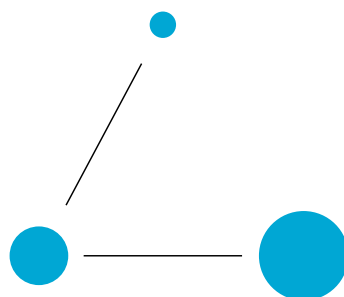


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ANNALES
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ON THE CENTER-FOCUS PROBLEM FOR THE EQUATION

$$\frac{dy}{dx} + \sum_{i=2}^n a_i(x)y^i = 0, 0 \leq x \leq 1$$

WHERE a_i ARE POLYNOMIALS

SUR LE PROBLÈME DU CENTRE-FOYER POUR

L'ÉQUATION $\frac{dy}{dx} + \sum_{i=2}^n a_i(x)y^i = 0, 0 \leq x \leq 1$

OÙ a_i SONT DES POLYNÔMES

ABSTRACT. — This is an extended version of two lectures given during the Zagreb Dynamical Systems Workshop, October 22-26, 2018.

RÉSUMÉ. — Dans le résumé en français, les détails ne rentrent pas plus, mais ce n'est pas pour cela qu'il faut s'en dispenser. (If you don't speak French, don't worry, the editors will translate the abstract for you)

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1. The center-focus problem

The plane differential system

$$(1.1) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

is said to have a *center* at the singular point $(0, 0)$, if in a sufficiently small neighbourhood of this point all orbits are closed. Consider the scalar differential equation

$$(1.2) \quad \frac{dy}{dx} + f(x, y) = 0, \quad x \in [0, 1]$$

in which $f(x, 0) = 0, \forall x \in [0, 1]$. The equation (1.2) is said to have a *center* at $y = 0$, if all solutions $y(x)$ starting near the origin, satisfy $y(0) = y(1)$ (the interval $[0, 1]$ can be replaced by any closed interval).

Note on the terminology. We do not specify here the category to which belong P, Q, f . They will be either analytic or polynomial, depending on the context. The base field will be either \mathbb{R} or \mathbb{C} depending on the context too. Most results will be valid for both. Thus, the definition of a center for (1.2) is the same in the real and in the complex case. In the case of an analytic complex plane vector field (1.1) the "complex" definition of a center is less straightforward. We say that the origin is a non-degenerate center, if the vector field has an analytic first integral with a Morse critical point at the origin. If this is the case, we shall also say that (1.1) has a Morse singular point, e.g. [6]. We recall therefore

DEFINITION 1.1. — *The analytic complex vector field (1.1) is said to have a Morse singular point, if it allows an analytic first integral in a neighbourhood of this point, which has a Morse type singularity.*

If (1.1) has a Morse singular point, then the linear part of (1.1) is diagonalisable with non-zero eigenvalues, that is to say the singular point of the vector field is non-degenerate.

An example is the saddle $x' = x, y' = -y$ which has an analytic first integral xy of Morse type, and hence a Morse critical point. Of course, it is linearly equivalent (over \mathbb{C}) to $x' = y, y' = -x$ with first integral $x^2 + y^2$ which is the usual linear real center. The advantage to study Morse critical points over \mathbb{C} is that we can use complex analysis and complex algebraic geometry. This is the point of view adopted in these notes.

The two equations (1.1) and (1.2) are closely related. First, a polar change of variables transforms a plane system (1.1) with a center to equivalent equation of the form (1.2) with a center along the interval $[0, 2\pi]$. Second, if the family of functions $f(\cdot, y), x \in [0, 1]$ is replaced by its Fourier series $\hat{f}(\cdot, y)$ (so $\hat{f}(x + 1, y) = \hat{f}(x, y)$) and the equation (1.2) has a center at $y = 0$, then the new system

$$(1.3) \quad \frac{dy}{dx} + \hat{f}(x, y) = 0, \quad (x, y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$$

will have all its orbits starting near the periodic solution $y = 0$ on the cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$, periodic too. Of course, if the smooth function f is non-periodic, then the function \hat{f} is only piece-wise smooth. The transport map of (1.2) along $[0, 1]$ becomes a return map for (1.3) and the definition of a limit cycle for (1.2) is straightforward

too. Actually, the scalar equation (1.2) in which f is a regular function, should be considered as a simplified model of the eventually singular equation

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}.$$

We resume the above considerations in the following definitions, which make sense both on \mathbb{R} or \mathbb{C} :

DEFINITION 1.2. — *Let $\varphi = \varphi(\cdot; x_0, y_0)$ be the general solution of the equation $dy + f(x, y)dx = 0$ on the interval $[x_0, x_1]$.*

- (i): *The solution $\varphi = \varphi(\cdot; x_0, y_0)$ is said to be periodic iff $\varphi(x_1; x_0, y_0) = y_0$*
- (ii): *The solution $\varphi = \varphi(\cdot; x_0, y_0)$ is said to be a limit cycle, provided that it is periodic and isolated, that is to say there is a neighbourhood of its orbit on $S^1 \times \mathbb{R}$ free of periodic solutions.*
- (iii): *the map $y \mapsto \varphi(x_1; x_0, y)$ is the first return map of (1.2) in a neighbourhood of $(x_0, y = y_0)$.*
- (iv): *The equation (1.2) defines a center in a neighbourhood of the periodic solution φ provided that the first return map is the identity map in a neighbourhood of y_0 . If the return map is not the identity map, then we say that (1.2) defines a focus at the periodic solution φ .*

The center focus-problem for the equation (1.2) or (1.1) is, roughly speaking, to distinguish between a center and a focus. The algebro-geometric content of the problem is as follows. Suppose, that (1.2) is polynomial, more precisely

$$(1.4) \quad \frac{dy}{dx} + \sum_{i=1}^p a_i(x)y^i, \quad a_i \in \mathbb{C}[x], \quad \deg a_i \leq n, \quad x \in [0, 1]$$

so that $y = 0$ is a periodic solution. As we shall see in the next section, the first return map $y \mapsto \varphi(y)$ is analytic near $y = 0$ and moreover

$$\varphi(y) = y + \sum_{n=1}^{\infty} c_n(a)y^{n+1}.$$

where the coefficients $c_n = c_n(a)$ are polynomials in the coefficients of $a_j = a_j(x)$, $j \leq n$. The condition $\varphi = id$ determines an infinite number of polynomial relations on the coefficients of a_j . By the Hilbert basis theorem, only a finite number of them are relevant, and they define an algebraic variety - the so called center variety \mathcal{C}_n - in the vector space of all coefficients of the polynomials a_j . The problem is therefore (as formulated by Lins Neto [21] in the context of a polynomial foliation induced by (1.1))

Describe the irreducible components of \mathcal{C}_n .

The content of the lectures is as follows. In section 2 we give a self-contained proof of an explicit formula, due to Brudnyi, for the solutions of the equation

$$\frac{dy}{dx} + \sum_{i=1}^n a_i(x)y^i = 0, \quad 0 \leq x \leq 1$$

in terms of iterated path integrals.

In section 3 we show how the perturbation theory of the Abel equation $\frac{dy}{dx} = a(x)y^2$ leads to the problem of vanishing of a suitable Abelian integral. The conditions for vanishing of this Abelian integral give rise to a "moment problem" which has an elegant solution, due to Christopher. The solution is based on the well known Lüroth theorem.

Our main results concern the irreducible components of the center set of the scalar Abel equation and are formulated in section 4. In section 4.1 we prove, that the set of scalar Abel equation with universal center (in the sense of Brudnyy) provide irreducible components of the Center set.

In sections 4.2, 4.3 we give a full description of the center set of two remarkable classical systems : quadratic vector fields and Liénard type equations. These results belong mainly to Dulac, Cherkas and Christopher, but we present them in the broader context of the present notes. In particular, the base field will be \mathbb{C} . All quadratic centers have a Darboux first integral, all Liénard centers come from "pull-back" (they are rationally reversible).

It is fascinating that the centers of these two examples are of quite different nature, which motivates the so called Composition Conjecture for the Abel equation (1.2) to be discussed in section 4.4. In this section we show that there are scalar Abel equations with a non-universal center (not related to a "pull back"). These equations have a Darboux type first integral, and among them we find the recent counter-example to the Composition Conjecture mentioned above, found by Giné, Grau and Santallusia [17].

2. The first return map and the Brudnyy formula

In this section we shall describe the return map of (1.4) as a power series involving iterated path integrals. We give an explicit formula due to Brudnyy [5], which amounts to solve the differential equation. The classical approach to do this is by the Picard iteration method. If y_0 is the initial condition at x_0 of the differential equation

$$dy = f(x, y)dx$$

then the Picard iteration is

$$y_{n+1}(x) = y_0 + \int_{x_0}^x y_n(t)dt$$

where y_n tends to the solution of the equation as $n \rightarrow \infty$. We illustrate this on the example $dy = ydx$. If y_0 is the initial condition at $x = 0$ then

$$\begin{aligned} y_1(x) &= y_0 + \int_0^x y_0 dt \\ y_2(x) &= y_0 + \int_0^x y_1(t)dt = y_0 + \int_0^x y_0 dt + \iint_{0 \leq t_2 \leq t_1 \leq x} y_0 dt_1 dt_2 \end{aligned}$$

As

$$\int \dots \int_{0 \leq t_n \leq \dots \leq t_1 \leq x} y_0 dt_1 \dots dt_n = y_0 \frac{x^n}{n!}$$

we get $y(x) = y_0 e^x$ as expected. The multiple (or iterated) integrals above appear in a similar way in the non-autonomous linear $dy = a(x)ydx$, or even non-linear case $dy = f(x, y)dx$. The non-linear case is more involved, it is reduced to the linear one, but after introducing infinitely many new variables y, y^2, y^3, \dots . To get around this reduction we shall use a simple Ansatz, for which we need a formal definition of iterated integral.

Let Ass_ω be the graded free associative algebra generated by the infinite dimensional vector space of differential one-forms $\omega = a(x, y)dx$, $a \in \mathbb{C}\{x, y\}$. Its elements are non-commutative polynomials in such one-forms. The differential operator

$$D : Ass_\omega^1 \rightarrow Ass_\omega^1$$

$$D(a(x, y)dx) = \frac{\partial}{\partial y} a(x, y)dx$$

induces a differential operator on Ass_ω which acts by the Leibnitz rule. The readers familiar with the Picard-Lefschetz theory will recognize in D an avatar of the covariant derivative of an Abelian integral along the level sets $\{y = const\}$.

