frac{\alpha}{2}\,;\, \frac{\alpha}{2} \geq |x^2|\}$ and $B = \{0 \leq x^1 - y \leq \alpha\,;\, \frac{\alpha}{2} \geq |x^2|\}$. Hence

$$\mathrm{Var}_\mu(f) \geq \mu(A) - (\mu(B))^2 \quad \text{ and } \quad \int |\nabla f|^2\, d\mu \leq \frac{4}{\alpha^2}\, (\mu(B) - \mu(A))\,.$$

In addition

$$\mu(A) = Z_1 \left(\int_y^{y+\frac{\alpha}{2}} e^{-u^2}\, du\right) \left(\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} e^{-v^2}\, dv\right)\,,\ \mu(B) = Z_1 \left(\int_y^{y+\alpha} e^{-u^2}\, du\right) \left(\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} e^{-v^2}\, dv\right)$$

so that

$$\begin{aligned}\frac{\mu(A)}{\mu(B)} &\geq \frac{\int_y^{y+\frac{\alpha}{2}} e^{-u^2}\,du}{\int_y^{y+\alpha} e^{-u^2}\,du} \geq \frac{\frac{y^2}{1+2y^2}\left(\frac{e^{-y^2}}{y} - \frac{e^{-(y+\frac{\alpha}{2})^2}}{y+\frac{\alpha}{2}}\right)}{\frac{1}{2y}\left(e^{-y^2} - e^{-(\alpha+y)^2}\right)}\\ &\geq \frac{2y^2}{1+2y^2}\,\frac{1-e^{-\alpha(y+\frac{\alpha}{4})}}{1-e^{-\alpha(2y+\alpha)}}\,, \end{aligned} \tag{5.11}$$

and

$$\begin{aligned}\frac{\mu(A)}{\mu(B)-\mu(A)} &\geq \frac{\int_y^{y+\frac{\alpha}{2}} e^{-u^2}\,du}{\int_{y+\frac{\alpha}{2}}^{y+\alpha} e^{-u^2}\,du} \geq \frac{\frac{y^2}{1+2y^2}\left(\frac{e^{-y^2}}{y} - \frac{e^{-(y+\frac{\alpha}{2})^2}}{y+\frac{\alpha}{2}}\right)}{\frac{1}{2y}\left(e^{-(y+\frac{\alpha}{2})^2} - e^{-(\alpha+y)^2}\right)}\\ &\geq \frac{2y^2}{1+2y^2}\,\frac{1-e^{-\alpha(y+\frac{\alpha}{4})}}{e^{-\alpha(y+\frac{\alpha}{4})}-e^{-\alpha(2y+\alpha)}}\\ &\geq \frac{2y^2e^{\alpha(y+\frac{\alpha}{4})}}{1+2y^2}\,\frac{1-e^{-\alpha(y+\frac{\alpha}{4})}}{1-e^{-\alpha(y+\frac{3\alpha}{4})}}\,. \end{aligned} \tag{5.12}$$

$\mu(B)$ goes to 0 as $y \to +\infty$ while there exists some constant $c$ such that $\mu(A) \geq c\,\mu(B)$, provided $\alpha$ is fixed and $y$ large enough (depending on $\alpha$), in particular as soon as $\alpha y \to +\infty$. As previously we thus have for $\alpha y$ large enough, $\mathrm{Var}_\mu(f) \geq \frac{1}{2}\,\mu(A)$. Gathering all previous results, we thus get $C_P(\mu) \geq \frac{1}{8}\,\frac{\mu(A)}{\mu(B)-\mu(A)}$ so that $C_P(\mu)$ explodes (at least) like $e^{\alpha y}$ if $\alpha y \to +\infty$. Hence, even a small non convex obstacle going to infinity, makes the Poincaré constant explode.

More precisely consider an infinite number of such obstacles $(O(y_j, \alpha_j))$ such that one more time the convex face of the obstacle is orthogonal to the line joining the origin to $y_j$. *If $\alpha_j \to 0$ but $\alpha_j\,|y_j| \to +\infty$, then the process is not exponentially ergodic.*

Actually it is not difficult to see, though the calculations are a little bit more intricate, that the previous situation is similar to the case of two touching balls as in Figure 4.

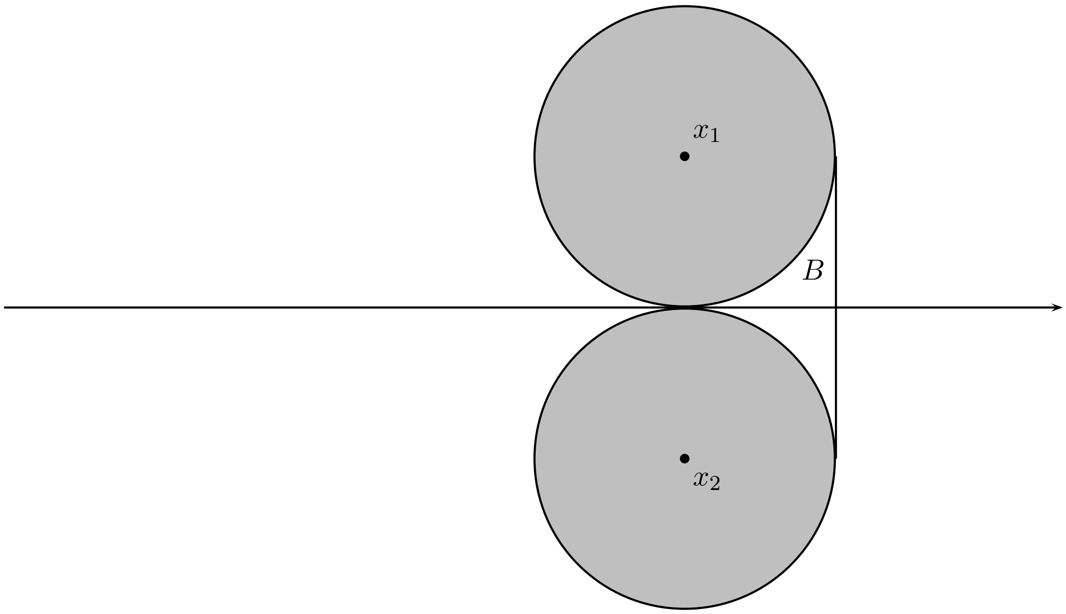


Figure 4. Touching balls

## Appendix A. Existence and uniqueness of the process.

The main (actually unique) result of this section is the following (recall that the notion of solution for a reflected system involves both $X$ and the local times $L$, see e.g. [IW81, Cat86])

**Theorem A.1.** *Assume* (1.1). *Then the system* (1.2) *has a unique (non explosive) strong solution for any allowed starting point $x$. In addition $\mu_{\lambda,\mathcal{X}}$ is the unique invariant (actually symmetric) probability measure.*

The remainder of this section is devoted to the proof of this result.

