On the Saito number of plane curves.

Yohann Genzmer and Marcelo E. Hernandes*

Abstract

In this work we study the *Saito number* of a plane curve and we present a method to determine the minimal Saito number for plane curves in a given equisingularity class, that gives rise to an actual algorithm. In particular situations, we also provide various formulas for this number. In addition, if ν_0 and ν_1 are two coprime positive integers and $N > 0$ then we show that for any $1 \leq k \leq \left[\frac{N\nu_0}{2}\right]$ there exits a plane curve equisingular to the curve

$$y^{N\nu_0} - x^{N\nu_1} = 0$$

such that its Saito number is precisely k.

Keywords – Saito number, Saito module, planar foliations, plane curves.
MSC 2024 – 14H20 ; 32S65, 14B05

Introduction

In his seminal article on the theory of logarithmic differential forms [18], K. Saito noticed that, in dimension 2, the module of logarithmic differential 1−forms leaving invariant a given germ of complex plane curve C is always free of rank 2. Nowadays, known as *the Saito module* of C and denoted by $\Omega^1(\log C)$, it appears somehow to encode a certain amount of information about the curve itself. Of special interest are the valuations of these logarithmic differential 1−forms and, in particular, the minimal valuation called *the Saito number* of C,

$$\mathfrak{s}(C) = \min_{\omega \in \Omega^1(\log C)} \nu(\omega).$$

In a series of articles, the first author studied the Saito number of a *generic* curve in its equisingularity class [8, 9, 10, 11], and as a product established various formulas or algorithms to provide the number of moduli of C. In [12], the two authors

*The first-named author was partially supported by the *Réseau Franco-Brésilien en Mathématiques* (GDRI-RFBM). The second-named author was partially supported by CNPq-Brazil.

highlighted a link between the Tjurina number of C and its Saito number, still in a generic situation. Recently, in [4], F. Cano, N. Corral and D. Senovilla-Sanz derived from an algorithm introduced by Delorme [7, 15], a procedure to construct a basis of the Satio module for an irreducible curve whose has only one Puiseux pair. Moreover, in [1], for a *generic* irreducible curve, P. Fortuny Ayuso and J. Ribon proposed an algorithm that leads to the computation of a basis of the Saito module. Today, the situation is ripe for further study of the non-generic case and this is the main goal of this article.
In the first section, we study the minimal Saito number in a given equisingularity class $\mathrm{Top}(C)$, which somehow, corresponds to the *less generic* situation. We prove the following

Theorem 1. *Let C be a germ of complex curve in $\left(\mathbb{C}^2, 0\right)$. Let b be the number of punctual blowing-ups in the minimal desingularization process of C. Then, there exists an algorithm whose complexity is $O\left(2^b\right)$ that provides the following minimum*

$$\min_{C' \in \mathrm{Top}(C)} \mathfrak{s}\left(C'\right)$$

and the topological data associated to a $1-$form reaching the above lower bound.

The equisingularity class $\mathrm{Top}(C)$ of C is naturally stratified by the Saito number of its curves. In the second section, we prove that, in the one Puiseux pair case, as well as in some non-irreducible situations, the Saito numbers reach any value between its minimum and maximum value inside a given equisingularity class. As an example, we prove the following result

Theorem 2. *Let C be the curve given by the equation*

$$y^{N\nu_0} - x^{N\nu_1} = 0$$

where $1 \leq \nu_0 < \nu_1$ are coprime integers and $N\nu_0 > 1$. Then for any integer k such that

$$\min_{C' \in \mathrm{Top}(C)} \mathfrak{s}\left(C'\right) = 1 \leq k \leq \left\lfloor \frac{N\nu_0}{2} \right\rfloor = \max_{C' \in \mathrm{Top}(C)} \mathfrak{s}\left(C'\right)$$

there exists C_k equisingular to C such that

$$\mathfrak{s}\left(C_k\right) = k.$$

We conjecture that the result above holds for any topological class of plane curve.

1 Minimal Saito number in $\mathrm{Top}(C)$.

In this section, we prove Theorem 1 by identifying a foliation leaving invariant a curve C' equisingular to C and reaching the minimal Saito number.

1.1 Dual graph of foliations.

Let E be any process of blowing-ups. The dual graph of E can be numbered by the multiplicities of its irreducible components, defined as follows : the multiplicity $\rho(s)$ of the initial component s is 1 and inductively, if s results from the blowing-up of a point $p \in s'$ and $p \notin s''$ for any other component $s'' \neq s$, that is p is *a free point*, then $\rho(s) = \rho(s')$; if it results from the blowing-up of the point $p \in s' \cap s''$, that is p is *a satellite point*, then $\rho(s) = \rho(s') + \rho(s'')$. Following the definitions of [16], for a foliation $\mathcal{F}$, a regular curve C and a point p in C we consider the following indeces : the foliation being given by the $1-$form $a\mathrm{d}x + b\mathrm{d}y \in \Omega^1$, where $C = \{y = 0\}$, we set

- if C is invariant by $\mathcal{F}$, $\operatorname{Ind}(\mathcal{F}, C, p) = \nu(b(x, 0))$
- if C is not invariant by $\mathcal{F}$, $\operatorname{Tan}(\mathcal{F}, C, p) = \nu(a(x, 0))$.

Notice that if $\mathcal{F}$ is singular at p, in any case, both indeces above are strictly positive. A *double numbered colored graph* $\mathbb{A}$ is a graph where the vertices are colored in black or white and are double numbered. For any vertex s of $\mathbb{A}$ and $\epsilon \in \{1, 2\}$, $s(\epsilon)$ stands for the ϵ^{th} index of s.
The multiplicity of a double numbered colored graph $\mathbb{A}$ is defined by

$$\nu(\mathbb{A}) = -1 + \sum_{s \in \mathbb{A}} s(1)\, s(2).$$

Definition 1.1. *The dual graph of $\mathcal{F}$ with respect to E is the double numbered colored graph $\mathbb{A}[\mathcal{F}, E]$ defined as follows:*

- *The graph of $\mathbb{A}[\mathcal{F}, E]$ is the dual graph of E.*
- *If $E^\star\mathcal{F}$ is generically transverse to a component, the corresponding vertex is black. If not, the vertex is white.*
- *Each vertex s is double numbered by*
  - $\left(\rho(s), -\operatorname{val}_w(s) + \sum_{p \in s} \operatorname{Ind}(E^\star\mathcal{F}, s, p)\right)$ *if the vertex is white;*
  - $\left(\rho(s), 2 - \operatorname{val}_w(s) + \sum_{p \in s} \operatorname{Tan}(E^\star\mathcal{F}, s, p)\right)$ *if the vertex is black*

  where $\operatorname{val}_w(s)$ *stands for the number of white vertices attached to s.*

In this formalism, the next property is just Theorem 3 in [16].

Proposition 1.2 ([16])**.** *For any foliation $\mathcal{F}$ and any blowing-up process E, we get*

$$\nu(\mathcal{F}) = \nu(\mathbb{A}[\mathcal{F}, E]).$$

Example 1.3. *Let $\mathcal{F}$ be the foliation defined by the $1-$form*

$$\left(2x^2 + \frac{5}{2}y^3 - \frac{9}{2}x^3 y\right)\mathrm{d}y - \left(3xy - 3x^2y^2\right)\mathrm{d}x.$$

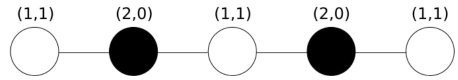

Figure 1: Dual graph of $\mathcal{F}$ with respect to the desingularization of C.

It leaves invariant the curve $C = \left\{\left(y^2 - x^3\right)\left(y^3 - x^2\right) = 0\right\}$ *and it is of multiplicity* $\nu(\mathcal{F})$ *is equal to* 2. *Now, if* E *refers to the minimal process of reduction of* C *then the graph* $\mathbb{A}[\mathcal{F}, E]$ *is in Figure 1. In particular, its valuation satisfies*

$$\begin{aligned}\nu(\mathbb{A}[\mathcal{F}, E]) &= -1 + 1 \times 1 + 2 \times 0 + 1 \times 1 + 2 \times 0 + 1 \times 1 \\ &= 2 = \nu(\mathcal{F}).\end{aligned}$$

1.2 Construction of foliations with prescribed dual graphs.

Using a method to construct germ of singular foliations in the complex plane introduced by A. Lins Neto [17], we can produce foliations with prescribed double numbered dual graphs.

Proposition 1.4. *Let* $\mathbb{A}$ *be the dual graph of a process of blowing-ups* E. *Consider a coloration of* $\mathbb{A}$ *and for any* $s \in \mathbb{A}$ *a double numbering* $(s(1), s(2))$ *such that* $s(1)$ *is the multiplicity* $\rho(s)$. *There exists a foliation* $\mathcal{F}$ *and a process of blowing-ups* E' *with the same dual graph as* E *such that*

$$\mathbb{A}[\mathcal{F}, E'] = \mathbb{A}$$

if and only the following properties are satisfied

a) for any white vertex s, $s(2) + \mathrm{val}_w(s) \geq 0$;

b) for any black vertex s, $s(2) + \mathrm{val}_w(s) \geq 2$;

c) for any connected component $\mathbb{K}$ *of* $\mathring{\mathbb{A}}$ *which is* $\mathbb{A}$ *minus the black vertices, there exists* $s \in \mathbb{K}$ *such that* $s(2) > 0$.