To save brackets, it is convenient to introduce the following notation

$$(2.1) \quad D\omega_1\omega_2 \dots \omega_n = D(\omega_1\omega_2 \dots \omega_n)$$

so that (using brackets)

$$D\omega_1\omega_2 = D(\omega_1\omega_2) = (D\omega_1)\omega_2 + \omega_1(D\omega_2).$$

and

$$D\omega_1 D\omega_2 = D(\omega_1 D\omega_2) = (D\omega_1)(D\omega_2) + \omega_1(D^2\omega_2).$$

If we use the notation

$$D^k\omega = \omega^{(k)}$$

then

$$D\omega_1\omega_2 = \omega_1'\omega_2 + \omega_1\omega_2'$$

and

$$D\omega_1 D\omega_2 = (\omega_1\omega_2')' = \omega_1'\omega_2' + \omega_1\omega_2''.$$

For $\omega_1\omega_2 \dots \omega_n \in Ass_\omega^n$, $\omega_k = \varphi_k(x, y)dx$, define the iterated integral $\int_{x_0}^x \omega_1\omega_2 \dots \omega_n$ of length n , as equal to

$$(2.2) \quad \int \dots \int_{x_0 \leq t_n \leq \dots \leq t_1 \leq x} \varphi_1(t_1, y) \dots \varphi_n(t_n, y) dt_1 \dots dt_n.$$

The iterated integral allows also a recursive definition (hence the name) :

$$(2.3) \quad \int_{x_0}^x \omega_n\omega_{n-1} \dots \omega_1 = \int_{x_0}^x (\varphi_n(t) \int_{x_0}^t \omega_{n-1} \dots \omega_1) dt$$

where in the case $n = 1$ we have the Riemann integral $\int_{x_0}^x \omega_1$. We note, that the usual notation for the multiple integral (2.2) is $\int_{x_0}^x \omega_n\omega_{n-1} \dots \omega_1$ on the place of $\int_{x_0}^x \omega_1\omega_2 \dots \omega_n$, see [7, Chen] or [18, Hain]. The reason to prefer the definition (2.3) is that it is better adapted to applications in differential equation, e.g. [14]. Recall in this context, that

$$\int_{x_0}^x \omega_n\omega_{n-1} \dots \omega_1 = (-1)^n \int_x^{x_0} \omega_1\omega_2 \dots \omega_n.$$

For a short summary of properties of iterated integrals which we use see [14, Appendix], [13, section 2].

THEOREM 2.1. — *With the notation (2.1), a first integral of the differential equation $dy + f(x, y)dx = 0$ is given by the following recursively defined convergent series*

$$(2.4) \quad \boxed{\varphi(x_0; x, y) = y + \int_{x_0}^x \omega + \int_{x_0}^x \omega D\omega + \int_{x_0}^x \omega D\omega D\omega + \dots}$$

where

$$\omega = f(x, y)dx.$$

The general solution of (1.2) with initial condition (x_0, y_0) is given by

$$y = \varphi(x; x_0, y_0).$$

Example 2.2. — In the linear case

$$y' + \alpha y = 0 \iff dy + \alpha y dx = 0$$

we obtain

$$\begin{aligned} \varphi(x_0; x, y) &= y(1 + \alpha \int_{x_0}^x dx + \alpha^2 \int_{x_0}^x dx \cdot dx + \dots) \\ &= y(1 + \alpha(x - x_0) + \alpha^2 \frac{(x - x_0)^2}{2} + \dots) \\ &= y e^{\alpha(x - x_0)} \end{aligned}$$

and the general solution is

$$y = \varphi(x; x_0, y_0) = y_0 e^{\alpha(x_0 - x)}.$$

In the quadratic case

$$dy + 2xy^2 dx = 0, \omega = 2xy^2 dx$$

we compute recursively

$$\begin{aligned} \int_{x_0}^x \omega &= \int_{x_0}^x 2xy^2 dx = x^2 - x_0^2 \\ \int_{x_0}^x \omega D\omega &= \int_{x_0}^x 2xy^2 dx \cdot 4xy dx = y^3(x^2 - x_0^2)^2 \\ \int_{x_0}^x \omega D\omega \dots D\omega &= (x^2 - x_0^2)^n \end{aligned}$$

Therefore we get the first integral

$$\varphi(x_0; x, y) = y + y^2(x^2 - x_0^2) + y^3(x^2 - x_0^2)^2 + \dots$$

and the corresponding general solution is

$$\begin{aligned} y = \varphi(x; x_0, y_0) &= y_0 + y_0^2(x_0^2 - x^2) + y_0^3(x_0^2 - x^2)^2 + \dots \\ &= \frac{y_0}{1 - y_0(x_0^2 - x^2)}. \end{aligned}$$

Proof of Theorem 2.1. We first verify, that for every fixed x_0 , the function $\varphi(x_0; x, y)$ is a first integral :

$$\begin{aligned} d\varphi(x_0; x, y) &= \frac{\partial}{\partial x}\varphi(x_0; x, y)dx + \frac{\partial}{\partial y}\varphi(x_0; x, y)dy \\ &= \omega + \omega \int_{x_0}^x D\omega + \omega \int_{x_0}^x D\omega D\omega + \omega \int_{x_0}^x D\omega D\omega D\omega + \dots \\ &+ (1 + \int_{x_0}^x D\omega + \int_{x_0}^x D\omega D\omega + \int_{x_0}^x D\omega D\omega D\omega + \dots)dy \\ &= (\omega + dy)\frac{\partial}{\partial y}\varphi(x_0; x, y)dy = 0. \end{aligned}$$

As $\varphi(x_0; x_0, y_0) = y_0$ then the level set $\{(x, y) : \varphi(x_0; x, y) = y_0\}$ contains both (x_0, y_0) and (x, y) . By symmetry

$$y = \varphi(x; x_0, y_0)$$

is the solution of (1.2) with initial condition $y(x_0) = y_0$. The convergency proof is by standard a priori estimates (omitted) \square

Note that for fixed x_0, x_1 the two return maps

$$y \mapsto \varphi(x_1; x_0, y), \quad y \mapsto \varphi(x_0; x_1, y)$$

are mutually inverse. Therefore $\varphi(x_1; x_0, \cdot) = id$ if and only if $\varphi(x_0; x_1, \cdot) = id$. Using Theorem 2.1 we can give explicit center conditions. Assume that

$$f(x, y) = \sum_{i=1}^{\infty} a_i(x)y^{i+1}.$$

and develop the return map $\varphi(x_0; x_1, y)$ as a power series in y

$$(2.5) \quad \varphi(x_0; x_1, y) = y + \sum_{n=1}^{\infty} c_n(a)y^{n+1}.$$

If we denote, by abuse of notations, $a_i = a_i(x)dx$ then we get for the first few coefficients $c_n(a)$

$$c_1(a) = \int_{x_0}^{x_1} a_1$$

$$c_2(a) = \int_{x_0}^{x_1} a_2 + 2a_1a_1$$

$$c_3(a) = \int_{x_0}^{x_1} a_3 + 2a_2a_1 + 3a_1a_2 + 6a_1^3$$

$$c_4(a) = \int_{x_0}^{x_1} a_4 + 2a_3a_1 + 3a_2^2 + 4a_1a_3 + 6a_2a_1^2 + 8a_1a_2a_1 + 12a_1^2a_2 + 24a_1^4$$

and so on. The general form of the coefficients $c_n(a)$ is found immediately from Theorem 2.1. We resume this in the following

THEOREM 2.3 (Brudnyi's formula). — *The coefficients $c_n(a)$ of the first return map (2.5) for the differential equation*

$$\frac{dy}{dx} + \sum_{i=1}^{\infty} a_i(x)y^{i+1} = 0, x \in [x_0, x_1]$$

are given by the formulae

$$c_n(a) = \sum_{i_1 + \dots + i_k = n} c_{i_1, \dots, i_k} \int_{x_0}^{x_1} a_{i_1} \cdots a_{i_k}$$

where

$$\begin{aligned} c_{i_1} &= 1 \\ c_{i_1, i_2} &= i_2 + 1 \\ c_{i_1, i_2, i_3} &= (i_3 + 1)(i_2 + 1) \\ &\vdots \\ c_{i_1, \dots, i_k} &= (i_k + 1)(i_k + i_{k-1} + 1) \dots (i_k + \dots + i_2 + 1) \end{aligned}$$

The above formula was deduced first by Brudnyi [5, p.422] under equivalent form, see also [4, Proposition 2.4] in the case (4.26).

COROLLARY 2.4. — *The equation (1.2) has a center on the interval $[x_0, x_1]$ if and only if $c_n(a) = 0$, for all integer $n \geq 1$.*

Example 2.5. — Suppose that the equation

$$\frac{dy}{dx} + a_1(x)y^2 + a_2(x)y^3 + \dots = 0$$

has a center on the interval $[x_0, x_1]$. Then, using as above the notation $a_i = a_i(x)dx$ we have

$$\begin{aligned} c_1 &= \int_{x_0}^{x_1} a_1 = 0 \\ c_2 &= \int_{x_0}^{x_1} a_2 + 2 \int_{x_0}^{x_1} a_1^2 = 0. \end{aligned}$$

The identity

$$2 \int_{x_0}^{x_1} a_1^2 = \left(\int_{x_0}^{x_1} a_1 \right)^2$$

implies then, that $\int_{x_0}^{x_1} a_2 = 0$. If we consider more specifically the Abel equation

$$(2.6) \quad \frac{dy}{dx} + a_1(x)y^2 + a_2(x)y^3 = 0$$

then taking into consideration that $\int_{x_0}^{x_1} a_1^3 = 0$ and

$$\int_{x_0}^{x_1} (a_2 a_1 + 3a_1 a_2) = \int_{x_0}^{x_1} a_1 a_2 = 0$$

we obtain $c_3 = \int_{x_0}^{x_1} a_1 a_2$. Therefore a necessary condition for the Abel equation (2.6) to have a center on $[x_0, x_1]$ is

$$(2.7) \quad \int_{x_0}^{x_1} a_1 = 0, \int_{x_0}^{x_1} a_2 = 0, \int_{x_0}^{x_1} a_1 a_2 = 0$$

If we suppose that a_1, a_2 are polynomials of degree at most two, these conditions are also sufficient. The case $\deg a_1, a_2 = 3$ can be studied similarly, see [?, 3].

In general, an obvious sufficient condition to have a center is therefore

$$(2.8) \quad \int_{x_0}^{x_1} a_{i_1} \cdots a_{i_k} = 0, \forall i_j, k \geq 1.$$

Centers with the property (2.8) were called *universal* in [5].

Consider, more specifically, the following equation with polynomial coefficients a_i

$$(2.9) \quad dy + \sum_{i=1}^n y^{i+1} a_i(x) dx = 0, a_i(x) \in K[x].$$

THEOREM 2.6 (Brudny, [5], Corollary 1.20). — *The polynomial equation (2.9) has an universal center on the interval $[x_0, x_1]$, if and only if, it is a pull back of some polynomial equation*

$$(2.10) \quad dy = \left(\sum_{i=1}^n b_i(\xi) y^{i+1} \right) d\xi, b_i(\xi) \in K[\xi].$$

via a suitable polynomial map $\xi = \xi(x)$ having the property $\xi(x_0) = \xi(x_1)$.

Not all centers of (2.9) are universal, as discovered recently in [17].