In the sequel we shall denote by $L$ the (formal) infinitesimal generator

$$L = \frac{1}{2}\,\Delta - \lambda\,\langle x, \nabla\rangle\,, \tag{A.2}$$

whose domain is some extension of the set of smooth functions $f$ compactly supported in $\bar{D}$ such that for all $i$,

$$\frac{\partial f}{\partial n_i}(y) = 0$$

at any $y$ such that $|y - x_i| = r_i$, where $n_i$ denotes the normal vector field on the sphere of center $x_i$ and radius $r_i$.
We shall denote by $\mathcal{D}(L)$ this core.

### A.1. **Finite number of obstacles.**

When $N$ is finite, existence of a unique (strong) solution of (1.2) starting from any point (belonging to $\bar{D}$ for (1.2)), up to the explosion time, is standard (see e.g. [Cat86] for references) at least when the boundary of the obstacles is smooth. That is why we have chosen to smooth the hypercubes when looking at this particular situation. The only point is to show that the explosion time is almost surely infinite.

To this end, define

$$d_N = \max_{i=1,\dots,N} |x_i| \quad , \quad r = \max_{i=1,\dots,N} r_i \, , \tag{A.3}$$

and choose a smooth function $h_N$ such that $h_N \geq 1$ everywhere,

$$h_N(x) = 1 \text{ if } |x| < d_N + 2r \quad , \quad h_N(x) = 1 + |x|^2 \text{ if } |x| > d_N + 3r + 1 \, . \tag{A.4}$$

It is enough to remark that $h_N \in \mathcal{D}(L)$ and satisfies

$$Lh_N \leq -\, 2\lambda \, h_N \quad , \text{ for } |x| > d_L = (d/2\lambda)^{\frac{1}{2}} \vee (d_N + 3r + 1) \, . \tag{A.5}$$

$h_N$ can thus play the role of a Lyapunov function for Hasminskii's non explosion test.

We can thus define for any $x$ in $\bar{D}$ the law $P_t(x, dy)$ of the process at time $t$, $X_t$ starting from $x$, as well as a Markov semi-group $P_t$ acting on continuous and bounded functions. It is known that, for all $t > 0$,

$$P_t(x, dy) = p_t(x, y) \, dy$$

where $p_t \in C^\infty(\bar{D})$ (see [Cat86, Cat87]). Furthermore, the density $p_t$ is everywhere positive. This is a consequence of (1.1) (which implies in particular that $D$ is path connected) and standard tools about the support of the law of the whole process.

$\mu_{\lambda,\mathcal{X}}$ is clearly a symmetric, hence invariant, probability measure. Uniqueness follows from the previous regularity and positivity as usual. Let us denote by $q_t$ the density of the law of $X_t$ w.r.t. $\mu_{\lambda,\mathcal{X}}$ i.e.

$$q_t(x, y) = p_t(x, y) \, \frac{dx}{d\mu_{\lambda,\mathcal{X}}} \, .$$

Application of the Chapman-Kolmogorov formula and standard regularization arguments yield

$$q_{2t}(x, x) = \int q_t(x, y) \, q_t(y, x) \, \mu_{\lambda,\mathcal{X}}(dy) = \int q_t^2(x, y) \, \mu_{\lambda,\mathcal{X}}(dy) \, , \tag{A.6}$$

thanks to symmetry, i.e. $q_t \in \mathbb{L}^2(\mu_{\lambda,\mathcal{X}})$.

### A.2. **Infinite number of obstacles.**

We now consider the case of infinitely many obstacles, still satisfying the non overlapping condition (1.1), for the locally finite collection $\mathcal{X}$. We can thus construct the process up to exit times of an increasing sequence of relatively compact open subsets $U_n$, each of which containing only a finite number of (closed) obstacles, the remaining (closed) obstacles being included into $(\bar{U_n})^c$. The sequence $T_n$ of exit times of $U_n$ is thus growing to the explosion time, but now it is much more difficult to control this limit.

A standard way is to use Dirichlet forms theory. Namely let us consider

$$\mathcal{E}(f) = \int |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} \tag{A.7}$$

defined for $f$ which are smooth, bounded with bounded derivatives.

Our goal is to show that $\mathcal{E}$ is a conservative local Dirichlet form, so that one can associate to $\mathcal{E}$ a stationary Hunt process $(Y_t)_{t\geq 0}$ which is a non exploding diffusion process. This process coincides with $X$ up to the exit time of $U_n$ for all $n$, provided $X_0$ has distribution $\mu_{\lambda,\mathcal{X}}$ (exit time can be equal to 0). But, since $Y_t - Y_0$ is an additive functional of finite energy, it can be decomposed (Lyons-Zheng decomposition) for $0 \le t \le T$ into

$$Y_t - Y_0 = M_t + RM_t^T$$

where $M_.$ (resp. $RM_.^T$) is a forward (resp. backward) $\mathbb{L}^2$ martingale with brackets $\langle M \rangle_t = \langle RM^T \rangle_t = t$, hence are Brownian motions. It follows that for any $K > 0$,

$$\begin{aligned} \mathbb{P}\left(\sup_{t\in[0;T]} |Y_t| \ge K\right) &\le \mathbb{P}\left(\sup_{t\in[0;T]} |Y_t - Y_0| \ge \frac{K}{2} \text{ or } |Y_0| \ge \frac{K}{2}\right) \\ &\le \mathbb{P}\left(\sup_{t\in[0;T]} |M_t| \ge \frac{K}{4}\right) + \mathbb{P}\left(\sup_{t\in[0;T]} |RM_t^T| \ge \frac{K}{4}\right) + \mathbb{P}\left(|Y_0| \ge \frac{K}{2}\right) \end{aligned}$$

and Doob's inequality allows us to conclude that the latter upper bound goes to 0 as $K$ goes to infinity. It follows that the supremum of the exit times of the balls $B(0, K)$ is almost surely infinite, hence so does the supremum of the $T_n$'s, implying that $Y$ and $X$ coincide up to any time and that $X$ does not explode, when the initial law is $\mu_{\lambda,\mathcal{X}}$.