Proof. The direct part of properties (a) and (b) follows from Definition 1.1. The direct part of the property (c) is a classical consequence of the fact that the intersection matrix of E is definite negative [19]. Indeed, if for any $s \in \mathbb{K}$, $s(2) = 0$ then

$$\forall s \in \mathbb{K}, \sum_{p \in s} \mathrm{Ind}(E^\star\mathcal{F}, s, p) = \mathrm{val}_w(s).$$

Since for any s, $s' \in \mathbb{K}$, $s \cap s'$ is a singular point of $E^\star\mathcal{F}$ for which the index $\mathrm{Ind}(E^\star\mathcal{F}, s, p)$ is bigger than 1, the above equalities ensures that along $\mathbb{K}$, the foliation $E^\star\mathcal{F}$ is singular only at the points of type $s \cap s'$, s, $s' \in \mathbb{K}$ with indeces equal to 1. The latter is impossible according to [5, Proposition 3.1].

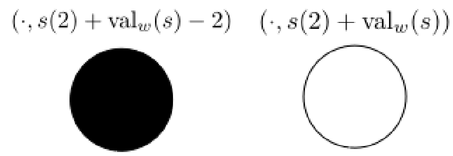

Figure 2: Local models of elementary foliations for black and white vertices.

Now, suppose that the properties $(a), (b)$ and (c) are satisfied. The statement relies upon a result of Lins Neto [17] of construction of singular foliations in dimension 2 from elementary elements which are reduced singularity. The only obstruction for such construction is the well-known Camacho-Sad relation [2]

$$\sum_{p\in s} \mathrm{CS}\left(E^{\star}\mathcal{F}, s, p\right) = -s\cdot s$$

where $s\cdot s$ is the self intersection of the component s and $\mathrm{CS}\left(\cdot\right)$ stands for the Camacho-Sad index of the foliation $E^{\star}\mathcal{F}$ with respect to s at p. More precisely, consider s in $\mathbb{A}$ with $s\cdot s = -k$. Locally around the irreducible component s, the total space of E is analytically equivalent to the neighborhood of $x_1 = 0$ in the following gluing model

$$\left(\mathbb{C}^2, (x_1, y_1)\right) \coprod \left(\mathbb{C}^2, (x_2, y_2)\right) \tag{1}$$

with the identification

$$y_2 = y_1^k x_1 \quad x_2 = \frac{1}{y_1}.$$

This neighborhood can be foliated by the foliation $\mathcal{R}_s$ given in the coordinates (x_1, y_1) by the 1−form

$$\omega = \mathrm{d}x_1 + \prod_{i=1}^{s(2)+\mathrm{val}_w(s)-2} \left(y_1 - i\right)\mathrm{d}y_1$$

which is completely transverse to $x_1 = 0$ except at the points $(0, i)\,, i = 1, \ldots, s\,(2) + \mathrm{val}_w\,(s) - 2$ where it is tangent at order 1. The foliation $\mathcal{R}_s$ is a local model for a foliation with double numbered black graph presented in Figure 2. The model (1) can also be foliated by $\mathcal{G}_s$ given by 1−form

$$\eta = \frac{\mathrm{d}x_1}{x_1} + \sum_{i=1}^{s(2)+\mathrm{val}_w(s)} \lambda_i^s \frac{\mathrm{d}y_1}{y_1 - i}. \tag{2}$$

with

$$\sum_{i=1}^{s(2)+\mathrm{val}_w(s)} \lambda_i^s = p. \tag{3}$$

By construction, for any i, one has

$$\mathrm{CS}\left(\mathcal{G}_s, \{x_1 = 0\}, i\right) = \lambda_i^s.$$

The foliation $\mathcal{G}_s$ is a local model for a foliation with double numbered white graph presented in Figure 2. All these local models can be glued together by gluing maps following the dual graph of $\mathbb{A}$. The property (c) ensures that, along any connected component $\mathbb{K}$ of the graph minus the black vertices, no incompatibility will occur between the relations (3) and the fact that, at any intersection point of two white components s and s', one should have

$$\mathrm{CS}\left(\mathcal{G}_s, s, s \cap s'\right) \cdot \mathrm{CS}\left(\mathcal{G}_{s'}, s', s \cap s'\right) = \lambda_{i_s}^s \cdot \lambda_{i_{s'}}^{s'} = 1. \tag{4}$$

Indeed, the union of the relations (3) and (4) yields a number of equations equal to

$$\sharp \mathrm{vertices}\left(\mathbb{K}\right) + \sharp \mathrm{edges}\left(\mathbb{K}\right).$$

However, the number of variables involved in the mentioned relations is

$$\sum_{s \in \mathbb{K}} s\left(2\right) + \mathrm{val}_w\left(s\right).$$

Following property (c) the above number of variables satisfies

$$\begin{aligned} \textstyle\sum_{s \in \mathbb{K}} s\left(2\right) + \mathrm{val}_w\left(s\right) \geq 1 + \textstyle\sum_{s \in \mathbb{K}} \mathrm{val}_w\left(s\right) &= 2 \cdot \sharp \mathrm{vertices}\left(\mathbb{K}\right) - 1 \\ &= \sharp \mathrm{vertices}\left(\mathbb{K}\right) + \sharp \mathrm{edges}\left(\mathbb{K}\right) \end{aligned}$$

since $\sharp \mathrm{vertices}\left(\mathbb{K}\right) = \sharp \mathrm{edges}\left(\mathbb{K}\right) + 1$. Therefore the system of equations (3) and (4) has always a solution - that can be chosen rational.
As a whole, the gluings lead to a foliation defined in a neighborhood of a divisor $\mathcal{D}$ with same intersection matrix as the one of the exceptional divisor of E. According to a classical result of H. Grauert [14], the neighborhood of $\mathcal{D}$ is analytically equivalent to the neighborhood of the exceptional divisor of some blowing-up process E' with same dual graph as E. The latter neighborhood is foliated by a foliation $\mathcal{F}'$ that can be contracted by E' in a foliation $\mathcal{F}$. By construction, one has

$$\mathbb{A}\left[\mathcal{F}, E'\right] = \mathbb{A}.$$

□

1.3 Minimal Saito number of a given topological class of curve.

Let C be a germ of curve in the complex plane defined by $f \in \mathbb{C}\{x, y\}$. The *Saito module* of C is the $\mathbb{C}\{x, y\}$-module

$$\Omega^1\left(\log C\right) = \{\omega \in \Omega^1 : \ \omega \wedge \mathrm{d}f \in \langle f \rangle\}.$$

Saito, in [18], showed that, in this context, $\Omega^1(\log C)$ is a free-module generated by a set of two elements $\{\omega_1, \omega_2\}$, called a *Saito basis* for C, and it is characterized by a property we refer to as the *Saito's criterion* : a set of two elements $\{\omega_1, \omega_2\}$ is a Saito basis if and only if for some unit $u \in \mathbb{C}\{x, y\}$, we get

$$\omega_1 \wedge \omega_2 = uf\mathrm{d}x \wedge \mathrm{d}y.$$

The *Saito number* of C is by definition

$$\mathfrak{s}(C) = \min\{\nu(\omega_1), \nu(\omega_2)\}.$$

It can be seen that $\mathfrak{s}(C)$ is also equal to $\min\{\nu(\mathcal{F}) : \ C \text{ is invariant by } \mathcal{F}\}$ and is an analytic invariant of C [8].
Let $\mathrm{Top}(C)$ be the topological (or equisingularity) class of C. In this section, as a consequence of the previous results, we present an algorithm to select a curve $C' \in \mathrm{Top}(C)$ and a foliation $\mathcal{F}'$ leaving invariant C' such that

$$\nu(\mathcal{F}') = \mathfrak{s}(C') = \min_{C'' \in \mathrm{Top}(C)} \mathfrak{s}(C'').$$

Let E be the desingularization process of C and $\mathbb{A}$ be the dual graph of E. The integer $n_s(C)$ stands for the number of components of the strict transform of C by E attached to the component s of the exceptional divisor.
Let $\mathcal{F}$ be a foliation whose C is an invariant curve and consider its dual graph $\mathbb{A}[\mathcal{F}, E]$. Suppose that s is a white vertex. Since s is invariant by $E^\star\mathcal{F}$, $E^\star\mathcal{F}$ is singular at any point p of s at which is attached a component of the strict transform of C. In particular, the index

$$\mathrm{Ind}\,(E^\star\mathcal{F}, s, p)$$

is strictly positive. Therefore, for any white vertex s, one has

$$s(2) \geq n_s(C).$$

From $\mathbb{A}[\mathcal{F}, E]$, we consider the double numbered colored graph $\mathbb{A}$ which is a copy of $\mathbb{A}[\mathcal{F}, E]$ except that we set

- for any white vertex $s(2) = n_s(C)$
- for any black vertex $s(2) = 2 - \mathrm{val}_w(s)$.

