A large deviation principle for empirical measures on Polish spaces

David García-Zelada

Aix-Marseille Université

November 23, 2020
Journées doctorants du GDR MEGA
Outline

1 Setting and question

2 Idea of the proof and theorem

3 Examples
Table of Contents

1 Setting and question
2 Idea of the proof and theorem
3 Examples
Guiding example

Suppose \((Z_1^{(n)}, \ldots, Z_n^{(n)}) \sim \mathbb{P}_n\) where

\[
d\mathbb{P}_n(z_1, \ldots, z_n) = \frac{1}{Z_n} \prod_{i<j} \|z_i - z_j\|^2 e^{-n \sum_{i=1}^n \|z_i\|^2} d\ell_{\mathbb{C}^n}(z_1, \ldots, z_n).
\]

Eigenvalues of a Gaussian random matrix, *Ginibre matrix*. 
Question

Study the limit behavior of \( \hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{Z_i^{(n)}} \). More precisely, find an LDP on the set of probability measures \( \mathcal{P}(\mathbb{C}) \).

[Hi, and Petz (1998)]
LDP: Find $\nu_n$ that goes to $\infty$ and $I : \mathcal{P}(\mathbb{C}) \to [0, \infty]$ such that

$$\mathbb{P}(\hat{\mu}_n \simeq \nu) = e^{-\nu_n(I(\nu)+o(1))}$$

for every $\nu \in \mathcal{P}(\mathbb{C})$. Equivalently,

$$\frac{1}{\nu_n} \log \mathbb{P}(\hat{\mu}_n \simeq \nu) = -I(\nu) + o(1).$$

A bit more precisely, for any measurable set $A \subset \mathcal{P}(\mathbb{C})$,

$$- \inf_{\nu \in A^c} I(\nu) \leq \lim_{n \to \infty} \frac{1}{\nu_n} \log \mathbb{P}(\hat{\mu}_n \in A) \leq - \inf_{\nu \in \bar{A}} I(\nu).$$
Laplace principle

LDP regularized version:

$$\lim_{n \to \infty} \frac{1}{\nu_n} \log \mathbb{E} \left[ e^{-\nu_n f(\hat{\mu}_n)} \right] = -\inf_{\nu \in \mathcal{P}(\mathbb{C})} \left\{ f(\nu) + l(\nu) \right\}$$

for every $f : \mathcal{P}(\mathbb{C}) \to \mathbb{R}$ continuous and bounded.
Restatement of the problem

Define

- \( G(z, w) = -2 \log \| z - w \| + \| z \|^2 + \| w \|^2 \);
- \( d\sigma(z) = \frac{e^{-\| z \|^2}}{\pi} d\ell_C(z) \).

We have

\[
\prod_{i<j}^{n} \| z_i - z_j \|^2 e^{-n \sum_{i=1}^{n} \| z_i \|^2} d\ell_C^n(z_1, \ldots, z_n)
\]

\[
= \pi^n \exp \left( - \sum_{i<j}^{n} G(z_i, z_j) \right) d\sigma^\otimes_n(z_1, \ldots, z_n).
\]
If we define

\[ H_n(z_1, \ldots, z_n) = \frac{1}{n^2} \sum_{i<j}^n G(z_i, z_j), \]

interpreted as the total (potential) energy of \( n \) particles, then

\[ d\mathbb{P}_n = \frac{1}{\tilde{Z}_n} \exp \left(-n^2 H_n\right) d\sigma^\otimes n. \]

How general can \( H_n \) be?
General setting

- \( M \) Polish space;
- \( \sigma \) probability measure on \( M \);
- \( H_n : M^n \to (-\infty, \infty] \) measurable bounded from below;
- \( \{\beta_n\}_n \) sequence of positive numbers.