Standard arguments (see [FOT94]) imply that there is no explosion starting from quasi every point $x$ (i.e. all $x$'s not belonging to some polar set $\mathcal{N}$, recall that here polar sets coincide with sets of zero capacity), though here we only need that this property holds for $\mu_{\lambda,\mathcal{X}}$ almost all $x$'s, which is an immediate consequence of disintegration of the measure.

Now let $x$ be some point in $D$, and choose a small ball $B(x, \varepsilon) \subset D$. If $\mathbb{P}_y$ denotes the law of $X$ starting from $y$ as usual, we have for all $z \in B(x, \varepsilon)$,

$$\mathbb{P}_z(\sup_n T_n < +\infty) = \int_{|y-x|=\varepsilon} \mathbb{P}_y(\sup_n T_n < +\infty)\, \eta_z(dy)$$

where $\eta_z$ denotes the $\mathbb{P}_z$ law of $X_\tau$ with $\tau$ the exit time of $B(x, \varepsilon)$ (that $\tau$ is almost surely finite is well known and actually follows from the arguments below).

Up to the exit time of $B(x, \varepsilon)$, $X$ is just an Ornstein-Uhlenbeck process, so that its law is equivalent to the one of the Brownian motion. For Brownian motion, it is well known that $\tau$ is a.s. finite, that the exit measure (starting from $z$) is simply the harmonic measure (related to $z$) on the sphere $S(x, \varepsilon)$, hence is equivalent to the surface measure $\sigma_x$. Thus the same properties hold true for our Ornstein-Uhlenbeck process.

It follows that $\eta_z$ is equivalent to the surface measure $\sigma_x$ on the sphere $S(x, \varepsilon)$, so that $\eta_z$ and $\eta_x$ are equivalent.

(One can see e.g [Cat91] theorem 4.18 for much more sophisticated situations).

Choose $z \notin \mathcal{N}$. The previous formula shows that for $\eta_z$ almost all $y \in S(x, \varepsilon)$, $\mathbb{P}_y(\sup_n T_n < +\infty) = 0$, so that the same holds $\eta_x$ almost surely and finally $\mathbb{P}_x(\sup_n T_n < +\infty) = 0$, showing that no explosion occurs starting from any point.

It remains to show that $\mathcal{E}$ is a conservative and local Dirichlet form. To this end introduce the truncated form

$$\mathcal{E}_n(f) = \frac{1}{\mu_{\lambda,\mathcal{X}}(U_n)} \int_{U_n} |\nabla f|^2 \, d\mu_{\lambda,\mathcal{X}} \tag{A.8}$$

corresponding to the reflected O-U process in $U_n$ with hard obstacles. It is not difficult to see that we can build the open sets $U_n$ in such a way that $\partial U_n$ is smooth. It thus follows that $\mathcal{E}_n$ is a conservative and local Dirichlet form, to which is associated a non-exploding process

$X^n$. The same reasoning as before shows that we can start from any point $x \in U_n$. We use the superscript $n$ for the Markov law corresponding to $\mathcal{E}_n$

Let $\tau_K$ be the exit time from the ball $B(0,K)$ and let $n_K$ be such that for $n \geq n_K$, $B(0,K) \subset U_n$. All processes $X^n$ ($n \geq n_K$), starting from $x \in B(0,K)$, coincide up to time $\tau_K$ (and coincide with the possibly exploding $X$). Now choose some initial measure $\nu(dy) = u(y)dy$ where $u$ is bounded and has compact support included in $B(0,R)$. Then $\nu$ is absolutely continuous with respect to $\mu^n_{\lambda,r}$ and one can find some constant $C(K,\nu)$ such that

$$\| \frac{d\nu}{d\mu^n_{\lambda,\mathcal{X}}} \|_\infty \leq C(K,\nu) \quad \text{ for all } n \geq n_K\,.$$

For any $T>0$, it yields, using the Lyons-Zheng decomposition as before

$$\begin{aligned}
\mathbb{P}_\nu\left(\sup_{t\in[0;T]} |X^n_t| \geq K\right) &\leq C(K,\nu)\,\mathbb{P}_{\mu^n_{\lambda,r}}\left(\sup_{t\in[0;T]} |X^n_t - X^n_0| \geq \frac{K}{2} \text{ or } |X^n_0| \geq \frac{K}{2}\right)\\
&\leq C(K,\nu)\left(\mathbb{P}_{\mu^n_{\lambda,r}}\left(\sup_{t\in[0;T]} |M^n_t| \geq \frac{K}{4}\right) + \mathbb{P}_{\mu^n_{\lambda,r}}\left(\sup_{t\in[0;T]} |RM^{T,n}_t| \geq \frac{K}{4}\right)\right) +\\
&\quad + C(K,\nu)\,\mathbb{P}_{\mu^n_{\lambda,r}}\left(|X^n_0| \geq \frac{K}{2}\right)\\
&\leq C(K,\nu)\left(C_1\, e^{-C_2\,K^2/T} + \mu^n_{\lambda,\mathcal{X}}(B^c(0,K/2))\right)\\
&\leq C(K,\nu)\left(C_1\, e^{-C_2\,K^2/T} + \frac{\mu_{\lambda,\mathcal{X}}(B^c(0,K/2))}{\mu_{\lambda,\mathcal{X}}(U_n)}\right)
\end{aligned}$$

for well chosen universal constants $C_1$, $C_2$. It immediately follows that $\mathbb{P}_\nu(\tau_K \leq T)$ (here we consider the process $X$) goes to 0 as $K$ goes to $+\infty$, so that the process starting from $\nu$ does not explode. This is of course sufficient for our purpose, since conservativeness follows by choosing a sequence $\nu_j$ converging to $\mu_{\lambda,\mathcal{X}}$.

**Remark A.9.** Once the non explosion is proven, standard arguments show that the process is Feller. Hence compact sets are closed petite sets in the terminology of [DMT95, DFMS04]. We refer to the latter reference for a complete discussion. ♢

## Appendix B. Useful estimates for exponential moments of hitting times.

In this section we shall recall some estimates of exponential moments of hitting times for some special linear processes. Denotes by $S(r)$ the first exit time of the symmetric interval $[-r,r]$ for a one dimensional process.