If doing so, the property (c) of Proposition 1.4 is not satisfied for some component $\mathbb{K}$, we set $s(2) = 1$ for the vertex s of $\mathbb{K}$ for which $\rho(s)$ is minimal. Applying Proposition 1.4 yields a foliation $\mathcal{F}'$ that let invariant a curve C' which is topologically equivalent to C. Indeed, for any component s, either s is black, $\mathcal{F}'$ generically transverse to s and we choose arbitrarily $n_S(C)$ regular and transverse invariant curves attached to s, or s is white and $\mathcal{F}'$ is locally given by (2) and leaves invariant $s(2) = n_s(C)$ regular and transverse curves still attached to s. The union of all these curves yields a curve C' whose desingularization process has the dual graph of E. Since $n_s(C') = n_s(C)$, then C' and C are topologically equivalent [20].

Proposition 1.5 (Algorithm 1)**.** *We have*

$$\nu\left(\mathcal{F}'\right)\leq\nu\left(\mathcal{F}\right).$$

Proof. Let $\mathcal{K}_1$ the set of components $\mathbb{K}$ of $\mathring{\mathbb{A}}$ for which one has $n_s\left(C\right)=0$ for all $s\in\mathbb{K}$ and $\mathcal{K}_2$ the other set of components. If $\mathbb{K}\in\mathcal{K}_1$, we denote by $s_{\mathbb{K}}$ a vertex for which $s_{\mathbb{K}}\left(2\right)>0$, that exists according to Proposition 1.4.
We have

$$\begin{aligned}\nu\left(\mathcal{F}\right)&=-1+\sum_{s\in\mathbb{A}[\mathcal{F},E]}s\left(1\right)s\left(2\right)\\&=-1+\sum_{s\text{ black}}\rho\left(s\right)\left(2-\mathrm{val}_w\left(s\right)+\sum_{p\in s}\mathrm{Tan}\left(E^{\star}\mathcal{F},s,p\right)\right)\\&\quad+\sum_{\mathbb{K}\in\mathcal{K}_1}\sum_{s\in\mathbb{K}}\rho\left(s\right)s\left(2\right)+\sum_{\mathbb{K}\in\mathcal{K}_2}\sum_{s\in\mathbb{K}}\rho\left(s\right)s\left(2\right).\end{aligned}$$

Now, for $\mathbb{K}\in\mathcal{K}_1$, we can give the following lower bound

$$\begin{aligned}\sum_{s\in\mathbb{K}}\rho\left(s\right)s\left(2\right)&\geq\rho\left(s_{\mathbb{K}}\right)s_{\mathbb{K}}\left(2\right)+\sum_{s\in\mathbb{K}\setminus\{s_{\mathbb{K}}\}}\rho\left(s\right)s\left(2\right)\\&\geq\min_{s\in\mathbb{K}}\rho\left(s\right).\end{aligned}$$

For $\mathbb{K}\in\mathcal{K}_2$, the following lower bound occurs

$$\sum_{s\in\mathbb{K}}\rho\left(s\right)s\left(2\right)\geq\sum_{s\in\mathbb{K}}\rho\left(s\right)n_s\left(C\right).$$

Finally, if s is black, then

$$\rho\left(s\right)\left(2-\mathrm{val}_w\left(s\right)+\sum_{p\in s}\mathrm{Tan}\left(E^{\star}\mathcal{F},s,p\right)\right)\geq\rho\left(s\right)\left(2-\mathrm{val}_w\left(s\right)\right).$$

Combining all these inequalities leads to

$$\begin{aligned}\nu\left(\mathcal{F}\right)\geq&-1+\sum_{s\text{ black}}\rho\left(s\right)\left(2-\mathrm{val}_w\left(s\right)\right)\\&+\sum_{\mathbb{K}\in\mathcal{K}_1}\min_{s\in\mathbb{K}}\rho\left(S\right)+\sum_{\mathbb{K}\in\mathcal{K}_2}\sum_{s\in\mathbb{K}}\rho\left(s\right)n_s\left(C\right)=\nu\left(\mathcal{F}'\right).\end{aligned}$$

□

Propostion 1.5 provides a simple algorithm that determines the minimal Saito number of a given equisingularity class of plane curves. Consider E the desingularization process of C and $\mathbb{A}$ its dual graph. Choose any coloration of the graph among the finite set of such coloration and set $s\left(2\right)$ as in Proposition 1.5 : the value of $s\left(2\right)$

depends only on C and on the chosen coloration. Each such choice leads to a certain multiplicity of foliation. The smallest multiplicity among them is the desired number. The complexity of this algorithm is $O(2^b)$ where b is the length of the desingularization process. As a consequence, Theorem 1 stated in the introduction is proved.
In the sections below, we treat some examples, for which, beyond the algorithm, some formula can be established.

1.3.1 Irreducible curve.

Let C be an irreducible plane curve given by a parametrization

$$\psi(t)=\left(t^{\nu_0},\sum_{i\geq\beta_1}a_it^i\right).$$

The equisingularity class $\mathrm{Top}(C)$ of C can be totally determined by the dual graph of the desingularization process E of C, equivalently by its characteristic exponents $\beta_0=\nu_0=\nu(C),\beta_1=\nu_1,\beta_2,\ldots,\beta_g$ or by its value semigroup $\Gamma=\langle\nu_0,\nu_1,\ldots,\nu_g\rangle$ (see [21] for instance).
Notice that C can be defined by the minimal polynomial $f\in\mathbb{C}\{x\}[y]$ of

$$\sum_{i\geq\beta_1}a_ix^{\frac{i}{\nu_0}}.$$

Let $f_g\in\mathbb{C}\{x\}[y]$ be the minimal polynomial of the function

$$\sum_{\beta_1\leq i<\beta_g}a_ix^{\frac{i}{\nu_0}},$$

according to [21] we get $\nu(f_g)=\frac{\nu_0}{e_{g-1}}$ where

$$e_{g-1}=gcd(\beta_0,\ldots,\beta_{g-1})=\gcd(\nu_0,\ldots,\nu_{g-1})$$

and the intersection multiplicity with f is $\mathrm{I}(f,f_g)=\nu_g$.
Consider the differential $1-$form $\omega=\nu_0x\mathrm{d}f_g-\nu_gf_g\mathrm{d}x$. The foliation associated to ω leaves invariant the curves $x=0$ and $f_g=0$ whose strict transform are attached respectively in the first and last *extreme*[1] *component* of E (see Theorem 3.7 in [6]). In addition, it is purely dicritical along the *central component*, that is the one to which is attached the strict transform of C. In such component, the foliation admits infinite many invariant curves C_u given by

$$\psi_u(t)=\left(t^{\nu_0},\sum_{\beta_1\leq i<\beta_g}a_it^i+ut^{\beta_g}+\sum_{i>\beta_g}r(u)t^i\right)$$

[1]A extreme component of E is an irreducible component that intersects only one other irreducible component of E.

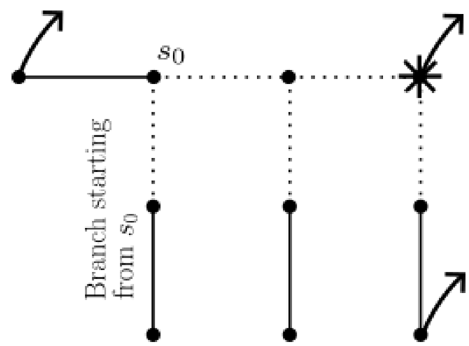

Figure 3: Desingularization process for a irreducible plane curve and $\mathcal{F}_{\min}$.

for every $u \in \mathbb{C} \setminus \{0\}$ and $r(u) \in \mathbb{C}(u)$. Consequently, $C_u \in \mathrm{Top}(C)$ for any $u \in \mathbb{C} \setminus \{0\}$. In addition, we get

$$\nu(\omega) = \frac{\nu_0}{e_{g-1}}.$$

On another hand, the simple structure of E allows us to follow Algorithm 1 described in Proposition 1.5 *by hand*. Indeed, the desingularization process of C is shown in Figure 3. Consider a foliation $\mathcal{F}_{\min}$ purely dicritical along the central component. Moreover, $\mathcal{F}_{\min}$ leaves invariant two regular curve attached to the first and the last extreme component of E of respective multiplicities 1 and $\frac{\nu(C)}{e_{g-1}}$. Applying Proposition 1.2 in that situation yields the formula

$$\nu\left(\mathcal{F}_{\min}\right) = \frac{\nu_0}{e_{g-1}}.$$

This value is also the minimum value for a Saito number in the equisingularity class of C. Indeed, consider a foliation $\mathcal{F}$ tangent to C. Proposition 1.2 is written

$$\nu\left(\mathcal{F}\right) = -1 + \sum_{s \in \mathbb{A}[\mathcal{F},E]} s(1)s(2).$$