For a further use, note that an obvious consequence from $F(x, y) \equiv y$ is that

$$\int_{x_0}^{x_1} a_1(x) dx = \int_{x_0}^{x_1} a_2(x) dx = 0.$$

This will be used when studying the center problem for the so called degenerate Abel equation of first kind

$$\frac{dy}{dx} = a_1(x)y^2 + a_2(x)y^3.$$

3. Bifurcation functions and a Theorem of Christopher

In this section we study the following perturbed Abel differential equation on the interval $[0, 1]$

$$y' = a(x)y^2 - \sum_{j=1}^{\infty} \varepsilon^j (y^2 p_j(x) + y^3 q_j(x))$$

or equivalently

$$(3.1) \quad \frac{dy}{y^2} = a(x) dx - \varepsilon \omega_1 - \varepsilon^2 \omega_2 - \dots$$

where

$$\omega_j = (p_j(x) + y q_j(x)) dx$$

and $a = a(x), p_j = p_j(x), q_j = q_j(x)$ are polynomials of degree

$$\deg a = n, \deg p_j \leq n, \deg q_j \leq n$$

and ε is a small parameter, see [1, 2, 3, 8, 27]. For $\varepsilon = 0$ (3.1) has a first integral

$$H(x, y) = \frac{1}{y} + A(x), \quad A(x) = \int a(x) dx.$$

How many limit cycles has the perturbed system (3.1) on the interval $[0, 1]$?

Recall from the preceding section that a solution $y(x)$ such that $y(0) = y(1)$ is called periodic on $[0, 1]$. A limit cycle of (3.1) on $[0, 1]$ is therefore an isolated periodic solution on $[0, 1]$.

The number of the limit cycles in a compact set are bounded by the number of the zeros of the so called *bifurcation function*, which we define bellow. A limit cycle which remains bounded when $\varepsilon \rightarrow 0$, tends to a periodic solution of the non perturbed system. If the non-perturbed system ($\varepsilon = 0$) has a periodic solution, then necessarily $A(0) = A(1)$, which already implies that it has a center. For this reason we assume from now on that $A(0) = A(1) = 0$, so that

$$dy = a(x)y^2 dx \Leftrightarrow dH = 0$$

has a center along $0 \leq x \leq 1$. The perturbed equation can be written

$$(3.2) \quad dH - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \dots = 0.$$

For a solution $y(x)$, let \mathcal{P}_ε be the first return map which sends the initial condition $y_0 = y(0)$ to $y_1 = y(1)$. We parameterise \mathcal{P}_ε by $h = \frac{1}{y} = H(0, y) = H(1, y)$ and note that \mathcal{P}_ε is analytic both in h and ε (close to zero). We have therefore for the first return map

$$(3.3) \quad \mathcal{P}_\varepsilon(h) - h = \varepsilon^k M_k(h) + O(\varepsilon^{k+1}), \quad M_k \neq 0$$

The function M_k is the *bifurcation function*, associated to the equation (3.1). It is also known as "first non-zero Melnikov function". The reader may compare this to (2.4) which is another representation of the first return map, defined for small y . As we shall see, the bifurcation function is globally defined. Therefore for every compact set K , $[0, 1] \subset K \subset \mathbb{R}^2$ and all sufficiently small $|\varepsilon|$, the number of the limit cycles of (3.1) in K is bounded by the number of the zeros of the bifurcation function M_k (counted with multiplicity).

M_k allows an integral representation

$$M_k(h) = \int_{\{H=h\}} \Omega_k$$

where the integration is along the level set

$$\{H = h\} = \{(x, y) : 1/y + A(x) = h, 0 \leq x \leq 1\}.$$

The differential form Ω_k is computed by the classical Françoise's recursion formula [12, 20, 25] as follows:

If $k = 1$ then $\Omega_1 = \omega_1$, otherwise

$$(3.4) \quad \Omega_m = \omega_m + \sum_{i+j=m} r_i \omega_j, \quad 2 \leq m \leq k$$

and the functions r_i , $1 \leq i \leq k - 1$ are determined successively from the identities $\Omega_i = dR_i + r_i dH$.

Note that neither r_i nor R_i are uniquely defined. The integrals $M_i(h)$ are, however, defined unambiguously.

The first order Melnikov function M_1 is computed in [3], and probably earlier, e.g. Lins Neto [22, section 3]. We have

$$\begin{aligned} M_1(h) &= \int_{\{H=h\}} \omega_1 \\ &= \int_{\{H=h\}} p_1(x)dx + yq_1(x)dx \\ &= \int_0^1 p_1(x)dx + \int_0^1 \frac{q_1(x)}{h - A(x)}dx \\ &= \int_0^1 p_1(x)dx + \sum_{k=0}^{\infty} h^{-k-1} \int_0^1 q_1(x)A^k(x)dx. \end{aligned}$$

M_1 vanishes identically if and only if

$$\int_0^1 p_1(x)dx = 0, \int_0^1 q_1(x)A^k(x)dx = 0, k = 0, 1, 2, \dots$$

which is the content of the polynomial moment problem for q_1 and A . If $M_1 = 0$ by the above formula we get

$$M_2(h) = \int_{\{H=h\}} r_1\omega_1 + \int_{\{H=h\}} \omega_2$$

where r_1 is computed from the identity $\omega_1 = dR_1 + r_1dH$. As $d\omega_1 = dr_1 \wedge dH$ then $dr_1 = \omega'_1 = \frac{d\omega_1}{dH}$ is the Gelfand-Leray form of ω_1 . From the identity $H(x, y(x, h)) \equiv h$ we have $\frac{\partial y}{\partial h} = -y^2$ and hence

$$r(x, y) = \int_0^x \omega'_1 = - \int_0^x y^2 q(x)dx.$$

We conclude

PROPOSITION 3.1 (formula (2.8) in [14]). — *Under the hypothesis $M_1 = 0$ the second Melnikov function reads*

$$(3.5) \quad M_2(h) = \int_{\{H=h\}} \omega_1\omega'_1 + \int_{\{H=h\}} \omega_2$$

where

$$\omega_1 = p_1(x)dx + yq_1(x)dx, \omega'_1 = -y^2q_1(x)dx, \omega_2 = p_2(x)dx + yq_1(x)dx.$$

The hypothesis $M_1 = 0$ is of interest for us, as it will allow to compute the tangent space to the center set at the point $(a, 0)$, see the next section. In full generality this vanishing problem is solved by Pakovich and Muzychuk [24]. For our purposes however, the polynomial $a(x)$ can be taken in a general position, as in the following

THEOREM 3.2 (Christopher [8]). — *Assume that A, q are complex univariate polynomials, such that*

$$A(0) = A(1) = 0, A'(0) \neq 0, A'(1) \neq 0.$$

The multivalued transcendental function

$$(3.6) \quad I(h) = \int_0^1 \frac{q(x)}{h - A(x)}dx$$

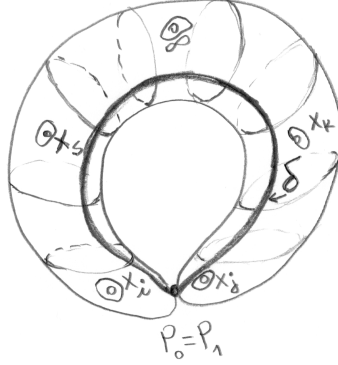


Figure 3.1. The singularized algebraic curve Γ_h^{sing} .

vanishes identically, if and only if the polynomials $Q = \int q$ and A satisfy the following "Polynomial Composition Condition" (PCC) :

There exist polynomials \tilde{Q}, \tilde{A}, W , such that

$$A = \tilde{A} \circ W, Q = \tilde{Q} \circ W, W(0) = W(1).$$

Before recalling the elegant proof of Christopher, we put I in the broader context of the Picard-Lefschetz theory.

The function $I(h)$ is well defined for sufficiently big h , and has an analytic continuation in a complex domain to certain multivalued function. It is in fact an Abelian integral depending on a parameter. More precisely, consider the genus zero affine curve

$$\Gamma_h = \{(x, y) \in \mathbb{C}^2 : \frac{1}{y} + A(x) = h\}, \quad A(0) = A(1) = 0.$$

It is a Riemann sphere with $n+2$ removed points, provided that $h \neq 0$. The removed points correspond to $(x = x_i(h), y = \infty)$, where $A(x_i(h)) \equiv h$, and to $(x = \infty, y = 0)$. Given a divisor $m = P_0 + P_1$ on Γ_h , where

$$P_0 = (0, \frac{1}{h}), \quad P_1 = (1, \frac{1}{h})$$

we define a singularized algebraic curve Γ_h^{sing} , see figure 3.1 As a topological space it is just the curve Γ_h with the two points P_0 and P_1 identified to a point m . The structural sheaf of Γ_h^{sing} is the same as the structural sheaf of Γ_h , except at the point $m \in \Gamma_h^{sing}$. A function f on Γ_h^{sing} is said to be regular, if it is regular on Γ_h , and moreover $f(P_0) = f(P_1)$. The path $[0, 1]$ connecting the points $x = 0$ and $x = 1$ closes on the singular algebraic curve Γ_h^{sing} . The function $I(h)$ defined in (3.6) is an Abelian integral on Γ_h^{sing}

$$I(h) = \int_{\delta(h)} yq(x)dx, \quad y = \frac{1}{h - A(x)}$$

where $\delta(h) \in H_1(\Gamma_h^{sing}, \mathbb{Z})$ is represented by the closed loop on Γ_h^{sing} corresponding to the interval $[0, 1]$.

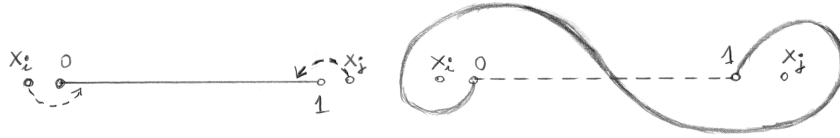


Figure 3.2. The monodromy of $x_i(h), x_j(h)$ and the loop $\delta(h)$.

We note that given an arbitrary effectif divisor $m = P_0 + P_1 + \dots + P_k$ on Γ_h , one constructs in a similar way a singularized curve Γ_h^{sing} , which is the natural framework of the generalized center problem for the Abel equation, see [2, Conjecture 1.7] and [4].

Proof of Theorem 3.2. —

The homology group $H_1(\Gamma_h^{sing}, \mathbb{Z})$ is of dimension $n + 2$. It is generated by $n + 1$ simple closed loops $\gamma_i = \gamma_i(h)$ which make one turn around the $n + 1$ punctures $x_i(h)$ on Γ_h , $A(x_i(h)) - h = 0$, as well the loop $\delta(h)$ connecting 0 and 1 on the singularized curve Γ_h^{sing} . The monodromy of the loop $\delta(h)$ is shown on the figure 3.2. As the integral $I(h)$ is constant, it follows that

$$\int_{\gamma_i(h)-\gamma_j(h)} yq(x)dx \equiv 0$$

where $A(x_i(0)) = A(x_j(0)) = 0$ and $x_i(0) = 0, x_j(0) = 1$. We have also

$$\begin{aligned} \int_{\gamma_i(h)-\gamma_j(h)} yq(x)dx &= \int_{\gamma_i(h)-\gamma_j(h)} \frac{q(x)}{h - A(x)} dx \\ &= -2\pi i \left(\frac{q(x_i(h))}{A'(x_i(h))} - \frac{q(x_j(h))}{A'(x_j(h))} \right) \\ &= -2\pi i [q(x_i(h))x'_i(h) - q(x_j(h))x'_j(h)] \\ &= -2\pi i \frac{d}{dh} [Q(x_i(h)) - Q(x_j(h))] \end{aligned}$$

where

$$Q(x) = \int q(x)dx$$

is a primitive of q , and $x_i(h)$ are the roots of the polynomial $A(x) - h$ (we used that $A'(x_i(h))x'_i(h) \equiv 1$). We denote,

$$(3.7) \quad J(h) = \int_{x_i(h)-x_j(h)} Q = Q(x_i(h)) - Q(x_j(h))$$

and call J an Abelian integral of dimension zero along the zero-cycle

$$x_i(h) - x_j(h) \in H_0(\{A(x) = h\}, \mathbb{Z})$$

([16], Definition 1). If the Abelian integral $I(h)$ vanishes identically, then the same holds true for $J'(h)$, hence $J(h) = const.$ and it is easy to check that the constant is zero, $J(h) \equiv 0$.