For the linear Brownian motion it is well known, (see [RY91] Exercise 3.10) that

$$E_0\left(e^{\theta\,S(r)}\right) = \frac{1}{\cos(r\,\sqrt{2\theta})} < +\infty$$

if and only if

$$\theta \leq \frac{\pi^2}{8\,r^2}\,.$$

Surprisingly enough (at least for us) a precise description of the Laplace transform of $S(r)$ for the O-U process is very recent: it was first obtained in [GJY03]. A simpler proof is contained in [GJ08] Theorem 3.1. The result reads as follows

**Theorem B.1.** {***see*** [GJY03, GJ08]} *If $S(r)$ denotes the exit time from $[-r, r]$ of a linear O-U process with drift $-\lambda x$ ($\lambda > 0$), then for $\theta \geq 0$,*

$$E_0\left(e^{-\theta\, S(r)}\right) = \frac{1}{{}_1F_1\left(\frac{\theta}{2\lambda}\,,\,\frac{1}{2}\,,\,\lambda\, r^2\right)}\,,$$

*where ${}_1F_1$ denotes the confluent hypergeometric function.*

The function ${}_1F_1$ is also denoted by $\Phi$ (in [GJY03] for instance) or by $M$ in [AS72] (where it is called Kummer function) and is defined by

$$ {}_1F_1(a,b,z) = \sum_{k=0}^{+\infty} \frac{(a)_k}{(b)_k}\,\frac{z^k}{k!} \quad \text{where} \quad (a)_k = a(a+1)...(a+k-1)\,,\,(a)_0 = 1\,. \tag{B.2}$$

In our case, $b = \frac{1}{2}$, so that ${}_1F_1$ is an analytic function, as a function of both $z$ and $\theta$. It follows that $\theta \mapsto E_0\left(e^{-\theta\, S(r)}\right)$ can be extended, by analytic continuation, to $\theta < 0$ as long as $\lambda r^2$ is not a zero of ${}_1F_1(\frac{\theta}{2\lambda}\,,\,\frac{1}{2}\,,\,.)$.

The zeros of the confluent hypergeometric function are difficult to study. Here we are looking for the first negative real zero. For $-1 < a < 0$, $b > 0$, it is known (and easy to see) that there exists only one such zero, denoted here by $u$. Indeed ${}_1F_1(a, b, 0) = 1$ and all terms in the expansion (B.2) are negative for $z > 0$ except the first one, implying that the function is decaying to $-\infty$ as $z \to +\infty$. However, an exact or an approximate expression for $u$ are unknown (see the partial results of Slater in [Sla56, AS72], or in [Gat90]). Our situation however is simpler than the general one, and we shall obtain a rough but sufficient bound.

First, comparing with the Brownian motion, we know that for all $\lambda > 0$ we must have

$$\frac{-\theta}{\lambda} \leq \frac{\pi^2}{8(r\sqrt{\lambda})^2}\,.$$

So, if $\lambda\, r^2 > \pi^2/8$ and $-\theta/2\lambda \geq 1/2$, the Laplace transform (or the exponential moment) is infinite. We may thus assume that $-\theta/2\lambda < 1/2$.
Hence, for ${}_1F_1\left(\frac{\theta}{2\lambda}\,,\,\frac{1}{2}\,,\,\lambda\, r^2\right)$ to be negative it is enough that

$$\begin{aligned} 1 \ &< \ \frac{-\theta}{\lambda}\left((\lambda\, r^2) + \sum_{k=2}^{+\infty} \frac{(1+\frac{\theta}{2\lambda})(2+\frac{\theta}{2\lambda})...(k-1+\frac{\theta}{2\lambda})}{(1+\frac{1}{2})(2+\frac{1}{2})...(k-1+\frac{1}{2})}\,\frac{(\lambda\, r^2)^k}{k!}\right) \\ &< \ \frac{-\theta}{\lambda}\left(\sum_{k=1}^{+\infty} \frac{(\lambda\, r^2)^k}{k!}\right)\,, \end{aligned}$$

i.e.

$$\text{as soon as} \quad \beta = -\theta > \frac{\lambda}{e^{\lambda\, r^2} - 1} \quad \text{then} \quad \mathbb{E}_0\left(e^{\beta\, S(r)}\right) = +\infty\,. \tag{B.3}$$

So there is a drastically different behavior between both processes.

Finally we shall also need estimates for a general CIR process or generalized squared radial Ornstein-Uhlenbeck process, i.e. the solution of

$$dU_t = 2\sqrt{U_t}dB_t + (\delta + 2\beta\, U_t)\, dt$$

when $\beta > 0$ and $\delta > 0$. According to [GJY03] Theorem 3, for $\theta > 0$,

$$E_0\left(e^{-\,\theta\, S(u)}\right) = \frac{e^{\beta\, u}}{{}_1F_1\left(\frac{(\theta+\beta\delta)}{2\beta}\, ,\, \frac{\delta}{2}\, ,\, \beta\, u\right)}\, . \tag{B.4}$$

It follows that for $0 < \theta < \beta\,\delta$, $E_0\left(e^{\theta\, S(u)}\right) < +\infty$.

## Appendix C. The case $N = 1$. Another estimate for a general $y$ using decomposition of variance.

A very usual method to deal with dimension controls is the decomposition of variance. This method can be used here in order to transfer the results of Proposition 2.1 to a non centered obstacle. Though the results are non optimal in many directions, the method contains some interesting features.

In this section for simplicity we will first assume that $\lambda = 1$, and second that $d \geq 3$. Recall that we are looking here at the case of an unique spherical obstacle $B(y, r)$, so that we simply denote by $\mu_{d,r}$ the restricted gaussian measure $\mu_{\lambda,\mathcal{X}}$. Since we will use an induction procedure on the dimension $d$ we explicitly make it appear in the notation.

Using rotation invariance we may also assume that $y = (a, 0)$ for some $a \in \mathbb{R}^+$, 0 being the null vector of $\mathbb{R}^{d-1}$. So, writing $x = (u, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$,

$$\mu_{d,r}(du, d\bar{x}) = \nu^0_{d-1,R(u)}(d\bar{x})\, \mu_1(du)\, ,$$

where $\nu^0_{d-1,R(u)}(d\bar{x})$ is the $d-1$ dimensional gaussian measure restricted to $B^c(0, R(u))$ as in section 2.1 with $R(u) = \sqrt{\left((r^2 - (u - a)^2)_+\right)}$ and $\mu_1$ is the first marginal of $\mu_{d,r}$ given by

$$\mu_1(du) = \frac{\gamma_{d-1}(B^c(0, R(u)))}{\gamma_d(B^c(y, r))}\, \gamma_1(du)\, ,$$

$\gamma_n$ denoting the $n$ dimensional gaussian measure $c_n\, e^{-|x|^2}\, dx$.