In view of the expression of $s(2)$, a term with a negative contribution in the above sum may appear only if $\mathcal{F}$ is dicritical along some component s_0 of valence 3 and any component attached to s_0 is not dicritical. However, doing so, along the branch of the tree attached to s_0 there must be some component s_1 such that $s_1(2) > 0$. Therefore, the contribution of both components s_0 and s_1 is written

$$s_0(1)(-1 + \alpha) + s_1(1)s_1(2)$$

with $\alpha \geq 0$. Since, $s_1(1) \geq s_0(1)$, as the whole, the contribution keeps on being positive : as a consequence, it is *useless* for $\mathcal{F}$ to be dicritical along s_0 in order to reach the desired minimum and the valuation of $\mathcal{F}$ is bigger than the valuation of $\mathcal{F}_{\min}$. Finally, we recover the result of [5], that is

$$\min_{C'\in \mathrm{Top}(C)} \mathfrak{s}\left(C'\right) = \nu(\omega) = \nu\left(\mathcal{F}_{\min}\right) = \frac{\nu_0}{e_{g-1}}.$$

Remark 1.6. *The Saito numbers of the curves in* $\mathrm{Top}(C)$ *where* C *is an irreducible curve for which* $e_{g-1} = 2$*, are constant. Indeed, from the result above and [8], we have*

$$\min_{C'\in \mathrm{Top}(C)} \mathfrak{s}\left(C'\right) = \max_{C'\in \mathrm{Top}(C)} \mathfrak{s}\left(C'\right) = \frac{\nu_0}{2}.$$

1.3.2 Non-irreducible Curves.

A curve with two components. As a generalization of Example 1.3 let us consider C be the plane curve defined by

$$f = \left(y^{\nu_0} - x^{\nu_1}\right)\left(x^{\nu_0} - y^{\nu_1}\right) = 0$$

where $1 = gcd(\nu_0, \nu_1) < \nu_0 < \nu_1$. It can be seen that the two following differential $1-$forms

$$\begin{array}{rcl} \omega_1 & = & \left(\nu_1^2 x^{\nu_1-\nu_0}(y^{\nu_1} - x^{\nu_0}) - \nu_0^2(y^{\nu_0} - x^{\nu_1})\right) dx \\ & & +\nu_0\nu_1 x y^{\nu_0-1}(1 - x^{\nu_1-\nu_0} y^{\nu_1-\nu_0}) dy \\ \omega_2 & = & \left(\nu_1^2 y^{\nu_1-\nu_0}(y^{\nu_0} - x^{\nu_1}) - \nu_0^2(y^{\nu_1} - x^{\nu_0})\right) dy \\ & & -\nu_0\nu_1 x^{\nu_0-1} y(1 - x^{\nu_1-\nu_0} y^{\nu_1-\nu_0}) dx \end{array}$$

satisfy the Saito criterion, that is

$$\omega_1 \wedge \omega_2 = (\nu_1 - \nu_0)(\nu_1 + \nu_0)(\nu_0^2 - \nu_1^2 x^{\nu_1-\nu_0} y^{\nu_1-\nu_0}) f \mathrm{d}x \wedge \mathrm{d}y$$

and thus $\{\omega_1, \omega_2\}$ is a Saito basis for C. As a consequence, we find

$$\mathfrak{s}(C) = \nu_0.$$

Since the latter is also an upper bound for the maximum Saito number in the associated equisingularity class, we obtain

$$\max_{C'\in \mathrm{Top}(C)} \mathfrak{s}(C') = \nu_0.$$

On the other hand, the minimal Saito number is reached for the foliation depicted in Figure 4, and thus

$$\min_{C'\in \mathrm{Top}(C)} \mathfrak{s}(C') = 2.$$

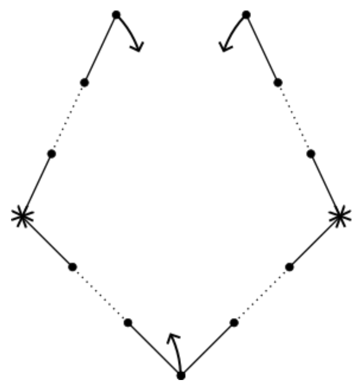

Figure 4: Topology of the foliation with the minimal Saito number for C : $(y^{\nu_0} - x^{\nu_1})(x^{\nu_0} - y^{\nu_1}) = 0$.

Curves with many components. Let C be a curve desingularized by E. Let $\mathbb{A}$ be the dual graph of E. We say that C *has a lot of components* if

$$\forall s \in \mathbb{A},\ n_s(C) \geq 2 + \sum_{s', s\cap s' \neq \emptyset} \frac{\rho(s')}{\rho(s)}.$$

The curves with a lot of components are of special interest because the foliation associated to their minimal Saito number happens to be *absolutely dicritical* as defined in [3].

Proposition 1.7. *If C has a lot of components then the minimal Saito number in* $\mathrm{Top}(C)$ *is equal to*

$$\min_{C' \in \mathrm{Top}(C)} \mathfrak{s}(C') = -1 + 2\sum_{s\in\mathbb{A}} \rho(s).$$

and is reached by an absolutely dicritical foliation with respect to E.

In particular, the minimal Saito number of a topological class of curve with a lot of components does not depends on C anymore but only on E.

Proof. Let $\mathcal{F}$ be a foliation leaving invariant C' and reaching the minimum above. According to Proposition 1.5, we can suppose that $\mathcal{F}$ is constructed by gluing local models of Proposition 1.4. Consider now $s \in \mathbb{A}[\mathcal{F}, E]$ and suppose that s is white. Using still the Lins Neto's argument, we can construct a foliation $\mathcal{F}_s$ such that $\mathbb{A}[\mathcal{F}_s, E]$ has the same coloration as $\mathbb{A}[\mathcal{F}_s, E]$ except that s is black in $\mathbb{A}[\mathcal{F}_s, E]$. In particular, following Proposition 1.2

$$\begin{aligned} \nu(\mathcal{F}) &= -1 + \rho(s) n_s(C) \\ &\quad + \sum_{s', s\cap s' \neq \emptyset} \delta_{s'} \rho(s') n_{s'}(C) + (1-\delta_{s'})\rho(s')(2 - \mathrm{val}_w(s')) \\ &\quad + (\text{does not depend on } s) \end{aligned}$$

and

$$\begin{aligned}\nu(\mathcal{F}_s) &= -1+\rho(s)(2-\mathrm{val}_w(s)) \\ &\quad + \sum_{s', s\cap s'\neq\emptyset} \delta_{s'}\rho(s')n_{s'}(C) + (1-\delta_{s'})\rho(s')(3-\mathrm{val}_w(s')) \\ &\quad +(\text{does not depend on } s)\end{aligned}$$

where $\delta_s = 1$ if s is white, 0 otherwise. Therefore, we get

$$\nu(\mathcal{F})-\nu(\mathcal{F}_s) = \rho(s)(n_s(C)-2+\mathrm{val}_w(s)) - \sum_{s', s\cap s'\neq\emptyset}(1-\delta_{s'})\rho(s').$$

It can be seen that the latter is a positive expression under the assumption of the proposition. As a consequence, we can always decrease the multiplicity of a foliation leaving invariant a curve $C' \in \mathrm{Top}(C)$ by making black any of component $\mathbb{A}[\mathcal{F}, E]$. In the end, the obtained foliation is *absolutely dicritical* as defined in [3] with respect to E and its multiplicity is

$$-1+\sum_{s\in\mathbb{A}}\rho(s)(2-\mathrm{val}_w(s)) = -1+2\sum_{s\in\mathbb{A}}\rho(s).$$

□

2 Range of the Saito function on given topological classes.

Let us consider C a germ of plane curve. In [11], it is proved that the maximum Saito number along the topological class Top (C) is given by

$$\max_{C'\in\mathrm{Top}(C)} \mathfrak{s}(C') = \begin{cases} \frac{\nu(C)}{2}-1 & \text{if } C \text{ is } \textit{radial} \text{ and } \nu(C) \text{ is even} \\ \left\lfloor \frac{\nu(C)}{2}\right\rfloor & \text{if not,}\end{cases}$$

radial being defined in [11]. Actually, the above maximum is reached for a generic element C' in the equisingularity class of C. Moreover, in the previous section, we provide an algorithm (see Proposition 1.5) to compute the minimum $\min_{C'\in\mathrm{Top}(C)} \mathfrak{s}(C')$. It is of natural interest to look at the integers between these two bounds that are reached as a Saito number of a certain analytical class of curves in the given equisingularity class.

For that purpose, let us consider the curve C_{N,ν_0,ν_1} given by

$$f_{N,\nu_0,\nu_1} = y^{N\nu_0} - x^{N\nu_1} = 0$$

where $N > 0$ and $\nu_0 \leq \nu_1$ are relatively prime. Since, considering the two $1-$forms

$$\omega = \nu_1 y\mathrm{d}x - \nu_0 x\mathrm{d}y \text{ and } \eta = \mathrm{d}f_{N,\nu_0,\nu_1}$$

leads to the Saito criterion

$$\omega \wedge \eta = N\nu_0\nu_1 f_{N,\nu_0,\nu_1} \mathrm{d}x \wedge \mathrm{d}y,$$

we obtain

$$\min_{C \in \mathrm{Top}\left(C_{N,\nu_0,\nu_1}\right)} \mathfrak{s}(C) = \min\{1, N\nu_0 - 1\},$$

which is equal to 1 except when $N\nu_0 = 1$, that is when the curve C_{N,ν_0,ν_1} is regular, for which the minimum in 0. As a consequence of the computations in [8, Proposition 8], it appears that the curve C_{N,ν_0,ν_1} is radial if and only if $\nu_1 = 1$ and $N \geq 3$. In particular, if $\nu_1 > 1$ then

$$\max_{C \in \mathrm{Top}\left(C_{N,\nu_0,\nu_1}\right)} \mathfrak{s}(C) = \left\lfloor \frac{N\nu_0}{2} \right\rfloor.$$

The goal of the next subsections is to show that the range of the map

$$C \in \mathrm{Top}\left(C_{N,\nu_0,\nu_1}\right) \mapsto \mathfrak{s}(C)$$

is the whole set of integers between the two above extrema.