The set of rational functions Q such that $Q(x_i(h)) \equiv Q(x_j(h))$ is a subfield of the field of all rational functions $\mathbb{C}(x)$. By the Lüroth theorem this subfield is of the form $\mathbb{C}(W)$ for suitable rational function W . It follows that $Q = \tilde{Q} \circ W, A = \tilde{A} \circ W$ and it is easily seen in this case that W is a polynomial. If $W(x_i(h)) \equiv W(x_j(h))$, then clearly $W(0) = W(1)$ and the Proposition is proved. If $W(x_i(h)) \not\equiv W(x_j(h))$, then we denote $w_i(h) = W(x_i(h)), w_j(h) = W(x_j(h))$ and note that

$$\tilde{A}(w_i(h)) \equiv A(w_j(h)) \equiv h, \tilde{Q}(w_i(h)) = \tilde{Q}(w_j(h)).$$

We may reason then by induction on \tilde{A}, \tilde{Q} which have a smaller degree than A, Q respectively. Thus this process must stop and we get W with $W(x_i(h)) \equiv W(x_j(h))$, and hence $W(0) = W(1)$. \square

4. Irreducible components of the Center set

An affine algebraic variety V in \mathbb{C}^n is the common zero locus of a finite collection of polynomials $f_i \in \mathbb{C}[z_1, \dots, z_n]$. The variety V is said to be irreducible, if for any pair of closed varieties V_1, V_2 such that $V = V_1 \cup V_2$, either $V_1 = V$ or $V_2 = V$. Of course, it might happen that a variety V is reducible $V = V_1 \cup V_2$, where $V_1, V_2 \neq V$. In this case we may ask whether V_1 and V_2 are further reducible and so on. It is a basic fact of commutative algebra that in this way only a finitely many irreducible subvarieties $V_i \subset V$ can be found, and more precisely

Any variety V can be uniquely expressed as a finite union of irreducible varieties V_i with $V_i \not\subseteq V_j$ for $i \neq j$. [19]

The varieties V_i which appear in the finite decomposition

$$V = \cup_i V_i$$

are the *irreducible components* of V .

Let $W \subset V$ be another algebraic variety. Is W an irreducible component of V ? It is usually easy to verify, whether W is irreducible. It is much harder to check that W is an irreducible component of V . Indeed, it might happen that $W \subsetneq V_i$ where V_i is an irreducible component of V . To verify this, one may compare the dimensions of the tangent spaces $T_x W$ and $T_x V$ at some smooth point $x \in V \cap W$ (one point x is enough!). Then $W \subsetneq V_i$ if and only if $T_x W \subsetneq T_x V$. Of course, there might be no way to know that x is a smooth point, in which case we use the tangent cones $TC_x W$ and $TC_x V$. For every $x \in W$ on an irreducible variety W holds $\dim TC_x W = \dim W$. Thus, for irreducible varieties $W \subset V$ holds

$$\dim TC_x W < \dim TC_x V \Leftrightarrow W \subsetneq V.$$

The choice of $x \in W$ is irrelevant, which allows a great flexibility.

The above approach will be applied in the case when V is the center set of the equation (1.2), and W is a subset of equations with a center. In the planar case (1.1) this approach was developed by Movasati [23]. He observed that the vanishing of the first Melnikov function, related to one-parameter deformations (arcs) of systems (1.1) with a center, provides equations for the tangent space $T_x W$, while the vanishing of the second Melnikov function provides equations for the tangent cone $TC_x W$. This

remarkable connection between algebraic geometry and dynamics will allow us to go farther in the description of irreducible components of the center set. We adapt the approach of Movasati [23] and Zare [28] to (1.2) in the context of the set \mathcal{A}_n of Abel differential equations

$$(4.1) \quad \frac{dy}{dx} = a(x)y^2 + b(x)y^3$$

parameterised by the polynomials $a(x), b(x)$ of degree at most n . They form therefore a vector space of dimension $2n + 2$, and consider the subset $\mathcal{C}_n \subset \mathcal{A}_n$ of Abel differential equations having a center on the interval $0 \leq x \leq 1$. As we saw in the preceding section, \mathcal{C}_n is defined by finitely many polynomial relations $c_n(a, b) = 0$ and therefore is an algebraic set.

4.1. Universal centers define irreducible components of the center set

If the integer $k > 1$ divides $n + 1$, then we denote by $\mathcal{U}_{n/k} \subset \mathcal{C}_n \subset \mathcal{A}_n$ the algebraic closure of the set of pairs of polynomials (a, b) (or Abel equations (4.1)), such that the following Polynomial Composition Condition (PCC) is satisfied

There exist polynomials \tilde{A}, \tilde{B}, W of degrees $(n + 1)/k, (n + 1)/k, k$, such that

$$(PCC) \quad A = \tilde{A} \circ W, B = \tilde{B} \circ W, W(0) = W(1).$$

The differential form associated to (4.1)

$$dy - (a(x)y^2 + b(x)y^3)dx = dy - y^2dA(x) - y^3dB(x)$$

is a pull back of the differential form

$$(4.2) \quad dy - (\tilde{A}'(w)y^2 + \tilde{B}'(w)y^3)dw = dy - y^2d\tilde{A}(w) - y^3d\tilde{B}(w)$$

under the map $(x, y) \rightarrow (w, y)$, where $w = W(x)$. In other words the equation (4.1) is obtained from

$$\frac{dy}{dw} = \tilde{A}'(w)y^2 + \tilde{B}'(w)y^3$$

via the substitution $w = W(x)$. This, combined to $W(0) = W(1)$ implies that the set of Abel equations $\mathcal{U}_{n/k}$ have a center at $y = 0$ along $[0, 1]$. Of course one could check directly that the center conditions $c_n(a) = 0$ are satisfied for all n (Theorem 2.3). Indeed, the iterated integrals $\int_{x_0}^{x_1} a_{i_1} \cdots a_{i_k}$ vanish, because they are pull backs under W of iterated integrals along an interval, contractible to the point $W(x_0) = W(x_1)$. Following Brudnuyi [5], we say that (4.1) determines an *universal center* if and only if

$$\int_{x_0}^{x_1} a_{i_1} \cdots a_{i_k} = 0, \forall i_j \in \mathbb{N}.$$

It is shown then that a center is universal, if and only if the corresponding equation (4.1) is a pull back under an appropriate polynomial as above, see Brudnuyi [5, Corollary 1.20]. Thus, the universal centers are exactly those, obtained by a polynomial pull back in the sense (4.2), see the Polynomial Composition Condition (PPC).

Note that the universal center set $\mathcal{U}_{n/k}$ is an irreducible algebraic variety, as a Zariski open subset of it is parametrized by the polynomials \tilde{A}, \tilde{B}, W of degrees respectively $(n+1)/k, (n+1)/k, k$. The main result of the section is

THEOREM 4.1. — *The algebraic sets $\mathcal{U}_{n/k}$ are irreducible components of the center set \mathcal{C}_n of the Abel equation*

$$\frac{dy}{dx} = a(x)y^2 + b(x)y^3, \deg a, \deg b \leq n.$$

We shall illustrate first the idea of the proof of Theorem 4.1 on the rather elementary case $k = n + 1$. The closure of the universal center set $\mathcal{U}_{n/n+1}$ consists of Abel equations (4.1) such that

$$\deg a, \deg b \leq n, \int_0^1 a(x)dx = \int_0^1 b(x)dx = 0$$

and moreover the polynomials $a(x), b(x)$ are co-linear. Thus, $\mathcal{U}_{n/n+1}$ is identified to the vector space of pairs of polynomials $(a(x), b(x))$ with the above properties, and is therefore of dimension $n + 1$. Consider now the point $(a(x), 0) \in \mathcal{U}_{n/n+1}$ where $a(x)$ is a degree n polynomial.

PROPOSITION 4.2. — *The tangent space $T_{(a,0)}\mathcal{U}_{n/n+1}$ is a vector space of dimension $n + 1$, which consists of pairs of polynomials (p, q) of degree at most n , such that q and a are co-linear polynomials, and $\int_0^1 p(x)dx = 0$*

The proof is left to the reader. Next, we compute the tangent cone $TC_{(a,0)}\mathcal{C}_n$ at $(a, 0)$ to the center set \mathcal{C}_n . To avoid complications, we choose a to be a non-composite polynomial.

PROPOSITION 4.3. — *Let a be a non-composite polynomial of degree n , such that $a(0) \neq 0, a(1) \neq 0$. Then*

$$TC_{(a,0)}\mathcal{C}_n = T_{(a,0)}\mathcal{U}_{n/n+1}$$

The above implies that *the algebraic set $\mathcal{U}_{n/n+1}$ is an irreducible component of the center set \mathcal{C}_n .*

Proof of Proposition 4.3. Consider a one-parameter deformation

$$(4.3) \quad \varepsilon \rightarrow (a - \varepsilon p + \dots, -\varepsilon q + \dots)$$

of (4.1) at the point $(a, 0)$. For $\varepsilon = 0$ the equation is

$$\frac{dy}{y^2} = a(x)dx$$

and has a first integral $H(x, y) = \frac{1}{y} + A(x)$ where A is a primitive of a , $A(0) = A(1)$. The perturbed equation is

$$dH - \varepsilon[p(x) + yq(x)]dx + \dots = 0.$$

We parameterize the cross-sections $\{x = 0\}, \{x = 1\}$ by $h = H(0, y) = H(1, y) = 1/y$ and write for the return map φ_ε

$$\varphi_\varepsilon(h) = h + \varepsilon M_1(h) + O(\varepsilon^2).$$

The Melnikov function M_1 , according to section 3, is computed to be

$$M_1(h) = \int_{\{H=h\}} p(x)dx + yq(x)dx = \int_0^1 p(x)dx + \int_0^1 \frac{q(x)}{h - A(x)}dx.$$

Assuming that for all sufficiently small ε the deformed Abel equation belongs to the center set \mathcal{C}_n , implies $M_1 = 0$, which on its turn imposes rather severe conditions on the polynomials p, q . First, $\int_0^1 p(x)dx = 0$ as follows already from (2.7). The second condition

$$\int_0^1 \frac{q(x)}{h - A(x)}dx \equiv 0$$

is well studied in a number of articles, and is known as the *polynomial moment problem*, e.g. [4] and the references there. For the case of a general A , see the Addendum by Pakovich in [27]. As $a(0) \neq 0, a(1) \neq 0$ then by Theorem 3.2 we have that $\int_0^1 \frac{q(x)}{h - A(x)}dx \equiv 0$ if and only if the composition condition holds true. As A is supposed to be prime, this means that A and $Q = \int q$ are co-linear polynomials. This completes the proof of Proposition 4.3 and Theorem 4.1 in the case $k = n$.