Let \( \gamma_n \) be the finite measure given by

\[
\mathrm{d}\gamma_n = \exp(-n\beta_n H_n) \, \mathrm{d}\sigma^{\otimes n}.
\]

How the limiting behavior of \( \gamma_n \) depends on the limiting behavior of \( H_n \)?
Suppose that, for every $\mu \in \mathcal{P}(M)$, the following limit exists:

$$H(\mu) = \lim_{n \to \infty} \int_{M^n} H_n d\mu^\otimes n,$$

and define the relative entropy

$$\text{Ent}_\sigma(\mu) = \int_M \left( \frac{d\mu}{d\sigma} \right) \log \left( \frac{d\mu}{d\sigma} \right) d\sigma.$$

In the Ginibre case,

$$H(\mu) = \frac{1}{2} \int_{\mathbb{C} \times \mathbb{C}} G(z, w) d\mu(z) d\mu(w)$$

$$= - \int_{\mathbb{C} \times \mathbb{C}} \log ||z - w|| d\mu(z) d\mu(w) + \int_{\mathbb{C}} ||z||^2 d\mu(z).$$
For \( x = (x_1, \ldots, x_n) \) define

\[
\hat{\mu}_x = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}.
\]

**Goal : Laplace principle**

If \( \beta_n \to \beta \in (0, \infty] \), for \( f : \mathcal{P}(M) \to \mathbb{R} \) bounded continuous,

\[
\lim_{n \to \infty} \frac{1}{n \beta_n} \log \int_{M^n} e^{-n\beta_n f(\hat{\mu}_x)} d\gamma_n(x) = - \inf_{\mu \in \mathcal{P}(M)} \left\{ f(\mu) + H(\mu) + \frac{1}{\beta} \text{Ent}_\sigma(\mu) \right\}.
\]
<table>
<thead>
<tr>
<th>1</th>
<th>Setting and question</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Idea of the proof and theorem</td>
</tr>
<tr>
<td>3</td>
<td>Examples</td>
</tr>
</tbody>
</table>
Main step: *Dupuis* and *Ellis* approach to LDP

**Lemma (Legendre transform of the entropy)**

- $E$ measurable space,
- $\nu \in \mathcal{P}(E)$ and
- $g : E \to (-\infty, \infty]$ measurable bounded from below.

Then

$$\log \int_{E} e^{-g} d\nu = -\inf_{\tau \in \mathcal{P}(E)} \left\{ \int_{E} g d\tau + \text{Ent}_{\nu}(\tau) \right\}. $$

$$\frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n \left( f(\hat{\mu}_x) + H_n(x) \right)} d\sigma \otimes_n (x)$$

$$= -\inf_{\tau \in \mathcal{P}(M^n)} \left\{ \int_{M^n} \left( f(\hat{\mu}_x) + H_n(x) \right) d\tau(x) + \frac{1}{n\beta_n} \text{Ent}_{\sigma \otimes_n}(\tau) \right\}. $$
Setting and question
Idea of the proof and theorem
Examples

Laplace principle goal II

New goal: Convergence of the infima

If $\beta_n \to \beta \in (0, \infty]$, for $f : \mathcal{P}(M) \to \mathbb{R}$ bounded continuous,

$$
\inf_{\tau \in \mathcal{P}(M_n)} \left\{ \int_{M^n} \left( f(\hat{\mu}_x) + H_n(x) \right) d\tau(x) + \frac{1}{n\beta_n} \text{Ent}_{\sigma \otimes_n}(\tau) \right\}
$$

$$
\longrightarrow_{n \to \infty} \inf_{\mu \in \mathcal{P}(M)} \left\{ f(\mu) + H(\mu) + \frac{1}{\beta} \text{Ent}_\sigma(\mu) \right\}.
$$
Notion of limit

- Sequence $\{H_n\}_n$ uniformly bounded from below,
- $H : \mathcal{P}(M) \to (-\infty, \infty]$.

**Definition (Macroscopic limit)**

$H$ is the macroscopic limit of $H_n$ if

- $\forall \mu \in \mathcal{P}(M)$
  $$\lim_{n \to \infty} \int_{M^n} H_n \, d\mu^\otimes_n = H(\mu)$$
  and

- whenever $\frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \to \mu$
  $$\liminf_{n \to \infty} H_n(x_1, \ldots, x_n) \geq H(\mu).$$

This notion of convergence suffices!
A Laplace principle

**Theorem (Laplace principle for positive temperature)**

Suppose

- $H$ is the macroscopic limit of $H_n$ and
- $\beta_n$ converges to some $\beta \in (0, \infty)$.