The standard decomposition of variance tells us that for a nice $f$,

$$\text{Var}_{\mu_{d,r}}(f) = \int \left(\text{Var}_{\nu^0_{d-1,R(u)}}(f)\right)\, \mu_1(du) + \text{Var}_{\mu_1}(\bar{f})\, , \tag{C.1}$$

where

$$\bar{f}(u) = \int f(u, \bar{x})\, \nu^0_{d-1,R(u)}(d\bar{x})\, .$$

According to Proposition 2.1, on one hand, it holds for all $u$,

$$\text{Var}_{\nu^0_{d-1,R(u)}}(f) \leq \left(1 + \frac{(r^2 - (u - a)^2)_+}{d - 1}\right) \int |\nabla_{\bar{x}} f|^2\, d\nu^0_{d-1,R(u)}\, , \tag{C.2}$$

so that

$$\int \left(\text{Var}_{\nu^0_{d-1,R(u)}}(f)\right)\, \mu_1(du) \leq \left(1 + \frac{r^2}{d - 1}\right) \int |\nabla_{\bar{x}} f|^2\, d\mu_{d,r}\, . \tag{C.3}$$

On the other hand, $\mu_1$ is a logarithmically bounded perturbation of $\gamma_1$ hence satisfies some Poincaré inequality so that

$$\mathrm{Var}_{\mu_1}(\bar{f}) \leq C_1 \int \left|\frac{d\bar{f}}{du}\right|^2 d\mu_1\,. \tag{C.4}$$

So we have first to get a correct bound for $C_1$, second to understand what $\frac{d\bar{f}}{du}$ is.

C.1. **A bound for $C_1$.** Since $\mu_1$ is defined on the real line, upper and lower bounds for $C_1$ may be obtained by using Muckenhoupt bounds (see [ABC+00] Theorem 6.2.2). Unfortunately we were not able to obtain the corresponding explicit expression in our situation as $\mu_1$ is not sufficiently explicitly given to use Muckenhoupt criterion. So we shall give various upper bounds using other tools.

The usual Holley-Stroock perturbation argument combined with the Poincaré inequality for $\gamma_1$ imply that

$$C_1 \leq \frac{1}{2}\,\frac{\sup_u \{\gamma_{d-1}(B^c(0,R(u)))\}}{\inf_u \{\gamma_{d-1}(B^c(0,R(u)))\}} \leq \frac{1}{2}\,\frac{\int_0^{+\infty} \rho^{d-2}\,e^{-\rho^2}\,d\rho}{\int_r^{+\infty} \rho^{d-2}\,e^{-\rho^2}\,d\rho} = \frac{1}{2}\left(1+\frac{\int_0^{r} \rho^{d-2}\,e^{-\rho^2}\,d\rho}{\int_r^{+\infty} \rho^{d-2}\,e^{-\rho^2}\,d\rho}\right). \tag{C.5}$$

Using the first inequality and the usual lower bound for the denominator, it follows that

$$\text{for all } r>0, \quad C_1 \leq \pi^{(d-2)/2}\,\frac{e^{r^2}}{r^{d-3}}\,.$$

The function $\rho \mapsto \rho^{d-2}\,e^{-\rho^2}$ increases up to its maximal value which is attained for $\rho^2 = (d-2)/2$ and then decreases to 0. It follows, using the second form of the inequality (C.5) that

- if $r \leq \sqrt{\frac{d-2}{2}}$ we have $C_1 \leq \frac{1}{2} + r^2$, while
- if $r \geq \sqrt{\frac{d-2}{2}}$ we have

$$C_1 \leq \frac{1}{2} + \left(\frac{d-2}{2}\right)^{\frac{d-2}{2}} e^{-\frac{d-2}{2}}\,\frac{e^{r^2}}{r^{d-4}}\,.$$

These bounds are quite bad for large $r$'s but do not depend on $y$.

Why is it bad ? First for $a=0$ (corresponding to the situation of section 2.1) we know that $C_1 \leq 1+\frac{r^2}{d}$ according to Proposition 2.1 applied to functions depending on $x_1$. Actually the calculations we have done in the proof of proposition 2.1, are unchanged for $f(z)=z_1$, so that it is immediately seen that $C_1 \geq \max(\frac{1}{2}, \frac{r^2}{d})$.
Intuitively the case $a=0$ is the worst one, though we have no proof of this. We can nevertheless give some hints.

The natural generator associated to $\mu_1$ is

$$\begin{aligned} L_1 &= \frac{d^2}{du^2} - \left(u - \frac{d}{du}\log(\gamma_{d-1}(B^c(0,R(u))))\right)\frac{d}{du} \\ &= \frac{d^2}{du^2} - u\,\frac{d}{du} + \frac{(u-a)\,(R(u))^{d-3}\,e^{-R^2(u)}}{\int_{R(u)}^{+\infty} \rho^{d-2}\,e^{-\rho^2}\,d\rho}\;\mathbb{1}_{|u-a|\leq r}\,\frac{d}{du}\,. \end{aligned}$$

The additional drift term behaves badly for $a \leq u \leq a+r$, since in this case it is larger than $-u$, while for $u \leq a$ it is smaller. In stochastic terms it means that one can compare the induced process with the Ornstein-Uhlenbeck process except possibly for $a \leq u \leq a+r$. In analytic terms let us look for a Lyapunov function for $L_1$. As for the O-U generator the simplest one is $g(u) = u \mapsto u^2$ for which

$$L_1 g \leq 2 - 4u^2 + 4u(u-a)\, \mathbb{I}_{a \leq u \leq a+r}\,.$$

Remember that $a \geq r$ so that $-au \leq \frac{1}{2}\, u^2$. It follows

$$\text{provided } a \geq r, \quad L_1 g \leq 2 - 2g\,. \tag{C.6}$$

For $|u| \geq 2$ we then have $L_1 g(u) \leq -\, g(u)$, so that $g$ is a Lyapunov function outside the interval $[-\sqrt{2}, \sqrt{2}]$ and the restriction of $\mu_1$ to this interval coincides (up to the constants) with the gaussian law $\gamma_1$ hence satisfies a Poincaré inequality with constant $\frac{1}{2}$ on this interval. According to the results in [BBCG08] we recalled in the previous section, we thus have that $C_1$ is bounded above by some universal constant $c$.