Note that in full generality, a vector (p, q) which belongs to the tangent cone is a vector, such that there is a one-parameter deformation

$$\varepsilon \rightarrow (a + \varepsilon^k p + \dots, \varepsilon^k q + \dots)$$

at the point $(a, 0)$ which belongs to the center set \mathcal{C}_n . The same arguments give the same constraints to the vector (p, q) .

Proof of Theorem 4.1 in the general case. — Assume that the integer $k > 1$ divides $n + 1$ and consider the algebraic set $\mathcal{U}_{n/k}$ of Abel differential equations, at $y = 0$ along $[0, 1]$. The proof follows the same lines as the case $k = n$, with the notable difference that the second Melnikov function M_2 will be needed.

We compute first the tangent space to $\mathcal{U}_{n/k}$ at a general point $(a, 0)$. Consider for this purpose the one-parameter deformation

$$(4.4) \quad \mathcal{F}_\varepsilon : \frac{dy}{y^2} = a(x)dx - \varepsilon\omega_1 - \varepsilon\omega_2 - \dots$$

where

$$\omega_i = p_i(x)dx + yq_i(x)dx$$

are polynomial one-forms, $\deg p_i \leq n, \deg q_i \leq n$. As before we denote

$$A = \int a, P_i = \int p_i, Q_i = \int q_i$$

where

$$A(x) = \tilde{A}(W(x)), W(0) = W(1), P_i(0) = P_i(1), Q_i(0) = Q_i(1).$$

The point $(a, 0)$ belongs to $\mathcal{U}_{n/k}$ if and only if $A = \tilde{A} \circ W$ for some degree k polynomial W .

PROPOSITION 4.4. — *The tangent space $T_{(a,0)}\mathcal{U}_{n/k}$ is the vector space of polynomials (p_1, q_1) such that*

$$P_1(x) = \tilde{P}_1 \circ W(x) + R(x) \cdot \tilde{A}'(W(x)), Q_1(x) = \tilde{Q}_1(W(x))$$

where \tilde{P}_1, \tilde{Q}_1 are arbitrary polynomials of degree at most $(n+1)/k$ and $R = R(x)$ is any degree k polynomial, such that $R(0) = R(1)$.

The proof is straightforward, it suffices to consider the first order approximation in ε of the general deformation

$$\omega_1^\varepsilon = d[(\tilde{A} + \varepsilon\tilde{P}) \circ (W + \varepsilon R)(x)] + \varepsilon y d[\tilde{Q} \circ (W + \varepsilon R)(x)]$$

of $\omega_1^0 = adx$.

Next, we study the tangent cone $TC_{(a,0)}\mathcal{C}_n$. We need to compare the affine varieties $T_{(a,0)}\mathcal{U}_{n/k} \subset TC_{(a,0)}\mathcal{C}_n$.

PROPOSITION 4.5. — *In a sufficiently small neighbourhood of every general point $(p, q) \in T_{(a,0)}\mathcal{U}_{n/k}$ the tangent cones $TC_{(a,0)}\mathcal{C}_n$ and $T_{(a,0)}\mathcal{U}_{n/k}$ coincide.*

The above Proposition shows that there is no irreducible component of $TC_{(a,0)}\mathcal{C}_n$ which contains an irreducible component of $T_{(a,0)}\mathcal{U}_{n/k}$ of strictly smaller dimension. This would imply Theorem 4.1.

The first Melnikov function, as in the case $k = n$, is $M_1 = \int_0^1 p_1 dx + yq_1 dy$. By Christopher's theorem $M_1 = 0$ implies that q_1 satisfies the composition condition

$$Q_1(x) = \tilde{Q}_1(W(x)).$$

Additional obstructions on the form of p_1 will be found by inspecting the second Melnikov function M_2 . Under the condition that $M_1 = 0$ we find [14, formula (2.8)]

$$M_2 = \int_0^1 \omega_1 \omega_1' + \int_0^1 \omega_2$$

where the derivative $'$ is with respect to the parameter h . The identity $h = A(x) + \frac{1}{y}$ shows that $y' = -y^2$ and $\omega_1' = -y^2 dx$, it is clearly a covariant derivative in a cohomology bundle (although we do not need this interpretation here). Therefore, for the iterated integral of length two we find

$$\begin{aligned} \int_0^1 \omega \omega' &= - \int_{\{H=h\}} (p_1 dx + q_1 y dx)(y^2 q_1 dx) \\ &= - \int_{\{H=h\}} (p_1 dx)(y^2 q_1 dx) \\ &= \int_{\{H=h\}} y^2 q_1 P_1 dx \end{aligned}$$

where P_1 is a primitive of p_1 . Indeed, $M_1 = 0$ implies the composition condition for $Q_1 = \int q_1$ and A , that is to say the integral $\int_{\{H=h\}} yq_1 dx$ vanishes as a pull back. The same then holds true for its derivative $\int_{\{H=h\}} y^2 q_1 dx$ as well for the iterated integral $\int_{\{H=h\}} (yq_1 dx)(y^2 q_1 dx)$. Further, by the shuffle relation for iterated integrals

$$\begin{aligned} \int_{\{H=h\}} (p_1 dx)(y^2 q_1 dx) + \int_{\{H=h\}} (y^2 q_1 dx)(p_1 dx) \\ = \int_{\{H=h\}} p_1 dx \int_{\{H=h\}} y^2 q_1 dx = 0 \end{aligned}$$

Further, for $\int_0^1 \omega_2$ we find

$$\begin{aligned} \int_0^1 \omega_2 &= \int_0^1 (p_2 + yq_2)dx = \int_0^1 \frac{dQ_2}{h - A(x)} \\ &= - \int_0^1 \frac{Q_2 dA}{(h - A)^2} + \frac{Q_2}{h - A} \Big|_0^1 \\ &= - \int_0^1 y^2 Q_2 a dx. \end{aligned}$$

so that under the condition $M_1 = 0$ implies

$$M_2(h) = \int_{\{H=h\}} y^2 q_1 P_1 dx - y^2 Q_2 a dx = \int_0^1 \frac{q_1(x)P_1(x) - Q_2(x)a(x)}{(h - A(x))^2} dx.$$

We apply Christopher's theorem to M_2 and conclude that the primitive of the polynomial $q_1(x)P_1(x) - Q_2(x)a(x)$ is a composite polynomial, it can be expressed as a polynomial function in $W(x)$, and therefore

$$q_1(x)P_1(x) - Q_2(x)a(x) = P(W(x))W'(x)$$

or equivalently

$$\tilde{Q}'_1(W(x))P_1(x) - Q_2(x)\tilde{A}'(W(x)) = R_1(W(x))$$

for certain polynomial R_1 . Assuming that \tilde{Q}'_1 and \tilde{A}' are mutually prime, there exist polynomials R_2, R_3 such that

$$\tilde{Q}'_1(W)R_2(W) - \tilde{A}'(W)R_3(W) = R_1(W)$$

so

$$\tilde{Q}'_1(W(x))(P_1(x) - R_2(W(x))) - (Q_2(x) - R_3(W(x)))\tilde{A}'(W(x)) = 0.$$

This implies finally that $\tilde{A}'(W(x))$ divides $P_1(x) - R_2(W(x))$ and

$$P_1(x) = R_2(W(x)) + R(x)\tilde{A}'(W(x)).$$

Proposition 4.4, and hence Theorem 4.1 is proved. □

4.2. An example : the center set of plane quadratic vector fields

Let \mathcal{A}_n be here the set of all polynomial vector fields of degree at most n . The only (non-trivial) case in which the center set $\mathcal{C}_n \subset \mathcal{A}_n$ is completely known is the quadratic one, $n = 2$. For comprehensive description and historical comments concerning the center-focus problem in the quadratic case see Zoladek [29]. To the plane quadratic vector field (1.1) we associate a foliation $\mathcal{F}_\omega = \{\omega = 0\}$ on \mathbb{C}^2 , defined by the polynomial one-form

$$\omega = P(x, y)dy - Q(x, y)dx.$$

The leaves of the foliation are the orbits of the plane vector field (1.1), and the restriction of the one-form ω on the leaves of \mathcal{F}_ω vanishes identically.

In this section we assume that the polynomials P, Q are of degree at most two, and the system has a center. As the foliation is over \mathbb{C} we must be more careful

in the definition. We shall say that a singular point is a center, if the point is non-degenerate, and has a local holomorphic first integral with a Morse critical point. Thus, in a neighbourhood of such a point, and up to a complex affine change of the variables, the system can be written in the form

$$\dot{x} = x + P_2(x, y), \quad \dot{y} = -y + Q_2(x, y)$$

for some homogeneous polynomials P_2, Q_2 . The following classical result is implicit in Zoladek [29, Theorem 1], and explicit in [21, Theorem 1.1], [11, Theorem 9]

THEOREM 4.6. — *The center set \mathcal{C}_2 of plane polynomial quadratic systems with a Morse center has four irreducible components.*

The result follows essentially from the Dulac's classification [10] of such Morse centers in a complex domain. We sketch a proof.

Proof. — The foliation \mathcal{F}_ω , respectively the vector field (1.1), is said to be logarithmic, if

$$(4.5) \quad P(x, y)dy - Q(x, y)dx = f_1 \dots f_k \sum_{i=1}^k \lambda_i \frac{df_i}{f_i}, \quad f_i \in K[x, y], \lambda_i \in K$$

for suitable polynomials f_i and exponents λ_i . As

$$\sum_{i=1}^k \lambda_i \frac{df_i}{f_i} = d \log \prod_{i=1}^k f_i^{\lambda_i}$$

then the logarithmic foliation \mathcal{F}_ω has a first integral of Darboux type

$$\prod_{i=1}^k f_i^{\lambda_i}.$$

Let $\mathcal{L}(d_1, d_2, \dots, d_k)$ denotes the set of such logarithmic foliations (or plane vector fields) with

$$\deg f_1 \leq d_1, \deg f_2 \leq d_2, \dots, \deg f_k \leq d_k.$$

For generic polynomials f_i of degree d_i the degree of the associated vector field is $\sum d_i - 1$. Therefore $\mathcal{L}(d_1, d_2, \dots, d_k)$ is quadratic, provided that $d_1 = 3$ or $d_1 = 1, d_2 = 2$ or $d_1 = d_2 = d_3 = 1$. This defines three large irreducible sets of quadratic systems with a Morse center, $\mathcal{L}(3), \mathcal{L}(1, 2), \mathcal{L}(1, 1)$ respectively. The irreducibility of these algebraic sets follows from the fact that they are parameterised by the coefficients of the polynomials f_i and by the exponents λ_i . We have one more exceptional irreducible set of systems with a Morse center which is

$$\mathcal{Q}_4 = \mathcal{L}(2, 3) \cup \mathcal{A}_2.$$

Here $\mathcal{L}(2, 3)$ is the set of polynomial foliations as above, with a first integral f_2^3/f_3^2 where $\deg f_2 = 2, \deg f_3 = 3$. Generically such a foliation is of degree four, but it happens that its intersection \mathcal{Q}_4 with the space \mathcal{A}_2 of quadratic foliations is non empty and it is an irreducible algebraic set. The notation \mathcal{Q}_4 is introduced by Zoladek [29], the index 4 indicates the co-dimension of the set in \mathcal{A}_2 . To complete the proof we carefully investigate all cases of the Dulac's classification as reproduced in [6, Theorem 7]. It is seen there that in all ten cases found by Dulac, the corresponding

quadratic system with a Morse center is either in one of the four cases above, or in their closures

$$\overline{\mathcal{L}(3)}, \overline{\mathcal{L}(2, 1)}, \overline{\mathcal{L}(1, 1, 1)}, \overline{\mathcal{L}(3, 2)} \cap \mathcal{A}_2.$$

To illustrate the last claim, consider a quadratic foliation defined by the closed one form

$$\omega_0 = p_1 p_2 \cdot \eta_0, \quad \eta_0 = \lambda_1 \frac{dp_1}{p_1} + \lambda_2 \frac{dp_2}{p_2} + dq$$

where p_1, p_2, q are linear functions. The system has a first integral $p_1^{\lambda_1} p_2^{\lambda_2} e^q$, and hence generically a Morse center, although it does not belong to any $\mathcal{L}(d_1, d_2, \dots, d_k)$.