Then, for every bounded continuous $f : \mathcal{P}(M) \to \mathbb{R}$,

$$
\lim_{n \to \infty} \frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f(\hat{\mu}_x)} \, d\gamma_n(x) = -\inf_{\mu \in \mathcal{P}(M)} \left\{ f(\mu) + H(\mu) + \frac{1}{\beta} \text{Ent}_\sigma(\mu) \right\}.
$$

If $\iota = \inf \left( H + \frac{1}{\beta} \text{Ent}_\sigma \right) < \infty$, it implies an LDP with

rate function $H + \frac{1}{\beta} \text{Ent}_\sigma - \iota$ and speed $n\beta_n$.
Suppose that

\[(X_1^{(n)}, \ldots, X_n^{(n)}) \sim \frac{\gamma_n}{\gamma_n(M^n)}.\]

Under the conditions of the preceding theorem:

**Theorem (Limit of empirical measures)**

*If* \(H + \frac{1}{\beta} \operatorname{Ent}_\sigma\) *has a unique minimizer* \(\mu_{\text{eq}}\),

\[
\frac{1}{n} \sum_{i=1}^{n} \delta_{X_i^{(n)}} \xrightarrow{\text{a.s.}} \mu_{\text{eq}}.
\]
What happens when $\beta_n \to \infty$? Two more conditions.

- **$\{H_n\}_n$ confining**: Let $x_n = (x_1, \ldots, x_n)$.

\[
\liminf_{n \to \infty} H_n(x_n) < \infty \implies \{\hat{\mu}_{x_n}\}_n \text{ is precompact in } \mathcal{P}(M).
\]

- **$H$ regular**: If $H(\mu) < \infty$, there exists $\mu_n \to \mu$ such that

\[
\forall n, \text{Ent}_\sigma(\mu_n) < \infty \quad \text{and} \quad \lim_{n \to \infty} H(\mu_n) = H(\mu).
\]
Another Laplace principle

**Theorem (Laplace principle for zero temperature)**

**Suppose**
- $H$ is the macroscopic limit of $H_n$,
- $\beta_n$ tends to infinity,
- $\{H_n\}_n$ is confining and
- $H$ is regular.

*Then, for every bounded continuous $f : \mathcal{P}(M) \to \mathbb{R}$,*

$$\lim_{n \to \infty} \frac{1}{n \beta_n} \log \int_{M^n} e^{-n \beta_n f(\hat{\mu}_x)} d\gamma_n(x)$$

$$= - \inf_{\mu \in \mathcal{P}(M)} \{ f(\mu) + H(\mu) \}.$$
Zero temperature case limit

Suppose that

\[(X_1^{(n)}, \ldots, X_n^{(n)}) \sim \frac{\gamma_n}{\gamma_n(M^n)}.\]

Under the conditions of the preceding theorem:

**Theorem (Limit of empirical measures)**

*If $H$ has a unique minimizer $\mu_{eq}$,*

\[\frac{1}{n} \sum_{i=1}^{n} \delta_{X_i^{(n)}} \xrightarrow{\text{a.s.}} n \to \infty \mu_{eq}.\]
Theorem (Γ-convergence)

Suppose

- $H$ is the macroscopic limit of $H_n$ and
- $\{H_n\}_n$ is confining.

Then, for every bounded continuous function $f : \mathcal{P}(M) \to \mathbb{R}$

$$\lim_{n \to \infty} \inf_{x_n \in M^n} \{f(\hat{\mu}_{x_n}) + H_n(x_n)\}$$

$$= \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + H(\mu)\}.$$
Deterministic case limit

Suppose that

$$\lim_{n \to \infty} \left[ H_n(X_1^{(n)}, \ldots, X_n^{(n)}) - \inf H_n \right] = 0.$$ 

Under the conditions of the preceding theorem:

**Theorem (Limit of empirical measures)**

If $H$ has a unique minimizer $\mu_{eq}$,

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{X_i^{(n)}} \xrightarrow{n \to \infty} \mu_{eq}.$$
Two-body interaction

Suppose $H_n$ is given by

$$H_n(x_1, \ldots, x_n) = \frac{1}{n^2} \sum_{i<j} G(x_i, x_j)$$

for some $G : M \times M \to (-\infty, \infty]$.