2.1 Saito numbers in the topological class Top(C_{1,ν_0,ν_1}).

If $\nu_0 = 1$ then $\mathfrak{s}(C) = 0$, thus, in what follows, we suppose that $\nu_0 > 1$. In particular, we obtain

$$\min_{C \in \mathrm{Top}\left(C_{1,\nu_0,\nu_1}\right)} \mathfrak{s}(C) = 1 \quad \text{and} \quad \max_{C \in \mathrm{Top}\left(C_{N,\nu_0,\nu_1}\right)} \mathfrak{s}(C) = \left\lfloor \frac{\nu_0}{2} \right\rfloor.$$

Since $\mathfrak{s}(C_{1,\nu_0,\nu_1}) = 1$, to show that any $k \in \{1, \ldots, \left\lfloor \frac{\nu_0}{2} \right\rfloor\}$ is achieved as a Saito number for an element in $\mathrm{Top}(C_{1,\nu_0,\nu_1})$ it is sufficient to consider $k > 1$ and $\nu_0 > 3$. Given any irreducible plane curve C with parameterization $\psi(t) \in \mathbb{C}\{t\} \times \mathbb{C}\{t\}$ the semigroup Γ_C associated to C is

$$\Gamma_C = \{\nu(\psi^*(h));\ h \in \mathbb{C}\{x, y\} \text{ such that } \psi^*(h) \neq 0\}.$$

We can extend the valuation $\nu(\cdot)$ to a differential 1-form not tangent to C and we define the set

$$\Lambda_C = \{\nu(\psi^*(\eta)) + 1;\ \eta \text{ is a differential 1-form not tangent to } C\}.$$

The set Λ_C is an analytical invariant for C and it is a Γ_C-semimodule finitely generated [15], that is, there exist $\lambda_{-1}, \lambda_0, \ldots, \lambda_s \in \Lambda_C$ such that

$$\Lambda_C = \bigcup_{i=-1}^{s} (\Gamma_C + \lambda_i)$$

with $\lambda_j \notin \Lambda_{j-1} := \bigcup_{i=-1}^{j-1}(\Gamma_C + \lambda_i)$ for $0 \leq j \leq s$. In [15] we find an algorithm to compute a minimal system of generators of Λ_C for any irreducible plane curve C. Given $C \in \mathrm{Top}\,(C_{1,\nu_0,\nu_1})$, that is, $\Gamma_C = \langle \nu_0, \nu_1 \rangle$ there is another valuation associated to C called *divisorial valuation* given as the following. If $h = \sum_{i,j\geq 0} h_{ij}x^i y^j \in \mathbb{C}\{x,y\} \setminus \{0\}$ the divisorial valuation $\nu_D(h)$ of h is (see [4])

$$\nu_D\,(h) = \min\{\,\nu_0 i + \nu_1 j|\ \ h_{ij} \neq 0\} \tag{5}$$

and we extend it to a differential 1-form $A\mathrm{d}x + B\mathrm{d}y$ by

$$\nu_D\,(A\mathrm{d}x + B\mathrm{d}y) = \min\{\nu_D(A) + \nu_0, \nu_D(B) + \nu_1\}.$$

For $C \in \mathrm{Top}(C_{1,\nu_0,\nu_1})$, Cano, Corral and Senovilla-Sanz in [4] introduce a finite set of integers from which is derived a characterization of a Saito basis for C. In the following, we briefly describe this construction. Let $\lambda_{-1}, \lambda_0, \ldots, \lambda_s \in \Lambda_C$ be the miminal generators for Λ_C. Setting $t_0 = \lambda_0 = \nu_1$, for $1 \leq i \leq s+1$, define inductively the following data

$$\begin{array}{rcl} u_i^\star & = & \min\{\lambda_{i-1} + \nu_\star n \in \Lambda_{i-2};\ n \geq 1\} \\ t_i^\star & = & t_{i-1} + u_i^\star - \lambda_{i-1} \end{array} \tag{6}$$

where $\star = 0, 1$ and

$$t_i = \min\{t_i^0, t_i^1\} \quad \tilde{t_i} = \max\{t_i^0, t_i^1\}.$$

The mentioned above result is enunciated below.

Theorem 2.1 (Cano, Corral and Senovilla-Sanz)**.** *For $C \in \mathrm{Top}(C_{1,\nu_0,\nu_1})$, there exist two differential 1-forms ω_{s+1} and $\tilde{\omega}_{s+1}$ leaving C invariant such that*

$$\nu_D(\omega_{s+1}) = t_{s+1} \quad \text{and} \quad \nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}.$$

Moreover, for any pair of differential 1-forms as above, the set $\{\omega_{s+1}, \tilde{\omega}_{s+1}\}$ is a Saito basis for C.

For $\nu_0 > 3$ and any $1 < k \leq \left\lfloor \frac{\nu_0}{2} \right\rfloor$, let us consider the differential 1-form defined by

$$\omega = \nu_1 x^{k-1}(\nu_0 x dy - \nu_1 y dx) + \nu_0(\gamma - \nu_1)y^{\nu_0 - k}dy \tag{7}$$

with $\gamma := (\nu_0 - k + 1)\nu_1 - k\nu_0$.
Notice that for any plane curve $C \in \mathrm{Top}\,(C_{1,\nu_0,\nu_1})$ we get

$$\nu_D\,(\omega) = \nu_D\left(x^{k-1}\,(\nu_0 x\mathrm{d}y - \nu_1 y\mathrm{d}x)\right) = k\nu_0 + \nu_1.$$

Lemma 2.2. *There is a curve $C \in \mathrm{Top}(C_{1,\nu_0,\nu_1})$ invariant by ω and given by a parametrization of the form*

$$\psi(t) = \left(t^{\nu_0}, t^{\nu_1} + t^\gamma + \sum_{i \geq 2\gamma - \nu_1} a_i t^i\right). \tag{8}$$

Proof. Considering $\psi(t)$ as in the lemma, we have

$$\psi^*(\nu_0 x dy - \nu_1 y dx) = \nu_0(\gamma - \nu_1)t^{\gamma+\nu_0-1} + \sum_{i \geq 2\gamma-\nu_1} \nu_0(i-\nu_1)a_i t^{i+\nu_0-1}$$

and

$$\psi^*(y^{\nu_0-k+1}) = t^{(\nu_0-k+1)\nu_1} + (\nu_0-k+1)t^{\gamma+(\nu_0-k)\nu_1} + \sum_{j>(\nu_0-k)\nu_1+\gamma} Q_j t^j$$

where $Q_j \in \mathbb{C}[a_{2\gamma-\nu_1+1}, \ldots, a_{j-(\nu_0-k)\nu_1}]$. In this way, since

$$\gamma = (\nu_0-k+1)\nu_1 - k\nu_0$$

we get

$$\begin{aligned}
\psi^*(y^{\nu_0-k}dy) &= \nu_1 t^{(\nu_0-k+1)\nu_1-1} + (\gamma+(\nu_0-k)\nu_1)t^{\gamma+(\nu_0-k)\nu_1-1} \\
&\quad + \sum_{j>(\nu_0-k)\nu_1+\gamma} \frac{j}{\nu_0-k+1} Q_j t^{j-1} \\
&= \nu_1 t^{\gamma+k\nu_0-1} + (\gamma+(\nu_0-k)\nu_1)t^{2\gamma-\nu_1+k\nu_0-1} + \\
&\quad + \sum_{i>2\gamma-\nu_1} \frac{i+k\nu_0}{\nu_0-k+1} Q_{i+k\nu_0} t^{i+k\nu_0-1}
\end{aligned}$$

and

$$\begin{aligned}
\psi^*(\omega) &= \nu_0(\gamma-\nu_1)\left(2\nu_1 a_{2(\gamma-\nu_1)} - (\gamma+(\nu_0-k)\nu_1)\right) t^{\gamma+(\nu_0-k)\nu_1-1} \\
&\quad + \nu_0 \sum_{i>2\gamma-\nu_1} \left(\nu_1(i-\nu_1)a_i - \frac{(i+k\nu_0)(\gamma-\nu_1)}{\nu_0-k+1} Q_{i+k\nu_0}\right) t^{i+k\nu_0-1}.
\end{aligned}$$

From $k \leq \nu_0 - k$, we get $i + k\nu_0 - (\nu_0-k)\nu_1 < i$ and

$$Q_{i+k\nu_0} \in \mathbb{C}[a_{2\gamma-\nu_1}, \ldots, a_{i+k\nu_0-(\nu_0-k)\nu_1}] \subseteq \mathbb{C}[a_{2\gamma-\nu_1}, \ldots, a_{i-1}].$$

So, setting

$$a_{2\gamma-\nu_1} = \frac{\gamma+(\nu_0-k)\nu_1}{2\nu_1} \quad \text{and} \quad a_i = \frac{(\gamma-\nu_1)(i+k\nu_0)}{\nu_1(i-\nu_1)(\nu_0-k+1)} Q_{i+k\nu_0}$$

yields a parameterization $\psi(t)$ defining a plane curve $C \in \mathrm{Top}(C_{1,\nu_0,\nu_1})$ invariant by ω. □

We now prove the main result of this subsection.