However, the one-parameter family of one-forms

$$\omega_\varepsilon = p_1 p_2 (1 + \varepsilon q) \left(\lambda_1 \frac{dp_1}{p_1} + \lambda_2 \frac{dp_2}{p_2} + \frac{1}{\varepsilon} \frac{d(1 + \varepsilon q)}{1 + \varepsilon q} \right) \in \mathcal{L}(1, 1, 1)$$

tends to ω_0 , when the parameter ε tends to 0. This shows that $\omega_0 \in \overline{\mathcal{L}(1, 1, 1)}$. The missing details can be found in [11, Appendix]. \square

The exceptional set \mathcal{Q}_4 might look not quite explicit, we investigate it in details bellow.

The space \mathcal{A}_n of polynomial vector fields of degree at most n are identified to a vector space of dimension $(n + 1)(n + 2)$. On \mathcal{A}_n acts the affine group Aff_2 of affine transformations of K^2 (as usual $K = \mathbb{R}$ or $K = \mathbb{C}$), as well the multiplicative groupe K^* corresponding to "change of time", $\dim Aff_2 \times K^* = 7$. Therefore the minimal dimension of a component of the center set \mathcal{C}_n is 7. Such components, if exist, will be in a sense exceptional.

In the quadratic case $n = 2$ the dimension of the four components of \mathcal{C}_2 are easily found. For instance, in the case $\mathcal{L}(1, 1, 1) \subset \mathcal{A}_2$, and up to an affine changes of variables and time, one may suppose that the first integral is in the form $xy^\lambda(1 - x - y)^\mu$. Therefore the dimension of $\mathcal{L}(1, 1, 1)$ is $2 + 7 = 9$ and the codimension is $3 = 12 - 9$. We find similarly that $\dim \mathcal{L}(2, 1) = \dim \mathcal{L}(3) = 9$.

We describe now the last component \mathcal{Q}_4 . Let $[x : y : z]$ be homogeneous coordinates in \mathbb{P}^2

$$(4.6) \quad P_2(x, y, z) = a_2(x, y) + a_1(x, y)z + a_0(x, y)z^2$$

$$(4.7) \quad P_3(x, y, z) = b_3(x, y) + b_2(x, y)z + b_1(x, y)z^2 + b_0(x, y)z^3$$

be homogeneous polynomials in x, y, z of degree 2 and 3. The function

$$H = P_2^3 / P_3^2$$

is therefore rational on \mathbb{P}^2 and induces a foliation on \mathbb{P}^2

$$(4.8) \quad 3P_3(x, y, z)dP_2(x, y, z) - 2P_2(x, y, z)dP_3(x, y, z) = 0.$$

The corresponding affine foliation on the chart \mathbb{C}^2 defined by $z = 1$

$$(4.9) \quad 3P_3(x, y, 1)dP_2(x, y, 1) - 2P_2(x, y, 1)dP_3(x, y, 1) = 0$$

is of degree 4. We may obtain a plane polynomial foliation of degree 2 by imposing the following additional conditions.

Suppose first, that the infinite line $\{z = 0\}$ is invariant, that is to say (up to affine change)

$$(4.10) \quad H(x : y : 1) = \frac{a_2(x, y)^3}{b_3(x, y)^2} = 1.$$

This condition can be written as

$$P_2(x, y, z)^3 = P_3(x, y, z)^2 + O(z).$$

The foliation (4.8) takes the form

$$z[P(x, y, z)dx + Q(x, y, z)dy] + R(x, y, z)dz = 0$$

where $\deg P, \deg Q \leq 3$, so (4.9) is of degree 3. If we further suppose that z divides the homogeneous one form $3P_3dP_2 - 2P_2dP_3$ then (4.8) takes the form

$$z^2[P(x, y, z)dx + Q(x, y, z)dy] + zR(x, y, z)dz = 0$$

where $\deg P, \deg Q \leq 2$, so (4.9) is a plane quadratic foliation. The condition that z^2 divides $2P_3dP_2 - 3P_2dP_3$ can be written as

$$P_2(x, y, z)^3 = P_3(x, y, z)^2 + O(z^2)$$

or equivalently

$$(4.11) \quad a_2(x, y)^3 = b_3(x, y)^2$$

$$(4.12) \quad 3a_2(x, y)^2 a_1(x, y) = 2b_3(x, y)b_2(x, y).$$

These polynomial relations can be further simplified by affine changes of the variables x, y . First, (4.11) implies that a_2 is a square of a linear function in x, y which we may suppose equal to x , that is to say

$$a_2(x, y) = x^2, \quad b_3(x, y) = x^3.$$

The second condition (4.12) becomes $3xa_1 = 2b_2$ where we may put $a_1 = 2y$, and hence

$$a_1(x, y) = 2y, \quad b_2(x, y) = 3xy.$$

It is seen that the polynomial $P_3(x, y, 1)$ has a real critical point which we can put at the origin, so we shall also suppose that $b_1 = 0$. Using finally a "change of time" (the action of K^*) we assume that $b_0 = 1$ while $a_0 = \alpha \in K$ is a free parameter (modulus). The first integral takes therefore the form

$$(4.13) \quad H_\alpha(x, y) = \frac{(x^2 + 2y + \alpha)^3}{(x^3 + 3xy + 1)^2}$$

with induced quadratic foliation

$$(4.14) \quad (-\alpha x^2 - 2y^2 - \alpha y + x)dx + (xy - \alpha x + 1)dy.$$

This is the exceptional co-dimension four component of Q_4 .

The reader may check that the corresponding vector field

$$x' = xy - \alpha x + 1, \quad y' = \alpha x^2 + 2y^2 + \alpha y - x$$

has a Morse center at $x = 1/\alpha, y = 0$ which is moreover a usual real center for $\alpha \in (1, 0)$. The above computation is suggested by [21] where, however, the modulus α is wrongly fixed equal to $\alpha = \infty$). The foliation on \mathbb{P}^2 corresponding to

$$H_\infty(x, y) = \frac{(x^2 + 2y + 1)^3}{(x^3 + 3xy)^2}$$

has two invariant lines $\{x = 0\}$ and $\{z = 0\}$, in contrast to the general foliation defined by $dH_\alpha(x, y) = 0$ which has only one invariant line $\{z = 0\}$. We resume the above as follows

PROPOSITION 4.7. — *Every polynomial vector field having a rational first integral of the form*

$$H(x, y) = \frac{(a_0(x, y) + a_1(x, y) + a_2(x, y))^3}{(b_0(x, y) + b_1(x, y) + b_2(x, y) + b_3(x, y))^2}$$

where the homogeneous polynomials a_i, b_j of degrees $0 \leq i \leq 2, 0 \leq j \leq 3$ are subject to the relations

$$\begin{aligned} a_2(x, y)^3 &= b_3(x, y)^2 \\ 3a_2(x, y)^2 a_1(x, y) &= 2b_3(x, y) b_2(x, y) \end{aligned}$$

is of degree two. The set of such quadratic vector fields form the irreducible component \mathcal{Q}_4 of the center set \mathcal{C}_2 . Up to an affine change of the variables x, y the polynomial H can be assumed in the form $H(x, y) = \frac{(x^2+2y+\alpha)^3}{(x^3+3xy+1)^2}$ where α is a parameter.

We conclude this section with the following remarkable property of \mathcal{Q}_4 . One may check that general rational function of the form $H(x, y) = P_2^3/P_3^2$, where P_2, P_3 are bi-variate polynomials of degree two and three, defines a pencil of genus four curves $\Gamma_t = \{(x, y) : H(x, y) = t\}$ on \mathbb{C}^2 . However, the special rational function H_α (4.13) defines an elliptic pencil, that is to say the level sets $\Gamma_t = \{(x, y) : H_\alpha(x, y) = t\}$ on \mathbb{C}^2

$$\{(x, y) \in \mathbb{C}^2 : H_\alpha(x, y) = const\}$$

are genus one curves, see [15].

4.3. An example : the center set of the polynomial Liénard equation

Consider the following polynomial Liénard equation

$$(4.15) \quad \dot{x} = y, \quad \dot{y} = -q(x) - yp(x)$$

in which the origin $(0, 0)$ is an isolated singular point. Note that it is equivalent to the second order non-linear equation

$$\ddot{x} + p(x)\dot{x} + q(x) = 0$$

(a generalised Van der Pol oscillator). The description of the non-degenerate real analytic centers of (4.15) is due to Cherkas [26], and in the polynomial case Christopher [9] gave an interpretation, which we formulate bellow.

The Liénard equation (4.15) induces a polynomial foliation

$$(4.16) \quad ydy + (q(x) + yp(x))dx = 0.$$

Suppose that the following Polynomial Composition Condition (PCC) is satisfied

There exist polynomials \tilde{P}, \tilde{Q}, W such that

$$(PCC) \quad P = \tilde{P} \circ W, \quad Q = \tilde{Q} \circ W$$

where $P'(x) = p(x), Q'(x) = q(x)$.

Then the Liénard foliation takes the form

$$ydy + d\tilde{Q}(W) + yd\tilde{P}(W) = 0.$$

It is easy to see that if the origin $x = 0$ is a Morse critical point of W , and if $q'(0) > 0$, then the Liénard equation has a center at $(0, 0)$.

The Theorem of Cherkas-Christopher can be formulated as follows

THEOREM 4.8. — *The real polynomial Liénard equation (4.15) has a non-degenerate real center if and only if $q'(0) > 0$, and the polynomials p, q satisfy the above Polynomial Composition Condition, where the real polynomial W has a Morse critical point at the origin.*

A self-contained proof can be found in [9], it is based on the following simple observation due to Cherkas [26, Lemma 1]

LEMMA 4.9. — *The real analytic equation*

$$(4.17) \quad \dot{x} = y, \quad \dot{y} = -x + y \sum_{i=1}^{\infty} a_i x^i$$

has a center at the origin, if and only if $a_{2j} = 0, \forall j \geq 1$.