We may gather our results

**Lemma C.7.** *The following upper bound holds for $C_1$ :*

(1) *(small obstacle) if $r \leq \sqrt{\frac{d-2}{2}}$ we have $C_1 \leq \frac{1}{2} + r^2$,*
(2) *(far obstacle) if $|y| > r + \sqrt{2}$, $C_1 \leq c$ for some universal constant $c$,*
(3) *(centered obstacle) if $y = 0$, $C_1 \leq 1 + \frac{r^2}{d}$,*
(4) *in all other cases, there exists $c(d)$ such that $C_1 \leq c(d)\, \frac{e^{r^2}}{r^{d-3}}$ .*

*We conjecture that actually $C_1 \leq C(1+r^2)$ for some universal constant $C$.*

**Remark C.8.** In a recent preprint [KT13], the authors obtain a much better upper bound in case (4) (in fact a constant) when the origin belongs to the boundary of the ball and $d = 3$. ♢ ♢

C.2. **Controlling $\frac{d\bar{f}}{du}$.** It remains to understand what $\frac{d\bar{f}}{du}$ is and to compute the integral of its square against $\mu_1$.

Recall that

$$\bar{f}(u) = \int f(u, \bar{x})\, \nu^0_{d-1, R(u)}(d\bar{x})\,.$$

Hence

$$\begin{aligned} \bar{f}(u) \quad &= \quad \mathbb{I}_{|u-a|>r} \int f(u,\bar{x})\, \nu^0_{d-1,0}(d\bar{x}) \\ &\quad + \mathbb{I}_{|u-a|\leq r} \int_{\mathbb{S}^{d-2}} \int_{R(u)}^{+\infty} f(u, \rho\,\theta)\, \frac{\rho^{d-2}\, e^{-\rho^2}}{c(d)\, \gamma_{d-1}(B^c(0, R(u)))}\, d\rho\, d\theta\,, \end{aligned}$$

where $d\theta$ is the non-normalized surface measure on the unit sphere $\mathbb{S}^{d-2}$ and $c(d)$ the normalization constant for the gaussian measure. Hence, for $|u-a| \neq r$ we have

$$\begin{aligned}\frac{d}{du}\bar{f}(u) &= \int \frac{\partial f}{\partial x_1}(u,\bar{x})\,\nu^0_{d-1,R(u)}(d\bar{x}) \\ &\quad - \mathbb{1}_{|u-a|\leq r} \int f(u,\bar{x})\,\mathbb{1}_{|\bar{x}|>R(u)}\,\frac{\frac{d}{du}\,(\gamma_{d-1}(B^c(0,R(u))))}{\gamma^2_{d-1}(B^c(0,R(u)))}\,\gamma_{d-1}(d\bar{x}) \\ &\quad - \mathbb{1}_{|u-a|\leq r}\,\frac{R'(u)\,R^{d-2}(u)\,e^{-R^2(u)}}{c(d)\,\gamma_{d-1}(B^c(0,R(u)))}\int_{\mathbb{S}^{d-2}} f(u,R(u)\,\theta)\,d\theta\,.\end{aligned}$$

Notice that if $f$ only depends on $u$, $\bar{f} = f$ so that

$$\frac{d}{du}\bar{f}(u) = \frac{\partial f}{\partial x_1}(u) = \int \frac{\partial f}{\partial x_1}(u)\,\nu^0_{d-1,R(u)}(d\bar{x})\,,$$

and thus the sum of the two remaining terms is equal to 0. Hence in computing the sum of the two last terms, we may replace $f$ by $f - \int f(u,\bar{x})\,\nu^0_{d-1,R(u)}(d\bar{x})$ or if one prefers, we may assume that the latter $\int f(u,\bar{x})\,\nu^0_{d-1,R(u)}(d\bar{x})$ vanishes. Observe that this change will not affect the gradient in the $\bar{x}$ direction.
Assuming this, the second term becomes

$$- \mathbb{1}_{|u-a|\leq r}\,\frac{\frac{d}{du}\,(\gamma_{d-1}(B^c(0,R(u))))}{\gamma_{d-1}(B^c(0,R(u)))}\int f(u,\bar{x})\,\nu^0_{d-1,R(u)}(d\bar{x}) \;=\; 0\,.$$

We thus have (using Cauchy-Schwarz inequality) and a scale change

$$\begin{aligned}\int \left|\frac{d\bar{f}}{du}\right|^2 d\mu_1 &\leq 2\int \left|\frac{\partial f}{\partial x_1}\right|^2 (u,\bar{x})\,\mu_{d,r}(du,d\bar{x}) \\ &\quad + 2\int \left(\mathbb{1}_{|u-a|\leq r}\,\frac{R'(u)\,e^{-R^2(u)}}{c(d)\,\gamma_{d-1}(B^c(0,R(u)))}\int_{\mathbb{S}^{d-2}(R(u))} f(u,\,\theta)\,d\theta\right)^2 \mu_1(du)\,.\end{aligned}$$

Our goal is to control the last term using the gradient of $f$. One good way to do it is to use the Green-Riemann formula, in a well adapted form. Indeed, let $V$ be a vector field written as

$$V(\bar{x}) = -\,\frac{\varphi(|\bar{x}|)}{|\bar{x}|^{d-1}}\,\bar{x} \qquad \text{where } \varphi(R(u)) = R^{d-2}(u)\,. \tag{C.9}$$

This choice is motivated by the fact that the divergence, $\nabla.(\bar{x}/|\bar{x}|^{d-1}) = 0$ on the whole $\mathbb{R}^{d-1} - \{0\}$.
Of course in what follows we may assume that $R(u) > 0$, so that all calculations make sense.
The Green-Riemann formula tells us that, denoting $g_u(\bar{x}) = f(u,\bar{x})$, for some well choosen $\phi$

$$\begin{aligned}\int_{\mathbb{S}^{d-2}(R(u))} f(u,\,\theta)\,d\theta &= \int_{\mathbb{S}^{d-2}(R(u))} g_u\,\langle V, (-\bar{x}/|\bar{x}|)\rangle\,d\theta = \int \mathbb{1}_{|\bar{x}|\geq R(u)}\,\nabla.(g_u V)(\bar{x})\,d\bar{x} \\ &= -\int \mathbb{1}_{|\bar{x}|\geq R(u)}\,\langle \nabla g_u(\bar{x}), (\bar{x}/|\bar{x}|^{d-1})\rangle\,\varphi(|\bar{x}|)\,d\bar{x} \\ &\quad - \int \mathbb{1}_{|\bar{x}|\geq R(u)}\,g_u(\bar{x})\,(\varphi'(|\bar{x}|)/|\bar{x}|^{d-1})\,d\bar{x}\,.\end{aligned}$$

Now we choose $\varphi(s)=R^{d-2}(u)\,e^{R^2(u)}\,e^{-s^2}$ and recall that $R'(u)=-((u-a)/R(u))\,\mathbb{1}_{|u-a|\leq r}$.