Proposition 2.3. *For any* $1 \leq k \leq \left\lfloor \frac{\nu_0}{2} \right\rfloor$, *there exists* $C \in \mathrm{Top}\left(C_{1,\nu_0,\nu_1}\right)$ *such that* $\mathfrak{s}(C) = k$.

Proof. If $k = 1$ then considering C_{1,ν_0,ν_1} we get $\mathfrak{s}(C_{1,\nu_0,\nu_1}) = 1$. Therefore, suppose that $\nu_0 > 3$ and let k be an integer in $\left\{2, \ldots, \left\lfloor \frac{\nu_0}{2} \right\rfloor\right\}$. Let $C \in \mathrm{Top}(C_1, \nu_0, \nu_1)$ be the curve characterized in the previous lemma with parametrization $\psi(t)$ as (8). We will compute the minimal generators $\{\lambda_{-1}, \lambda_0, \ldots, \lambda_s\}$ for Λ_C using the algorithm developed in [15]. The first two generators are written

$$\begin{aligned} \lambda_{-1} &= \nu(\psi^*(\mathrm{d}x)) = \nu_0, \\ \lambda_0 &= \nu(\psi^*(\mathrm{d}y)) = \nu_1. \end{aligned}$$

The next generator of Λ_C will be equal to the valuation $\nu(\psi^*(\omega_1^{(i)}))$ where

$$\nu(\psi^*(\omega_1^{(i)})) \notin (\Gamma_C + \nu_0) \cup (\Gamma_C + \nu_1) \subset \Gamma_C$$

for $i \in \{1, 2\}$ and

$$\begin{aligned} \omega_1^{(1)} &= \nu_0 x \mathrm{d}y - \nu_1 y \mathrm{d}x + h_{11}\mathrm{d}x + h_{12}\mathrm{d}y \\ \omega_1^{(2)} &= \nu_0 y^{\nu_0 - 1}\mathrm{d}y - \nu_1 x^{\nu_1 - 1}\mathrm{d}x + h_{21}\mathrm{d}x + h_{22}\mathrm{d}y. \end{aligned}$$

The function $h_{ij} \in \mathbb{C}\{x, y\}$ will satisfy furthermore

$$\begin{aligned} \nu(\psi^*(h_{11}\mathrm{d}x + h_{12}\mathrm{d}y)) &\geq \nu(\psi^*(\nu_0 x \mathrm{d}y - \nu_1 y \mathrm{d}x)) \\ \nu(\psi^*(h_{21}\mathrm{d}x + h_{22}\mathrm{d}y)) &\geq \nu(\psi^*(\nu_0 y^{\nu_0 - 1}\mathrm{d}y - \nu_1 x^{\nu_1 - 1}\mathrm{d}x)). \end{aligned}$$

The valuation of $\psi^*(\nu_0 x \mathrm{d}y - \nu_1 y \mathrm{d}x)$ is equal to

$$(\nu_0 - k + 1)\nu_1 - (k - 1)\nu_0$$

and does not belong to $(\Gamma_C + \nu_0) \cup (\Gamma_C + \nu_1)$. Moreover, we get

$$\nu(\psi^*(\nu_0 y^{\nu_0 - 1}\mathrm{d}y - \nu_1 x^{\nu_1 - 1}\mathrm{d}x)) > \nu_0 \nu_1.$$

Therefore, we observe that

$$\nu(\psi^*(\omega_1^{(2)})) \in (\Gamma_C + \nu_0) \cup (\Gamma_C + \nu_1)$$

for any $h_{21}, h_{22} \in \mathbb{C}\{x, y\}$. Thus, denoting $\omega_1 = \nu_0 x \mathrm{d}y - \nu_1 y \mathrm{d}x$, we obtain one more minimal generator for Λ_C as

$$\lambda_1 = \nu(\psi^*(\omega_1)) = \gamma + \nu_0 = (\nu_0 - k + 1)\nu_1 - (k - 1)\nu_0.$$

Beyond λ_1 the next possible minimal generator for Λ_C is obtained considering

$$\begin{aligned} \omega_2^{(1)} &= \nu_1 x^{k-1}\omega_1 + \nu_0(\gamma - \nu_1) y^{\nu_0 - k}\mathrm{d}y + h_{11}\mathrm{d}x + h_{12}\mathrm{d}y + h_{13}\omega_1 \\ \omega_2^{(2)} &= y^{k-1}\omega_1 + (\gamma - \nu_1) x^{\nu_1 - k}\mathrm{d}x + h_{21}\mathrm{d}x + h_{22}\mathrm{d}y + h_{23}\omega_1 \end{aligned}$$

with $h_{ij} \in \mathbb{C}\{x, y\}$ and

$$\nu(\psi^*(h_{11}\mathrm{d}x + h_{12}\mathrm{d}y + h_{13}\omega_1)) \geq \nu(\psi^*(\nu_1 x^{k-1}\omega_1 + \nu_0(\gamma - \nu_1)y^{\nu_0-k}\mathrm{d}y))$$
$$\nu(\psi^*(h_{21}\mathrm{d}x + h_{22}\mathrm{d}y + h_{23}\omega_1)) \geq \nu(\psi^*(y^{k-1}\omega_1 + (\gamma - \nu_1)x^{\nu_1-k}\mathrm{d}x)).$$

Notice that $\nu_1 x^{k-1}\omega_1 + \nu_0(\gamma - \nu_1)y^{\nu_0-k}\mathrm{d}y$ is precisely the 1-form ω given in (7). This implies that $\psi^*(\omega) = 0$; so $\omega_2^{(1)}$ does not produce any new minimal generator for Λ_C.
On the other hand, we remark that any integer n such that

$$n \geq \nu(\omega_2^{(2)}) > (\nu_1 - k + 1)\nu_0$$

belongs to $\bigcup_{i=-1}^{1}(\Gamma_C + \lambda_i)$. Indeed, any integer n can be uniquely expressed as $n = \alpha\nu_1 + \beta\nu_0$ with $0 \leq \alpha < \nu_0$ and $\beta \in \mathbb{Z}$. If $\beta \geq 0$ then n belongs to $(\Gamma_C + \nu_0) \cup (\Gamma_C + \nu_1)$. If $\beta < 0$ then the condition $n > (\nu_1 - k + 1)\nu_0$ implies $\alpha \leq \nu_0 - k$ and $k - 1 + \beta > 1$. Consequently, there exists $\delta \in \mathbb{N}$ such that

$$\begin{aligned} n &= \alpha\nu_1 + \beta\nu_0 \\ &= \delta\nu_1 + (k - 1 + \beta)\nu_0 + (\nu_0 - k + 1)\nu_1 - (k - 1)\nu_0 \in \Gamma_C + \lambda_1. \end{aligned}$$

So, $\omega_2^{(2)}$ does not produce any new minimal generator for Λ_C and we conclude that the minimal generators for Λ_C are

$$\{\lambda_{-1} = \nu_0, \quad \lambda_0 = \nu_1, \quad \lambda_1 = \gamma + \nu_0 = (\nu_0 - k + 1)\nu_1 - (k - 1)\nu_0\}.$$

Computing the integers introduced at (6) we obtain $t_0 = \nu_1$ and

$$\begin{array}{ll} u_1^0 = \nu_0 + \nu_1 & u_1^1 = \nu_0\nu_1 \\ t_1^0 = t_1 = \nu_0 + \nu_1 & t_1^1 = \tilde{t}_1 = \nu_0\nu_1 \\ & \\ u_2^0 = (\nu_0 - k + 1)\nu_1 & u_2^1 = (\nu_1 - k + 1)\nu_0 \\ t_2^0 = t_2 = k\nu_0 + \nu_1 & t_2^1 = \tilde{t}_2 = k\nu_1 + \nu_0. \end{array}$$