Indeed, the truncated equation

$$(4.18) \quad \dot{x} = y, \quad \dot{y} = -x + y \sum_{j=0}^{\infty} a_{2j+1} x^{2j+1}$$

is reversible in x , and hence it has a center at the origin. In a sufficiently small neighbourhood of the origin, (4.18) is rotated with respect to the vector field (4.17), unless $a_{2j} = 0, \forall j \geq 1$. The final argument of Christopher is to use the Lüroth theorem, to deduce the PCC condition. This topological argument of Cherkas does not apply in a complex domain. We shall prove, however, the following more general

THEOREM 4.10. — *The (possibly complex) polynomial Liénard equation (4.15) has a Morse critical point at the origin, if and only if the polynomials p, q satisfy the above Polynomial Composition Condition, and W has a Morse critical point at the origin.*

The condition (4.16) to have a linear Morse critical point implies $q(0) = 0, q'(0) \neq 0$ and $p(0) = 0$. Therefore the polynomial $\frac{1}{2}y^2 + Q(x)$ has a Morse critical point at the origin and there exists a local bi-analytic change of the variable $x \rightarrow X$ such that $\frac{1}{2}y^2 + Q(x(X)) = \frac{1}{2}(y^2 + X^2)$. Thus

$$(4.19) \quad ydy + (q(x) + yp(x))dx = \frac{1}{2}d(y^2 + X^2) + ydP(x(X)).$$

We expand

$$dP(x(X)) = \left(\sum_{i=1}^{\infty} a_i X^i \right) dX$$

and obtain the equivalent foliation

$$(4.20) \quad \frac{1}{2}d(y^2 + X^2) + y \left(\sum_{i=1}^{\infty} a_i X^i \right) dX = 0.$$

By analogy to the Cherkas Lemma we shall prove

LEMMA 4.11. — *The foliation (4.20) has a Morse critical point at the origin if and only if $a_{2j} = 0, \forall j \geq 1$.*

Proof. — After rescaling $(X, y) \mapsto \varepsilon(X, y)$ the foliation takes the form

$$(4.21) \quad \mathcal{F}_\varepsilon : \frac{1}{2}d(y^2 + X^2) + y \left(\sum_{i=1}^{\infty} \varepsilon^i a_i X^i \right) dX = 0$$

and it suffices to prove that for sufficiently small ε it has a Morse critical point. Note first that the truncated foliation $\mathcal{F}_\varepsilon^t$

$$(4.22) \quad \mathcal{F}_\varepsilon^t : \frac{1}{2}d(y^2 + X^2) + y \left(\sum_{j=0}^{\infty} \varepsilon^{2j+1} a_{2j+1} X^{2j+1} \right) dX = 0$$

is a pullback of

$$(4.23) \quad \frac{1}{2}d(y^2 + \xi) + \frac{1}{2}y \left(\sum_{j=0}^{\infty} \varepsilon^{2j+1} a_{2j+1} \xi^j \right) d\xi = 0$$

under the map $\pi : (X, y) \mapsto (\xi, y), \xi = X^2$. The foliation (4.23) is regular at the origin and has a first integral

$$\frac{1}{2}(y^2 + \xi) + O(\varepsilon)$$

where $O(\varepsilon)$ is analytic in ε, ξ, Y , and vanishes as $\varepsilon = 0$. Thus $\mathcal{F}_\varepsilon^t$ has a first integral

$$H_\varepsilon(x, y) = \frac{1}{2}(y^2 + X^2) + O(\varepsilon)$$

where $O(\varepsilon)$ is analytic in ε, X^2, Y , and vanishes as $\varepsilon = 0$. This also shows that the origin is a Morse critical point.

As H_ε is a first integral of the truncated foliation $\mathcal{F}_\varepsilon^t$ then for every fixed ε we have

$$(1 + O(\varepsilon))dH_\varepsilon(x, y) = \frac{1}{2}d(y^2 + X^2) + y \left(\sum_{j=0}^{\infty} \varepsilon^{2j+1} a_{2j+1} X^{2j+1} \right) dX.$$

Suppose now that for some $j \geq 1$, $a_{2j} \neq 0$ and let $j = k$ be the smallest integer with this property. We have

$$(4.24) \quad \mathcal{F}_\varepsilon : (1 + O(\varepsilon))dH_\varepsilon(x, y) + \varepsilon^{2k} y a_{2k} X^{2k} dX + O(\varepsilon^{2k+1})dX = 0$$

where by abuse of notations $O(\varepsilon^{2k+1})$ denotes an analytic function in X, y, ε which is divisible by ε^{2k+1} . The origin is a Morse critical point if and only if the holonomy map of the two separatrices of \mathcal{F}_ε at the origin, are the identity maps. The holonomy map will be evaluated by the usual Poincaré-Pontryagin-Melnikov formula. The

separatrices are tangent to the lines $y \pm iX = 0$. We take a cross-section to one of the separatrices, parameterised by the restriction of $H_\varepsilon(x, y)$ on it. Let

$$\gamma_\varepsilon(h) \subset \{(x, y) : H_\varepsilon(x, y) = h\}$$

be a continuous family of closed loops vanishing at the origin as $h \rightarrow 0$. The holonomy map of \mathcal{F}_ε , corresponding to this closed loop is

$$\begin{aligned} h &\mapsto h + \frac{\varepsilon^{2k}}{1 + O(\varepsilon)} \left(\int_{\gamma_\varepsilon(h)} y a_{2k} X^{2k} dX + O(\varepsilon) dX \right) \\ &= h + \varepsilon^{2k} \int_{\gamma_\varepsilon(h)} y a_{2k} X^{2k} dX + O(\varepsilon^{2k+1}) dX \\ &= h + \varepsilon^{2k} a_{2k} \int_{\gamma_0(h)} y X^{2k} dX + O(\varepsilon^{2k+1}) d \end{aligned}$$

where

$$\gamma_0(h) \subset \{(x, y) : H_0(x, y) = h\} = \{(x, y) : \frac{1}{2}(y^2 + X^2) = h\}.$$

By homogeneity of the polynomials

$$\int_{\gamma_0(h)} y X^{2k} dX = h^{k+1} \int_{\gamma_0(1)} y X^{2k} dX.$$

As the homology of the algebraic curve $\{y^2 + X^2\} = 2h$ has one generator we can suppose that this generator is just the real circle $\gamma_0(1) = \{(y, X) \in \mathbb{R}^2 : y^2 + X^2 = 2\}$ and in this case

$$\int_{\gamma_0(1)} y X^{2k} dX = \iint_{y^2 + X^2 \leq 2} X^{2k} dX dy \neq 0.$$

We conclude that if the holonomy map is the identity map, then $a_{2k} = 0$ which is the desirable contradiction. Lemma 4.11 is proved. \square

Proof of Theorem 4.10. — Assuming that the Liénard equation has a Morse critical point, and hence $Q(x)$ has a Morse critical point at the origin, denote $x_1(h), x_2(h)$ the two roots of the polynomial $Q(x) - h$ which vanish at 0 as h tends to 0. We have obviously that $X(x_1(h)) = -X(x_2(h))$. By Lemma 4.11 the analytic function $P(x(X))$ is even in X , and hence $P(x_1(h)) \equiv P(x_2(h))$. Following an idea of Christopher (already used at the end of section 3), consider now the field $\mathcal{C} \subset \mathbb{C}(x)$ formed by all rational functions $R = R(x) \in \mathbb{C}(x)$ satisfying the identity

$$R(x_1(h)) \equiv R(x_2(h)).$$

According to the Lüroth theorem, every subfield of $\mathbb{C}(x)$ is of the form $\mathbb{C}(W)$ for some rational function $W = W(x)$. Thus we have $\mathcal{C} = \mathbb{C}(W)$ where $P, Q \in \mathcal{C}$. This already implies that P, Q satisfy (PCC) which completes the proof of the Theorem. \square

4.4. Abel equations with Darboux type first integral

The polynomial Liénard equation

$$(4.25) \quad \dot{x} = y, \quad \dot{y} = -q(x) - yp(x)$$

with associated foliation $ydy + (q(x) + yp(x))dx = 0$, after the substitution $y \rightarrow 1/y$, becomes the following Abel equation

$$(4.26) \quad \frac{dy}{dx} = y^2p(x) + y^3q(x).$$

Equivalently, we consider the foliation

$$(4.27) \quad dy = (y^2p(x) + y^3q(x))dx.$$

The classification of Morse critical points of the Liénard equation (4.15) obtained in section 4.3 suggests that a similar claim would hold true for the scalar Abel equation (4.27). This is the content of the following

Composition Conjecture [4, p.444]. *The Abel equation (4.27) has a center at the solution $y = 0$ along some fixed interval $[a, b]$ if and only if the following (PPC) condition holds true*

$$(PPC) \quad P = \tilde{P} \circ W, \quad Q = \tilde{Q} \circ W, \quad W(a) = W(b).$$

Note that the Cherkas-Christopher theorem is for non-degenerate centers. The Composition Conjecture missed the possibility for the Abel or Liénard equations to have a Darboux type first integral, with resonant saddle point and characteristic ratio $p : -q$ (instead of a non-degenerate center with $1 : -1$ ratio). Incidentally, Liénard equations with a Darboux type first integral will produce counter-examples to the Composition Conjecture, which is the subject of the present section. We explain in this context the recent counter-example of Giné, Grau and Santallusia [17].

The method of constructing such systems is based on the example of the co-dimension four center set \mathcal{Q}_4 for quadratic system, as explained in section 4.2. The general method is outlined in the Appendix.