We have finally obtained

$$\begin{aligned}\mathbb{1}_{|u-a|\leq r}\,\frac{R'(u)\,e^{-R^2(u)}}{c(d)\,\gamma_{d-1}(B^c(0,R(u)))}\,\int_{\mathbb{S}^{d-2}(R(u))} f(u,\theta)\,d\theta \,=\\ =\mathbb{1}_{|u-a|\leq r}\,(u-a)\,R^{d-3}(u)\,\int \langle\nabla_{\bar x}f(u,\bar x),(\bar x/|\bar x|^{d-1})\rangle\,\nu^0_{d-1,R(u)}(d\bar x)\\ -\mathbb{1}_{|u-a|\leq r}\,(u-a)\,R^{d-3}(u)\,2\int (f(u,\bar x)/|\bar x|^{d-3})\,\nu^0_{d-1,R(u)}(d\bar x)\,.\end{aligned}$$

To control the first term we use Cauchy-Schwarz inequality, while for the second one we use Cauchy-Schwarz and the Poincaré inequality for $\nu^0_{d-1,R(u)}$, since $\int f(u,\bar x)\,\nu^0_{d-1,R(u)}(d\bar x)=0$. This yields

$$\begin{aligned}\int \left|\frac{d\bar f}{du}\right|^2 d\mu_1 \;\leq\; & 2\int \left|\frac{\partial f}{\partial x_1}\right|^2 (u,\bar x)\,\mu_{d,r}(du,d\bar x)\\ & +4\int |\nabla_{\bar x}f|^2\,\mu_{d,r}(du,d\bar x)\,(A_1+4\,A_2)\end{aligned}$$

where

$$A_1=\int |u-a|^2\,\mathbb{1}_{|u-a|\leq r}\,R^{2d-6}(u)\,\left(\int |\bar x|^{4-2d}\,\nu^0_{d-1,R(u)}(d\bar x)\right)\,\mu_1(du)\,,$$

and

$$A_2=\int |u-a|^2\,\mathbb{1}_{|u-a|\leq r}\,R^{2d-6}(u)\,\left(1+\frac{R^2(u)}{d-1}\right)\,\left(\int |\bar x|^{6-2d}\,\nu^0_{d-1,R(u)}(d\bar x)\right)\,\mu_1(du)\,.$$

It is immediate (recall that the support of $\nu^0_{d-1,R(u)}$ is $|\bar x|\geq R(u)$) that

$$A_2\,\leq r^2\,\left(1+\frac{1}{d-1}\,\int_{R(u)>0} R^2(u)\,\mu_1(du)\right)\,.$$

If $r\leq\beta\sqrt{d-1}$ we thus have $A_2\leq(1+\beta^2)r^2$. In full generality it holds $A_2\leq r^2\,(1+(r^2/d-1))$.

This bound can be improved for large $r$'s provided $a$ is large too. Indeed, on $R(u)>0$,

$$\mu_1(du)\leq\frac{\gamma_{d-1}(B^c(0,R(u)))}{\gamma_d(B^c(0,r))}\,\gamma_1(du)\leq\,c\,\frac{e^{r^2-u^2}}{r}\,\left(\int_{R(u)}^{+\infty}\rho^{d-2}\,e^{-\rho^2}\,d\rho\right)\,du,$$

for some universal constant $c$. Using integration by parts we have, for $z>0$,

$$\begin{aligned}\int_z^{+\infty}\rho^{d-2}\,e^{-\rho^2}\,d\rho \;&\leq\; \frac12\,z^{d-3}\,e^{-z^2}+\frac{d-3}{2}\int_z^{+\infty}\rho^{d-4}\,e^{-\rho^2}\,d\rho\\ &\leq\;\frac12\,z^{d-3}\,e^{-z^2}+\frac{d-3}{2z^2}\int_z^{+\infty}\rho^{d-2}\,e^{-\rho^2}\,d\rho\,,\end{aligned}$$

so that, provided $z^2>d-1$,

$$\int_z^{+\infty}\rho^{d-2}\,e^{-\rho^2}\,d\rho\leq\frac{z^2}{2z^2-(d-3)}\,z^{d-3}\,e^{-z^2}\;\leq\;\frac{z^{d-1}}{2d+1}\,e^{-z^2}\,.$$

To bound $A_2$, we perform the integral on $R(u) \leq \sqrt{d-1}$ and $R(u) > \sqrt{d-1}$, so that using the previous bound we obtain

$$\begin{aligned} A_2 &\leq r^2 \left(2 + c\,\frac{r^d}{(d-1)(2d+1)} \int_{R(u)>\sqrt{d-1}} e^{r^2-u^2-R^2(u)}\, du\right) \\ &\leq r^2 \left(2 + c\,\frac{2\,r^{d+2}}{(d-1)(2d+1)}\, e^{r^2-(a-r)^2-(d-1)}\right), \end{aligned}$$

provided $a > r$. If $a > (2+\alpha)r$ for some $\alpha > 0$ we thus have,

$$A_2 \leq r^2 \left(2 + c\,\frac{2\,r^{d+2}}{(d-1)(2d+1)}\, e^{-\alpha\, r^2}\right) \leq C(\alpha)\, r^2\,,$$

for some $C(\alpha)$ that only depends on $\alpha$ (and not on $d$).

Finally we have obtained

(1) if for some $\alpha > 0$, $r < \alpha\sqrt{d-1}$ or $a > (2+\alpha)r$, $A_2 \leq C(\alpha)\, r^2$ ,
(2) in all cases $A_2 \leq r^2\,(1 + (r^2/d-1))$.