Therefore, we get $\nu_D(\omega) = k\nu_0 + \nu_1 = t_2$. By Theorem 2.1, there exists a differential 1-form $\tilde{\omega}$ with $\nu_D(\tilde{\omega}) = \tilde{t}_2 = k\nu_1 + \nu_0$ such that $\{\omega, \tilde{\omega}\}$ is a Saito basis for C.
Suppose that in the Taylor expansion of $\tilde{\omega}$ there exists a term of the form

$$a_{ij}x^i y^j \mathrm{d}x \quad \text{or} \quad a_{ij}x^i y^j \mathrm{d}y$$

such that $a_{ij} \neq 0$ and $\nu\left(x^i y^j\right) = i + j \leq k - 1$. Then, we can see that

$$\max\{\nu_D\left(x^i y^j \mathrm{d}x\right), \nu_D\left(x^i y^j \mathrm{d}y\right)\} \leq (i + j + 1)\,\nu_1 \leq k\nu_1 < k\nu_1 + \nu_0 = \tilde{t}_2$$

which is impossible. Therefore, we get $\nu(\tilde{\omega}) \geq k$ and

$$\mathfrak{s}(C) = \nu(\omega) = k.$$

□

2.2 Saito numbers in the topological class Top$(C_{N,1,1})$

Among the curves C_{N,ν_0,ν_1}, the curves $C_{N,1,1}$ with $N \geq 3$ are the only radial ones, that is, the maximum Saito number is generically realized by a dicritical differential $1-$form [11].
Denoting by $\mathfrak{M}_N$ the number $\max_{C \in \mathrm{Top}(C_{N,1,1})} \mathfrak{s}(C)$, we get

$$\mathfrak{M}_N = \begin{cases} 0 & N = 1 \\ 1 & N = 2, 3, 4 \\ \frac{N-1}{2} & N \geq 5 \text{ and } N \text{ odd} \\ \frac{N}{2} - 1 & N \geq 5 \text{ and } N \text{ even.} \end{cases}$$

Proposition 2.4. *For any $N > 1$ and any k in $\{1, \ldots, \mathfrak{M}_N\}$, there exists C in* Top $(C_{N,1,1})$ *such that*

$$\mathfrak{s}(C) = k.$$

Proof. If $k = \mathfrak{M}_N$ or if $N = 2, 3$ or 4 the statement is obvious. Suppose that $N \geq 5$ and let k be an integer in $\{1, \ldots, \mathfrak{M}_N - 1\}$. Let us consider the differential $1-$forms

$$\omega_1 = y\mathrm{d}x - x\mathrm{d}y, \ \omega_2 = \omega_1 + \mathrm{d}f$$

where f is the function

$$f = x^{N-2k+2} + y^{N-2k+2}.$$

The couple $\{\omega_1, \omega_2\}$ is a Saito basis for $\{f = 0\}$ since it satisfies the Saito criterion,

$$\omega_1 \wedge \omega_2 = (N - 2k + 2) f \mathrm{d}x \wedge \mathrm{d}y.$$

Since $N \geq 5$, the multiplicity of $\mathrm{d}f$ is bigger than 2 and after one blowing-up, both ω_i's are dicritical. Let us consider $l_1^{(i)} = 0, \ldots, l_{k-1}^{(i)} = 0$ be $k-1$ smooth and transversal curves tangent to ω_i. We suppose moreover that these curves are transversal at a whole and transversal to $f = 0$. Now writing

$$\prod_{i=1}^{k-1} l_i^{(2)} \omega_1 \wedge \prod_{i=1}^{k-1} l_i^{(1)} \omega_2 = (N - 2k + 2) \prod_{i=1}^{k-1} l_i^{(2)} l_i^{(1)} f \mathrm{d}x \wedge \mathrm{d}y \tag{9}$$

yields a Saito relation for the curve $C = \left\{ \prod_{i=1}^{k-1} l_i^{(2)} l_i^{(1)} f = 0 \right\}$, a curve which consists in the union of N smooth and transversal curves. Thus, C is equisingular to $C_{N,1,1}$ and, following (9), one has

$$\mathfrak{s}(C) = k.$$

□

2.3 Saito numbers in the topological class Top(C_{N,ν_0,ν_1}).

Let C_{N,ν_0,ν_1} be the curve given by the equation

$$f_{N,\nu_0,\nu_1} = y^{N\nu_0} - x^{N\nu_1} = 0$$

where $N > 0$ and $\nu_0 \leq \nu_1$ are relatively prime. Considering the divisorial valuation defined in (5) we get the following result.

Lemma 2.5. *Let ω be the $1-$form be defined by $\omega = \nu_1 y\mathrm{d}x - \nu_0 x\mathrm{d}y$. Suppose that $h = x^i y^j$. If $\nu_D(h) \leq N\nu_0\nu_1 - 1 - \nu_0 - \nu_1$ then $\mathrm{d}f_{N,\nu_0,\nu_1} + h\omega$ is dicritical along the central component of the desingularization process of C_{N,ν_0,ν_1}.*

Proof. Let u, v such that $u\nu_0 - v\nu_1 = 1$. The local coordinates in the neighborhood of the central component D can be written

$$x = x_D^{\nu_0 - v} y_D^{\nu_0} \quad y = x_D^{\nu_1 - u} y_D^{\nu_1}.$$

$y_D = 0$ being a local equation for D [13]. Computing the pullback of $\mathrm{d}f_{N,\nu_0,\nu_1} + h\omega$ in these coordinates yields

$$\begin{gathered} y_D^{N\nu_0\nu_1 - 1}\left(y_D(\cdots)\mathrm{d}x_D + x_D(\cdots)\mathrm{d}y_D\right) \\ +h\left(x_D^{\nu_0-v}y_D^{\nu_0}, x_D^{\nu_1-u}y_D^{\nu_1}\right)y_D^{\nu_0+\nu_1}x_D^{\nu_0-v+\nu_1-u-1}\mathrm{d}x_D. \end{gathered}$$

The hypothesis of the lemma ensures that the above $1-$form can be exactly divided by $y_D^{\nu_D(h)+\nu_0+\nu_1}$ and the $1-$form

$$\begin{gathered} y_D^{N\nu_0\nu_1 - 1 - \nu_0 - \nu_1 - \nu_D(h)}\left(y_D(\cdots)\mathrm{d}x_D + x_D(\cdots)\mathrm{d}y_D\right) \\ +\frac{h\left(x_D^{\nu_0-v}y_D^{\nu_0}, x_D^{\nu_1-u}y_D^{\nu_1}\right)}{y_D^{\nu_D(h)}}x_D^{\nu_0-v+\nu_1-u-1}\mathrm{d}x_D \end{gathered}$$

is generically transverse to $y_D = 0$. □

We are now in position to prove Theorem 2 stated in the introduction.

Theorem 3. *Let C_{N,ν_0,ν_1} be the curve defined by*

$$y^{N\nu_0} - x^{N\nu_1} = 0$$

where $N > 0$ and $\nu_0 \leq \nu_1$ are relatively prime with $N\nu_0 > 1$. Then for any $k \in \{1, \ldots, \left[\frac{N\nu_0}{2}\right]\}$ there exist $C \in \mathrm{Top}\left(C_{N,\nu_0,\nu_1}\right)$ such that

$$\mathfrak{s}(C) = k.$$

Proof. For $N = 1$ the result follows from Proposition 2.3. Suppose $N \geq 3$ and the result is true for any curve C_{N',ν_0,ν_1} with $1 \leq N' < N$. Let $k \in \{1, \ldots, \left[\frac{N\nu_0}{2}\right]\}$. For now, suppose that $k \geq \nu_0 + 1$. Let us consider $C' \in \mathrm{Top}(C_{N-2}, \nu_0, \nu_1)$ such that $\mathfrak{s}(C') = k - \nu_0$ and let $\{\omega, \eta\}$ be a Saito basis for C'. According to the Saito criterion, if f is a reduced equation of C' then

$$\omega \wedge \eta = uf \mathrm{d}x \wedge \mathrm{d}y$$

where $u \in \mathbb{C}\{x, y\}$ is a unit. Both $1-$forms ω and η can be supposed of multiplicity $k-\nu_0$ and are dicritical along the central component of the desingularization process of C', that is the same of C_{N-2}. Consider f_ω and f_η two reduced equations of analytical curves invariants for respectively ω and η such that after desingularization, $f_\omega = 0$ and $f_\eta = 0$ are attached to the common central component, transversal the one to the other and both transversal to C_{N-2}. By construction, these two curves are equisingular to $y^{\nu_0} - x^{\nu_1} = 0$. Now, we can write

$$f_\eta \omega \wedge f_\omega \eta = u f_\omega f_\eta f \mathrm{d}x \wedge \mathrm{d}y.$$

Since $f_\eta \omega$ and $f_\omega \eta$ leave both invariant the curve $f f_\eta f_\omega = 0$, the relation above is the Saito criterion for $f_\omega f_\eta f = 0$. Setting $C = \{f_\omega f_\eta f = 0\} \in \mathrm{Top}\left(C_{N,\nu_0,\nu_0}\right)$, we obtain

$$\mathfrak{s}(C) = \min\left\{\nu\left(f_\eta \omega\right), \nu\left(f_\omega \eta\right)\right\} = \nu_0 + k - \nu_0 = k.$$

