M2R Examination A8 Novembre the 27 th 2025,

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3 hours examination.
You can write in French or in English.

Exercice 1 (8 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $(B_t)_{t\geq 0}$ be a real valued Brownian motion starting at 0 defined on this probability space. Let $\mathcal{F}_t,\ 0 \leq t \leq \infty$ be its natural filtration completed. We assume that $\mathcal{F}_\infty \subset \mathcal{A}$, and we admit that $\mathcal{F}_t,\ 0 \leq t \leq \infty$ satisfies the usual conditions.

1. For $\alpha,\ \beta, x \in \mathbb{R}$, let $f(x) = \alpha x + \beta$, and $M_t = f(B_t)$ for $t \geq 0$. Show that M is martingale for the filtration $\mathcal{F}_t,\ 0 \leq t \leq \infty$.

2. Let $a < 0 < b$ be real numbers and $T_{a,b} = \inf\{t \geq 0,\ B_t \notin]a, b[\}$. We admit that $T_{a,b} < +\infty$ $\mathbb{P}$ almost surely. Show that $\mathbb{E}(B_{T_{a,b}}) = 0$.

3. Deduce from the previous question that

$$\mathbb{P}(B_{T_{a,b}} = a) = \frac{b}{b-a}, \quad \mathbb{P}(B_{T_{a,b}} = b) = \frac{-a}{b-a}.$$

4. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. We assume from now on that $M_t = g(B_t)$ is martingale for the filtration $\mathcal{F}_t,\ 0 \leq t \leq +\infty$. Show that $M_{T_{a,b}\wedge t}$ for $t \geq 0$, is a bounded martingale, where $x \wedge y = \min(x, y)$.

5. Deduce from the previous questions that for all $a < 0 < b$

$$\frac{b}{b-a} g(a) + \frac{-a}{b-a} g(b) = g(0).$$

6. Conclude that there exist $\alpha,\ \beta \in \mathbb{R}$, such $g(x) = \alpha x + \beta$.

Exercise 2 (8 points)

We assume that the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is endowed with a filtration $(\mathcal{F}_t)_{t\in[0,\infty]}$ that satisfies the usual conditions and such that $\mathcal{F}_\infty \subset \mathcal{A}$. For

all integers $n \geq 1$, let $M^n = (M_t^n)_{t\geq 0}$ be a continuous local martingale for the filtration $(\mathcal{F}_t)_{t\in[0,\infty]}$ starting from 0. We assume in this exercise that $\lim_{n\to\infty} \langle M^n, M^n\rangle_\infty = 0$ with convergence in probability.

1. Let $\varepsilon > 0$ and $\forall n \geq 1$ let $T_\varepsilon^n = \inf\{t \geq 0 : \langle M^n, M^n\rangle_t \geq \varepsilon\}$ with the usual convention $\inf \varnothing = \infty$. Explain why $\forall n \geq 1,\ T_\varepsilon^n$ is a $\mathcal{F}_t$-stopping time and $M_t^{n,\varepsilon} = M_{t\wedge T_\varepsilon^n}^n$, is a bounded martingale in L^2.

2. Show that $\forall n \geq 1,\ E\left[\sup_{t\geq 0} (M_t^{n,\varepsilon})^2\right] \leq 4\varepsilon$.

3. Show that $\lim_{n\to\infty} \left(\sup_{t\geq 0} |M_t^n|\right) = 0$ with convergence in probability.

Exercise 3 (8 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $(B_t)_{t\geq 0}$ be a real valued Brownian motion starting at 0 defined on this probability space. Let $\mathcal{F}_t,\ 0 \leq t \leq \infty$ be its natural filtration completed. We assume that $\mathcal{F}_\infty \subset \mathcal{A}$, and we admit that $\mathcal{F}_t,\ 0 \leq t \leq \infty$ satisfies the usual conditions. Let us set $Z \overset{\text{def}}{=} \{(t,\omega)/B_t(\omega) = 0\}$ and $z_\omega \overset{\text{def}}{=} \{0 \leq t < +\infty/B_t(\omega) = 0\}$.

1. Show that $Z \in \mathcal{B}([0,+\infty[) \otimes \mathcal{F}_\infty$.

2. Compute $(\lambda \otimes \mathbb{P})(Z)$ where λ is the Lebesgue measure on $[0,+\infty[$.

3. Deduce that $\mathbb{P}$ almost surely $\lambda(z_\omega) = 0$.

   Let $W_t = (W_{1,t}, W_{2,t})$ be a Brownian motion valued in $\mathbb{R}^2$ for the filtration $\mathcal{F}_t,\ 0 \leq t \leq \infty$ starting from $(1,0)$ and $R_t = \sqrt{W_{1,t}^2 + W_{2,t}^2}$.

4. Show that $\mathbb{P}$ almost surely $\lambda(\{t, 1/R_t = +\infty\}) = 0$.

5. We set for $i = 1,\ 2,\ B_{i,t} = \int_0^t \frac{W_{i,s}}{R_s} dW_{i,s}$ show that this process is square integrable martingale.

6. What is the distribution of the process $\tilde{B}_t = B_{1,t} + B_{2,t}$?

7. Let $Y_t = R_t^2$, show that $Y_t = 1 + 2\sum_{i=1}^2 \int_0^t W_{i,s} dW_{i,s} + 2t$.