Let

$$\begin{aligned} P_2 &= a_0(x) + a_1(x)y + a_2(x)y^2 \\ Q_2 &= b_0(x) + b_1(x)y + b_2(x)y^2 \end{aligned}$$

where a_i, b_j are polynomials, such that $P_2^p = Q_2^q + O(y^3)$, where p, q are positive relatively prime integers. This implies that the corresponding one-form

$$pQ_2dP_2 - qP_2dQ_2$$

is divisible by y^2 , and then the associated reduced foliation (after division by y^2) is of degree two in y , and moreover $\{y = 0\}$ is a leaf. Therefore the foliation is defined as

$$(4.28) \quad (r_1y + r_2)dy = y(r_3y + r_4)dx = 0, \quad r_i \in \mathbb{C}[x]$$

where

$$\begin{aligned} r_1 &= 2(p - q)a_2b_2 \\ r_2 &= (p - 2q)a_1b_2 - (q - 2p)b_1a_2 \\ r_3 &= pa'_2b_2 - qb'_2a_2 \\ r_4 &= pa'_1b_2 - qb'_1a_2 \end{aligned}$$

Note that if $a_2 = \text{const.} \neq 0$, $b_2 = \text{const.} \neq 0$ the foliation takes the Liénard form

$$(4.29) \quad (r_1y + r_2)dy = yr_4dx, r_1 = \text{const.}$$

Of course, it is not clear, whether such polynomials exist. To verify this we have to solve the equation

$$(a_0(x) + a_1(x)y + a_2(x)y^2)^p = (b_0(x) + b_1(x)y + b_2(x)y^2)^q \pmod{y^3}$$

assuming that $a_i(x), b_j(x)$ are polynomials, and a_2, b_2 are constants. A first condition is given by

$$a_0^p = b_0^q$$

which implies

$$\left(1 + \frac{a_1(x)}{a_0(x)}y + \frac{a_2(x)}{a_0(x)}y^2\right)^p = \left(1 + \frac{b_1(x)}{b_0(x)}y + \frac{b_2(x)}{b_0(x)}y^2\right)^q \pmod{y^3}$$

or equivalently

$$\begin{aligned} p \frac{a_1(x)}{a_0(x)} &= q \frac{b_1(x)}{b_0(x)} \\ p \frac{a_2(x)}{a_0(x)} + \frac{p(p-1)}{2} \left(\frac{a_1(x)}{a_0(x)}\right)^2 &= q \frac{b_2(x)}{b_0(x)} + \frac{q(q-1)}{2} \left(\frac{b_1(x)}{b_0(x)}\right)^2. \end{aligned}$$

Thus a_i, b_j are polynomials which satisfy the following redundant system of equations

$$\begin{aligned} a_0(x)^p &= b_0(x)^q \\ p \frac{a_1(x)}{a_0(x)} &= q \frac{b_1(x)}{b_0(x)} \\ p \frac{a_2(x)}{a_0(x)} - \frac{p}{2} \left(\frac{a_1(x)}{a_0(x)}\right)^2 &= q \frac{b_2(x)}{b_0(x)} - \frac{q}{2} \left(\frac{b_1(x)}{b_0(x)}\right)^2. \end{aligned}$$

It follows that for some polynomial R ,

$$a_0(x) = R(x)^q, \quad b_0(x) = R(x)^p$$

and moreover

$$pa_2R(x)^{-q} - qb_2R(x)^{-p}$$

is a square of a rational function, where we recall that $a_2 = \text{const.}, b_2 = \text{const.}$. It is easy to check that this is only possible if, say $p < q$, and $p = 2k - 1, q = 2k$ for an integer $k \geq 1$. With this observation the analysis of the system is straightforward and is left to the reader. We formulate the final result in the following

THEOREM 4.12. — For every integer $k \geq 1$ and polynomial $r(x)$ the function

$$H(x, y) = \frac{[(1 - r(x)^2)^{2k} + 2kr(x)(1 - r(x)^2)^k y + ky^2]^{2k-1}}{[(1 - r(x)^2)^{2k-1} + (2k - 1)r(x)(1 - r(x)^2)^{k-1} y + \frac{2k-1}{2}y^2]^{2k}}$$

is the first integral of a Liénard type equation of the form

$$\begin{aligned} \frac{dx}{dt} &= -y + r_2(x), \\ \frac{dy}{dt} &= yr_4(x), \end{aligned}$$

for suitable polynomials r_2, r_4 .

It is clear that the above Liénard system is a polynomial pull back under $x \rightarrow r(x)$ of a simpler *master Liénard system* with first integral

$$(4.30) \quad H_k(x, y) = \frac{[(1 - x^2)^{2k} + 2kx(1 - x^2)^k y + ky^2]^{2k-1}}{[(1 - x^2)^{2k-1} + (2k - 1)x(1 - x^2)^{k-1} y + \frac{2k-1}{2}y^2]^{2k}}$$

which can not be further reduced.

To the end of the section we consider in more detail the simplest cases $k = 1$ and $k = 2$. For $k = 1$ we get

$$H_1(x, y) = \frac{(1 - x^2)^2 + 2x(1 - x^2)y + y^2}{(1 - x^2 + xy + \frac{1}{2}y^2)^2}.$$

which is the first integral of the following cubic Liénard equation :

$$(4.31) \quad \begin{aligned} \frac{dx}{dt} &= y + 3x(1 - x^2) \\ \frac{dy}{dt} &= -y(1 + 3x^2) \end{aligned}$$

The characteristic ratios of the singular points $(0, 0), (\pm 1, 0)$ are equal to $3 : -1$ and $3 : -4$ and the characteristic values of $(+\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ and $(-\frac{1}{\sqrt{3}}, +\frac{1}{\sqrt{3}})$ equal -1 (so we have integrable cubic saddles, presumably new). The Liénard transformation $Y = 2y + 3x(1 - x^2)$ transforms the above to the standard form

$$\frac{dx}{dt} = Y, \quad \frac{dY}{dt} = p(x) + q(x)Y \Leftrightarrow Y \frac{dY}{dx} = p(x) + q(x)Y.$$

or also to the Abel type equation

$$\frac{d}{dx}(1/Y) = -q(x)\left(\frac{1}{Y}\right)^2 - p(x)\left(\frac{1}{Y}\right)^3$$

with respect to the variable $z = 1/Y$.

Assume that $k = 2$, the first integrals takes the form

$$H_2(x, y) = \frac{(y^2 - 2xy(1 - x^2)^2 + \frac{1}{2}(1 - x^2)^4)^3}{(y^2 - 2xy(1 - x^2) + \frac{2}{3}(1 - x^2)^3)^4}$$

while the corresponding foliation of Liénard type is defined by

$$(4.32) \quad (15x^4 - 6x^2 - 1)ydx - ((5x^2 - 3)(x^2 - 1)x + y)dy = 0.$$

Namely, the Liénard transformation

$$y \rightarrow -y - (5x^2 - 3)(x^2 - 1)x$$

transforms the equation (4.32) to

$$(4.33) \quad (q(x) + p(x)y)dx + ydy = 0$$

or equivalently to

$$\begin{aligned} \frac{dx}{dt} &= -y, \\ \frac{dy}{dt} &= q(x) + p(x)y, \end{aligned}$$

where

$$\begin{aligned} p(x) &= 2(20x^4 - 15x^2 + 1) \\ q(x) &= x(x-1)(x+1)(5x^2-3)(15x^4-6x^2-1). \end{aligned}$$

The first integral H_2 takes the form

$$\frac{(y^2 - 8x(1-x^2)(x^2 - \frac{1}{2})y + (1-x^2)^2(x^2 - \frac{1}{2})(15x^4 - 6x^2 - 1))^3}{(y^2 - 2x(1-x^2)(5x^2 - 2)y + \frac{1}{3}(1-x^2)^2(5x^2 - 2)(15x^4 - 6x^2 - 1))^4}.$$

However, after the substitution $1 \rightarrow 1/y$, the above Liénard equation is equivalent to the Abel equation

$$(4.34) \quad \frac{dy}{dx} = p(x)y^2 + q(x)y^3$$

with Darboux type first integral

$$H = \frac{y^2[1 - 8x(1-x^2)(x^2 - \frac{1}{2})y + (1-x^2)^2(x^2 - \frac{1}{2})(15x^4 - 6x^2 - 1)y^2]^3}{[1 - 2x(1-x^2)(5x^2 - 2)y + \frac{1}{3}(1-x^2)^2(5x^2 - 2)(15x^4 - 6x^2 - 1)y^2]^4}.$$

THEOREM 4.13 ([17]). — *The Abel equation (4.34) has a center at $y = 0$ along the interval $[-1, 1]$ but this center is not universal.*

Proof. — As $y = 0$ is a solution of (4.34) then for all sufficiently small $|\varepsilon|$ the solution $y = y(x)$ with initial condition $y(-1) = \varepsilon \neq 0$ exists along the compact interval $[-1, 1]$. The identity $H(\pm 1, y) = y^2$ shows that $y(1) = \varepsilon$ or $y(1) = -\varepsilon$ and it is easy to check that in fact $y(1) = \varepsilon$. Indeed, for real ε the solution $y(x)$ does not vanish and hence it has the same sign as ε . Therefore the transport map along the interval $[-1, 1]$ is the identity map, and the Abel equation has a center at the solution $y = 0$.

The polynomials $P = \int p$ and $Q = \int q$ are of degrees 5 and 10. Therefore if they had a common non-trivial composition factor, then the factor would be P and $Q = \tilde{Q} \circ P$ for suitable quadratic polynomial \tilde{Q} . It follows that $p = P'$ divides $q = Q'$ which is obviously not true. Thus P, Q can not have a common composition factor, and (by the Brudnyu's theorem) the center is not universal. \square

5. Appendix

Let K be a field and $A = K[[y]](x)$ be the $K(x)$ algebra of formal power series in y , with coefficients in the field of rational functions. To each fixed pair of mutually

prime positive integers $p, q^{(1)}$ we associate the functional equation

$$(5.1) \quad P^p = Q^q, \quad P, Q \in A.$$

PROPOSITION 5.1. — *Every solution of the functional equation (5.1) has the form $P = R^q, Q = R^p$ where*

$$R(x, y) = a_0(x)(1 + O(y))$$

$a_0(x)$ is a rational function in x , and $O(y) \in A$ is divisible by y .

The proof is straightforward. A allows an ascending filtration with respect to the powers of y , $K \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$ where

$$A_n = \left\{ \sum_{i=0}^n a_i(x)y^i : a_i \in K(x) \right\}.$$

For $P \in A$ we denote P_n the projection of P on A_n , that is to say

$$P = \sum_{i \geq 0} a_i y^i, \quad P_n = \sum_{i=0}^n a_i y^i.$$

Obviously $A_n \subset K[y](x)$ and to the pair of polynomials $P_n, Q_n \in K[x, y]$ we associate the logarithmic polynomial differential one-form

$$\omega = P_n Q_n d \log \frac{P_n^p}{Q_n^q} = p Q_n d P_n - q P_n d Q_n.$$

If $\omega = \alpha dx + \beta dy$, $\alpha, \beta \in K[y](x)$, then the degree $\deg_y \omega$ of ω (with respect to y) is

$$\max\{\deg_y \alpha, \deg_y \beta + 1\}.$$

For general $P, Q \in A$ of degree the degree of the associated ω equals $2n$, however

LEMMA 5.2. — *If the formal series P, Q satisfy the functional equation (5.1), then the associated one-form $\omega = p Q_n d P_n - q P_n d Q_n$ is divisible by y^n . The resulting reduced rational differential one-form $\omega_{red} = \omega / y^n$ is of degree n with respect to y , and moreover vanishes along the line $\{y = 0\}$.*

Proof. — Assume that $P^p = Q^q$. Without loss of generality we may suppose that y does not divide P or Q . As $P = P_n + O(y^{n+1}), Q = Q_n + O(y^{n+1})$ then $P_n^p = Q_n^q + O(y^{n+1})$ and hence $p P_n^{p-1} d P_n = q Q_n^{q-1} d Q_n + d O(y^{n+1})$. It follows that y^n divides the differential form

$$d \frac{P_n^p}{Q_n^q} = d \left(\frac{P_n^p}{Q_n^q} - 1 \right) = d \frac{P_n^p - Q_n^q}{Q_n^q}$$

and the reduced differential form $d \frac{P_n^p}{y^n Q_n^q}$ vanishes along $y = 0$. Therefore the same holds true for the one-form $p Q_n d P_n - q P_n d Q_n$. \square

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⁽¹⁾the case when $pq < 0$ is treated in a similar way

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