Superconcentration inequalities for centered Gaussian stationnary processes

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Outline

- What is superconcentration ?

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- Convergence of extremes (Gaussian case).
- Superconcentration inequality for stationary Gaussian sequences.
- Main result (abstract theorem)
- Tools and sketch of the proof.

- What is superconcentration ?

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In fact,

$$\mathrm{Var}(M_n) \leq \frac{C}{\log n}, \quad C > 0$$

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Lot of different models

- Largest eigenvalue in random matrix theory.

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- M random matrix from the GUE, namely

$$\mathbb{P}(dM) = Z_N^{-1} \exp\left(-Tr(M^2)/2\sigma^2\right) dM, \quad \sigma^2 = 1/4N$$

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- In fact, $\mathrm{Var}(\lambda_{\max}) \leq \frac{C}{N^{4/3}}$ [Ledoux-Rider '10].

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- In fact, $\mathrm{Var}(\max_\pi X_\pi) \leq C$ [Bramson-Ding-Zeitouni].

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- Stationary Gaussian sequences.

- What is superconcentration ?
- Convergence of extremes (Gaussian case).

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Proposition [Chatterjee '14]

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- Convergence of extremes (Gaussian case).
- Superconcentration inequality for stationary Gaussian sequences.

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- Reflects asymptotics Gumbel

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- Reflects Gumbel asymptotics.
- Implies $\mathrm{Var}(\max_i X_i) \leq \frac{C}{\log n}$ (optimal).

Tools

Proof ?

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Chatterjee's scheme of proof for the variance at the exponential level.

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Chatterjee's scheme of proof for the variance at the exponential level.

General theorem implies superconcentration inequality for Gaussian stationary sequences.

- What is superconcentration ?
- Convergence of extremes (Gaussian case).
- Superconcentration inequality for stationary Gaussian sequences.
- Main result (abstract theorem)

General theorem [T. '15]

$X = (X_1, \ldots, X_n) \sim \mathcal{N}(0, \Gamma)$ Assume that for some $r_0 \geq 0$, there exists a covering $\mathcal{C}(r_0)$ of $\{1, \ldots, n\}$ verifying :

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Indeed, if $\Gamma_{ij} > 0$ then $i = j$.

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$\mathbb{P}(I = i) = \mathbb{P}(X_i \geq X_j \, \forall j) = 1/n$.Then $\rho(r_0) \leq 1/n$.

General theorem [T. 15']

$X = (X_1, \dots, X_n) \sim \mathcal{N}(0, \Gamma)$ Assume that for some $r_0 \geq 0$, there exists a covering $\mathcal{C}(r_0)$ of $\{1, \dots, n\}$ verifying :

- for all $i, j \in \{1, \dots, n\}$ such that $\Gamma_{ij} \geq r_0$, there exists $D \in \mathcal{C}(r_0)$ such that $i, j \in D$;
- there exists $C \geq 1$ such that, *a.s.*, $\sum_{D \in \mathcal{C}(r_0)} 1_{\{I \in D\}} \leq C$, where $I = \mathrm{argmax}_i X_i$.

Let $\rho(r_0) = \max_{D \in \mathcal{C}(r_0)} \mathbb{P}(I \in D)$. Then, for every $\theta \in \mathbb{R}$,

$$\mathrm{Var}\left(e^{\theta M_n/2}\right) \leq C \, \frac{\theta^2}{4} \left(r_0 + \frac{1}{\log\left(1/\rho(r_0)\right)} \right) \mathbb{E}\left[e^{\theta M_n} \right].$$

Exponential Poincaré inequality

Conclusion ?

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$$\mathrm{Var}\left(\mathrm{e}^{\theta \mathrm{Z}/2}\right) \leq \frac{\theta^2}{4}\mathrm{K}\mathbb{E}\left[\mathrm{e}^{\theta \mathrm{Z}}\right], \quad \theta \in \mathbb{R}$$

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implies

$$\mathbb{P}\left(|Z - \mathbb{E}[Z]| \geq t\right) \leq 6e^{-ct/\sqrt{K}}, \quad t > 0$$

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(Stationary case : $K_{r_0} = 1/\log n$)

Recall

Theorem [T. '15]

$(X_i)_{i \geq 0}$ centered stationary Gaussian sequence, covariance function ϕ. Assume $\phi(n) = o(\log n)$ $(n \to \infty)$ and technicals hypothesis, then

$$\mathbb{P}\left(|M_n - \mathbb{E}[M_n]| \geq t\right) \leq 6e^{-ct\sqrt{\log n}}, \quad t \geq 0.$$

- What is superconcentration ?
- Convergence of extremes (Gaussian case).
- Superconcentration inequality for stationary Gaussian sequences.
- Main result (abstract theorem)
- Tools and sketch of the proof.

Key steps of the proof

Main steps

- Semigroup representation of the variance

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- Remember $\mathcal{C}(r_0)$ is "slightly bigger" than a partition of $\mathbb{R}^n$.

Conclusion

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- Gaussian stationary processes on $\mathbb{R}^n$.

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Hypercontractivity alone doesn't work .

Thank you for your attention.