The control of $A_1$ is also a little bit delicate. Indeed we have to split the integral in two parts, the first one corresponding to the $u$'s such that $R(u) \geq 1$ (if this set is not empty), the second one to the $u$'s such that $R(u) < 1$. Thus we have the following rough bound

$$A_1 \leq r^2 \left(1 + \int_{0<R(u)\leq 1} R^{2d-6}(u)\, \frac{\int_{\rho>R(u)} \rho^{2-d}\, e^{-\rho^2}\, d\rho}{\int_{\rho>R(u)} \rho^{d-2}\, e^{-\rho^2}\, d\rho}\, \mu_1(du)\right).$$

To bound the second term in the sum, we use, for $d > 3$,

$$\int_{\rho>R(u)} \rho^{2-d}\, e^{-\rho^2}\, d\rho \leq \int_{\rho>R(u)} \rho^{2-d}\, d\rho = \frac{R^{3-d}(u)}{d-3}$$

and

$$\int_{\rho>R(u)} \rho^{d-2}\, e^{-\rho^2}\, d\rho \geq \frac{R^{d-3}(u)}{2e}\,.$$

Combining these two bounds, we obtain

$$A_1 \leq r^2 \left(1 + \int_{0<R(u)\leq 1} \frac{2e}{d-3}\, \mu_1(du)\right) \leq r^2 \left(1 + \frac{2e}{d-3}\right),$$

provided $d > 3$.
If $d = 3$, we have

$$\int_{\rho>R(u)} \rho^{2-d}\, e^{-\rho^2}\, d\rho \leq \int_{1>\rho>R(u)} \rho^{-1}\, d\rho + \int_{\rho>1} e^{-\rho^2}\, d\rho \leq \log(1/R(u)) + \sqrt{\pi}\,.$$

It follows

$$A_1 \leq r^2 \left(1 + 2e\,\sqrt{\pi} + \int_{0<R(u)\leq 1} 2e\, \log(1/R(u))\, \mu_1(du)\right).$$

It remains to get an upper bound for

$$B_1 = \int_{0<R(u)\leq 1} \log(1/R^2(u))\, \mu_1(du)\,.$$

When $r \leq 1 (= (\sqrt{d-1}/\sqrt{2})$, we have for some universal constant $c$ that may vary from line to line,

$$\begin{aligned} B_1 &\leq -c \int_{a-r}^{a+r} \log(r^2-(u-a)^2)\, \frac{\gamma_2(B^c(0,R(u)))}{\gamma_3(B^c(0,r))}\, e^{-u^2}\, du \\ &\leq -c \int_{a-r}^{a+r} \log(r^2-(u-a)^2)\, e^{-u^2-R^2(u)}\, du \\ &\leq -c \int_{a-1}^{a+1} \log(1-(u-a)^2)\, du \\ &\leq c\,. \end{aligned}$$

When $r > 1$ the integral splits in two terms

$$\begin{aligned} B_1 &= -c \int_{a-r}^{a-\sqrt{r^2-1}} \log(r^2-(u-a)^2)\, \frac{e^{-u^2-R^2(u)+r^2}}{r}\, du \\ &\quad -c \int_{a+\sqrt{r^2-1}}^{a+r} \log(r^2-(u-a)^2)\, \frac{e^{-u^2-R^2(u)+r^2}}{r}\, du\,. \end{aligned}$$

Note that, provided $a > 2r$, $-u^2 - R^2(u) + r^2 = -a(2u-a) \leq -a(a-2r) \leq 0$ in the first integral while $-u^2 - R^2(u) + r^2 \leq -a^2 \leq 0$ for all $a$ in the second one. So we have, using the change of variable $u - a = -r + rv$ (resp. $u - a = r - rv$) and recalling that $c$ may vary but is still universal,

$$B_1 \leq -c \int_0^{(r-\sqrt{r^2-1})/r} \log(r)\, \log(v(2-v))\, dv \leq c\, \log(r)\,.$$

If we assume that $a > (2+\alpha)r$ for some $\alpha > 0$, one can improve the previous bound in $c(\alpha)$ independent of $r$.

Unfortunately, when $0 \leq a \leq 2r$ we only obtain $B_1 \leq c\, \log(r)\, e^{a(2r-a)}$.

We have thus obtained

(1) if $d > 3$ then $A_1 \leq c\, r^2$,
(2) for $d = 3$, if $r \leq 1$ or $a > (2+\alpha)r$, $A_1 \leq cr^2$,
(3) for $d = 3$, $r > 1$ and $a > 2r$, $A_1 \leq c\, r^2\, \log(r)$,
(4) for $d = 3$, $r > 1$ and $0 < a < 2r$, $A_1 \leq c\, r^2\, (1 + e^{a(2r-a)}) \leq c\, r^2\, e^{r^2}$.

Gathering together all we have done we have shown

**Theorem C.10.** *Assume $d \geq 3$. There exists a function $C(r,d)$ such that, for all $y \in \mathbb{R}^d$,*

$$C_P(1,y,r) \leq C(r,d)\,.$$

*Furthermore, there exists some universal constant $c$ such that*

$$C(r,d) \leq \left(1 + \frac{r^2}{d-1}\right) + C_1(r)\, \max\left(2\,, C_2(r)\right)\,,$$

*$C_1(r)$ being given in Lemma C.7 and $C_2(r)$ satisfying*

*(1) if $r \leq \sqrt{(d-1)/2}$ or $|y| > (2+\alpha)r$, $C_2(r) \leq c\, r^2$,*
*(2) if $d > 3$ or $d = 3$, $r \geq 1$ and $|y| > 2r$, $C_2(r) \leq c\, r^2 \left(1 + \frac{r^2}{d-1}\right)$,*
*(3) if $d = 3$, $r \geq 1$ and $0 \leq |y| \leq 2r$, $C_2(r) \leq c\, r^2\, \max\left(r^2\,, e^{|y|(2r-|y|)}\right)$.*

**Remark C.11.** The previous theorem is interesting as it shows that when $N = 1$, the Poincaré constant is bounded uniformly in $y$ and it furnishes some tractable bounds.

The method suffers nevertheless two defaults. First it does not work for $d = 2$, in which case the conditioned measure does no more satisfy a Poincaré inequality. More important for our purpose, the method does not extend to more than one obstacle, unless the obstacles have a particular location. $\diamondsuit$

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