- $H(\mu) = \frac{1}{2} \int_{M \times M} G(x, y)d\mu(x)d\mu(y)$.
- $G$ bounded from below $\implies \{H_n\}_n$ unif. bounded from below.
- $G$ lower semicont. $\implies H$ is the macroscopic limit of $H_n$.
- $G(x, y) \to \infty$ when $x, y \to \infty \implies \{H_n\}_n$ confining.
- $H$ regular : enough to ask $\mu_n \ll \sigma$ instead of $\text{Ent}_\sigma(\mu_n) < \infty$. 
k-body interaction

Let $G : M^k \to (-\infty, \infty]$ be lower semicontinuous and bounded from below.

Then

$$H_n(x_1, \ldots, x_n) = \frac{1}{n^k} \sum_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}} G(x_{i_1}, \ldots, x_{i_k}).$$

Macroscopic limit

$$H(\mu) = \frac{1}{k!} \int_{M^k} G \, d\mu \otimes_k.$$
Random polynomial energy term

$G : M \times M \to (-\infty, \infty]$ and $\nu \in \mathcal{P}(M)$

$$H_n(x_1, \ldots, x_n) = \frac{n + 1}{n^2} \log \left( \int_{M} e^{-\sum_{i=1}^{n} G(x_i, x)} d\nu(x) \right).$$

This term appears for Gaussian random polynomials!

Under some conditions, the macroscopic limit is

$$H(\mu) = - \inf_{x \in \text{supp} \; \nu} \left\{ \int_{M} G(x, y) d\mu(y) \right\}.$$
Paul Dupuis and Richard S. Ellis. *A weak convergence approach to the theory of large deviations.*

Paul Dupuis, Vaios Laschos and Kavita Ramanan. *Large deviations for configurations generated by Gibbs distributions with energy functionals consisting of singular interaction and weakly confining potentials.*

Robert J. Berman. *On large deviations for Gibbs measures, mean energy and gamma-convergence.*

G-Z. *A large deviation principle for empirical measures on Polish spaces : Application to singular Gibbs measures on manifolds.*
Thank you for your attention!
Concrete examples

**Ginibre ensemble.** \( \{X_{i,j}\}_{i,j \geq 1} \) i.i.d. complex standard Gaussians. Define

\[
\mathbf{X}_n = \left( \frac{X_{i,j}}{\sqrt{n}} \right)_{1 \leq i,j \leq n}.
\]

Law of eigenvalues of \( \mathbf{X}_n : M = \mathbb{C} \) with \( \beta_n = n \),

\[
G(z, w) = -2 \log \|z - w\| + \|z\|^2 + \|w\|^2
\]

and

\[
d\sigma(z) = \frac{e^{-\|z\|^2}}{\pi} d\ell_{\mathbb{C}}(z).
\]

LDP : [Hiai and Petz (1998)].
Spherical ensemble. $\tilde{X}_n \sim X_n$ independent. Define

$$Y_n = X_n \tilde{X}^{-1}_n.$$  

Law of eigenvalues of $Y_n$: $M = \mathbb{C}$ with $\beta_n = n$,

$$G(z, w) = -2 \log |z - w| + \log(1 + |z|^2) + \log(1 + |w|^2)$$

and

$$d\sigma(z) = \frac{1}{\pi(1 + |z|^2)^2} d\ell_{\mathbb{C}}(z).$$

LDP: [Hardy (2012)].
**Gaussian Kac polynomials.** \( \{a_i\}_{i \geq 0} \) i.i.d. complex standard Gaussians. Define 

\[
p_n(z) = \sum_{i=0}^{n} a_i z^i.
\]

Law of zeros of \( p_n \): \( M = \mathbb{C} \) with \( \beta_n = n \),

\[
G(z, w) = -2 \log |z - w| + 2 \log_+ |z| + 2 \log_+ |w|,
\]

\[
d\sigma(z) = \frac{1}{2\pi} \min(1, |z|^{-4})d\ell_{\mathbb{C}}(z),
\]

with the extra term for \( \nu \) the uniform measure on the unit circle. 

LDP: [Zeitouni and Zelditch (2010)].