Suppose now that, $k \leq \left[\frac{\nu_0}{2}\right]$. The initial assumptions ensure that $\nu_0 \geq 2$. Applying the result for $N = 1$ yields a curve $C' = \{f = 0\} \in \mathrm{Top}\left(C_{1,\nu_0,\nu_1}\right)$ such that $\mathfrak{s}(C') = k$ and a Saito basis $\{\omega, \eta\}$ with $\nu(\omega) = k \leq \nu(\eta)$ for C' such that the $1-$form ω is dicritical along the central component of the desingularization process of C', that is the same of C_{1,ν_0,ν_1}. Choose $N-1$ curves $f_1 = 0, \ldots, f_{N-1} = 0$ invariant for ω attached to the central component and transversal to C'. From the Saito criterion $\omega \wedge \eta = uf \mathrm{d}x \wedge \mathrm{d}y$ for C', we can write

$$\omega \wedge f_1 \cdots f_{N-1} \eta = u f_1 \cdots f_{N-1} f \mathrm{d}x \wedge \mathrm{d}y$$

which is the Saito criterion for $f_1 \cdots f_{N-1} f = 0$. As a consequence, setting $C = \{f_1 \cdots f_{N-1} f = 0\}$, we get $C \in \mathrm{Top}\left(C_N, \nu_0, \nu_1\right)$ and

$$\mathfrak{s}(C) = \min\left\{\nu(\omega), \nu\left(f_1 \cdots f_{N-1} \eta\right)\right\} = k.$$

Finally, suppose that $k \in \left\{\left[\frac{\nu_0}{2}\right] + 1, \ldots, \nu_0\right\}$. Consider the curve $C' = \{f = 0\}$ where $f = y^{2\nu_0} - x^{2\nu_1}$ and its Saito basis given by

$$\{\omega = \nu_1 y \mathrm{d}x - \nu_0 x \mathrm{d}y, \mathrm{d}f\}.$$

For any function h, we can write

$$\omega \wedge (\mathrm{d}f + h\omega) = 2\nu_0 \nu_1 f \mathrm{d}x \wedge \mathrm{d}y.$$

We choose $h = x^{k-1}$. In that case,

$$\nu_D(h) = \nu_0(k-1) \leq \nu_0(\nu_0 - 1) \leq 2\nu_1\nu_0 - 1 - \nu_1 - \nu_0.$$

Following Lemma 2.5, the $1-$form $\mathrm{d}f + h\omega$ is still dicritical along the central component D of the desingularization process of C'. We fix $N-2$ curves $f_1 = 0,\ f_2 =$

$0, \ldots, f_{N-2} = 0$ invarant for $\mathrm{d}f + h\omega$ attached to the central component D and transversal to C'. The Saito criterion leads to

$$f_1 \cdots f_{N-2}\omega \wedge (\mathrm{d}f + h\omega) = 2\nu_0\nu_1 f_1 \cdots f_{N-2} f \mathrm{d}x \wedge \mathrm{d}y$$

which is the Saito basis for $C = \{f_1 \cdots f_{N-2} f = 0\}$. Thus

$$\begin{aligned}\mathfrak{s}(C) &= \min\left\{\nu\left(f_1 \cdots f_{N-2}\omega\right), \nu\left(\mathrm{d}f + h\omega\right)\right\} \\ &= \min\left\{(N-2)\,\nu_0 + 1, 2\nu_0 - 1, k\right\} = k\end{aligned}$$

since $N \geq 3$.
It remains to treat the case $N = 2$. Suppose first $\nu_0 \geq 2$. If $k \leq \left[\frac{\nu_0}{2}\right]$ the same argument as before ensures the property. Suppose that $k \in \left\{\left[\frac{\nu_0}{2}\right] + 1, \ldots, \nu_0\right\}$. Notice that according to [9], the property is true for $k = \nu_0$. Thus we can suppose $k \leq \nu_0 - 1$. Consider the curve $C' = \{f = y^{\nu_0} - x^{\nu_1} = 0\}$ and its Saito basis given by

$$\{\omega = \nu_1 y \mathrm{d}x - \nu_0 x \mathrm{d}y, \mathrm{d}f\}.$$

For any function h, we get $\omega \wedge (\mathrm{d}f + h\omega) = \nu_0\nu_1 f \mathrm{d}x \wedge \mathrm{d}y$. In particular, for $h = x^{k-1}$ we have

$$\nu_D\left(h\right) = \nu_0\left(k-1\right) \leq \nu_0\left(\nu_0 - 2\right) \leq \nu_1\nu_0 - 1 - \nu_1 - \nu_0$$

which is true under the assumption the assumption that $\nu_0 \geq 2$. Following Lemma 2.5, the $1-$form $\mathrm{d}f + h\omega$ is dicritical along the central component D of the desingularization process of C'. We choose one curve $f_1 = 0$ invariant for $\mathrm{d}f + h\omega$ attached to D and transversal to C'. The Saito criterion leads to

$$f_1\omega \wedge (\mathrm{d}f + h\omega) = \nu_0\nu_1 f_1 f \mathrm{d}x \wedge \mathrm{d}y$$

which $\{f_1\omega, \mathrm{d}f + h\omega\}$ is the Saito basis for $C = \{f_1 f = 0\}$. Expanding the expression of the form $\mathrm{d}f + h\omega$, we get

$$\nu\left(\mathrm{d}f + h\omega\right) = k$$

and thus

$$\begin{aligned}\mathfrak{s}(C) &= \min\left\{\nu\left(f_1\omega\right), \nu\left(\mathrm{d}f + h\omega\right)\right\} \\ &= \min\left\{\nu_0 + 1, k\right\} = k.\end{aligned}$$

For $N = 2$ and $\nu_0 = 1$, then the set $\{1, \ldots, \left[\frac{N\nu_0}{2}\right]\}$ reduces to integer 1 and the property is clear regarless the value of ν_1, which concludes the proof of the theorem. □

References

[1] P. Fortuny Ayuso and J. Ribón. Construction of a basis of Kähler differentials for generic plane branches. *arXiv*, (2405.09684), 2024.

[2] C. Camacho, A. Lins Neto, and P. Sad. Topological invariants and equidesingularization for holomorphic vector fields. *J. Differential Geom.*, 20(1):143–174, 1984.

[3] F. Cano and N. Corral. Absolutely dicritical foliations. *Int. Math. Res. Not. IMRN*, 8:1926–1934, 2011.

[4] F. Cano, N. Corral, and D. Senovilla-Sanz. Computing a Saito basis from a standard basis. *arXiv*, (2404.00316), 2024.

[5] J. Cano, P. Fortuny Ayuso, and J. Ribón. The local Poincaré problem for irreducible branches. *Rev. Mat. Iberoam.*, 37(6):2229–2244, 2021.

[6] N. Corral, M. E. Hernandes, and M. E. Rodrigues Hernandes. Dicritical foliation and semiroots of plane branches. *ArXiv*, (2304.01047), 2024.

[7] C. Delorme. Sur les modules des singularités des courbes planes. *Bulletin de la Société Mathématique de France*, 106:417–446, 1978.

[8] Y. Genzmer. Dimension of the moduli space of a germ of curve in $\mathbb{C}^2$. *Int. Math. Res. Not. IMRN*, 5:3805–3859, 2022.

[9] Y. Genzmer. Number of moduli for a union of smooth curves in $(\mathbb{C}^2, 0)$. *J. Symb. Comput.*, 113:148–170, 2022.

[10] Y. Genzmer. The Saito vector field of a germ of complex plane curve. *arXiv*, (2403.06587), 2024.

[11] Y. Genzmer. The Saito module and the moduli of a germ of curve in $\left(\mathbb{C}^2, 0\right)$. *Annales de l'Institut Fourier*, 2024.

[12] Y. Genzmer and M. E. Hernandes. On the Saito basis and the Tjurina number for plane branches. *Trans. Amer. Math. Soc.*, 373(5):3693–3707, 2020.

[13] Y. Genzmer and E. Paul. Moduli spaces for topologically quasi-homogeneous functions. *Journal of Singularities*, (14):3–33, 2016.

[14] H. Grauert. Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen. *Inst. Hautes Études Sci. Publ. Math.*, 5:233–292, 1960.

[15] A. Hefez and M.E. Hernandes. Standard bases for local rings of branches and their modules of differentials. *Journal of Symbolic Computation*, 42(1):178–191, 2007. Effective Methods in Algebraic Geometry (MEGA 2005).

[16] C. Hertling. Formules pour la multiplicité et le nombre de Milnor d'un feuilletage sur $(\mathbb{C}^2, 0)$. *Ann. Fac. Sci. Toulouse Math. (6)*, 9(4):655–670, 2000.

[17] A. Lins Neto. Construction of singular holomorphic vector fields and foliations in dimension two. *J. Differential Geom.*, 26(1):1–31, 1987.

[18] K. Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.

[19] M. Toma. A short proof of a theorem of Camacho and Sad. *Enseign. Math. (2)*, 45(3-4):311–316, 1999.

[20] O. Zariski. On the topology of algebroid singularities. *Amer. J. Math.*, 54(3):453–465, 1932.

[21] O. Zariski. *Le problème des modules pour les branches planes*. Hermann, Paris, second edition, 1986. Course given at the Centre de Mathématiques de l'École Polytechnique, Paris, October–November 1973, With an appendix by Bernard Teissier.

Yohann Genzmer
Université Paul Sabatier
Institut de Mathématiques de Toulouse
118 route de Narbonne
F-31062, Toulouse Cedex 9
France.
yohann.genzmer@math.univ-toulouse.fr

Marcelo Escudeiro Hernandes
Universidade Estadual de Maringá
Departamento de Matemática
Av. Colombo 5790
Maringá-PR 87020-900
Brazil.
mehernandes@uem.br