Logarithmic Sobolev Inequalities

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Logarithmic Sobolev inequalities

what they are, some history

analytic, geometric, optimal transportation proofs

last decade developments

at the interface between

analysis, probability, geometry
what are logarithmic Sobolev inequalities?
Sobolev inequalities

\[ f : \mathbb{R}^m \rightarrow \mathbb{R} \text{ smooth, compactly supported} \]

\[
\left( \int_{\mathbb{R}^m} |f|^p \, dx \right)^{2/p} \leq C_m \int_{\mathbb{R}^m} |\nabla f|^2 \, dx
\]

\[ p = \frac{2m}{m-2} \quad (> 2) \quad (m \geq 3) \]

sharp constant \[ C_m = \frac{1}{\pi m (m-2)} \left( \frac{\Gamma(m)}{\Gamma\left(\frac{m}{2}\right)} \right)^{2/m} \]
\[ \left( \int_{\mathbb{R}^m} |f|^p \, dx \right)^{2/p} \leq C_m \int_{\mathbb{R}^m} |\nabla f|^2 \, dx \]

\[ \frac{2}{p} \log \left( \int_{\mathbb{R}^m} |f|^p \, dx \right) \leq \log \left( C_m \int_{\mathbb{R}^m} |\nabla f|^2 \, dx \right) \]

Assume \[ \int_{\mathbb{R}^m} f^2 \, dx = 1 \]

Jensen's inequality for \[ f^2 \, dx \]

\[ \log \left( \int_{\mathbb{R}^m} |f|^p \, dx \right) = \log \left( \int_{\mathbb{R}^m} |f|^{p-2} f^2 \, dx \right) \geq \int_{\mathbb{R}^m} \log \left( |f|^{p-2} \right) f^2 \, dx \]

\[ \frac{p - 2}{p} \int_{\mathbb{R}^m} f^2 \log f^2 \, dx \leq \log \left( C_m \int_{\mathbb{R}^m} |\nabla f|^2 \, dx \right) \]
\[
\frac{p - 2}{p} \int_{\mathbb{R}^m} f^2 \log f^2 \, dx \leq \log \left( C_m \int_{\mathbb{R}^m} |\nabla f|^2 \, dx \right), \quad \int_{\mathbb{R}^m} f^2 \, dx = 1
\]

form of logarithmic Sobolev inequality

formally come back to Sobolev (worse constants)

issue on sharp constants

\[f : \mathbb{R}^n \to \mathbb{R} \text{ smooth, } \int_{\mathbb{R}^n} f^2 \, dx = 1\]

\[f^{\otimes kn} : \mathbb{R}^{kn} \to \mathbb{R}, \quad m = kn, \quad k \to \infty\]

\[
\int_{\mathbb{R}^n} f^2 \log f^2 \, dx \leq \frac{n}{2} \log \left( \frac{2}{n\pi e} \int_{\mathbb{R}^n} |\nabla f|^2 \, dx \right), \quad \int_{\mathbb{R}^n} f^2 \, dx = 1
\]

sharp (Euclidean) logarithmic Sobolev inequality

used by G. Perelman (2002)
(Euclidean) **logarithmic Sobolev inequality**

\[ \int_{\mathbb{R}^n} f^2 \log f^2 \, dx \leq \frac{n}{2} \log \left( \frac{2}{n\pi e} \int_{\mathbb{R}^n} |\nabla f|^2 \, dx \right), \quad \int_{\mathbb{R}^n} f^2 \, dx = 1 \]

\[ dx \rightarrow d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]

\(\mu\) standard Gaussian probability measure on \(\mathbb{R}^n\)

change \(f^2\) into \(f^2 e^{-|x|^2/2}\)

\(f : \mathbb{R}^n \to \mathbb{R}\) smooth, \(\int_{\mathbb{R}^n} f^2 \, d\mu = 1\)

\[ \int_{\mathbb{R}^n} f^2 \log f^2 \, d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu \]
(Gaussian) logarithmic Sobolev inequality

\[ \int_{\mathbb{R}^n} f^2 \log f^2 \, d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu, \quad \int_{\mathbb{R}^n} f^2 \, d\mu = 1 \]

\[ d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]

Sobolev type inequality \ (for \ \mu) \n
constant is sharp \n
constant independent of \ n \ (stability by product) \n
extension to infinite dimensional Wiener space \n
Gibbs measures, models from statistical mechanics
(Gaussian) logarithmic Sobolev inequality

\[ \int_{\mathbb{R}^n} f^2 \log f^2 \, d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu, \quad \int_{\mathbb{R}^n} f^2 \, d\mu = 1 \]

\[ d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]

different forms

Sobolev type inequality

information theory

PDE formulation
information theory description

\[ f \to \sqrt{f}, \quad f > 0, \quad \int_{\mathbb{R}^n} f \, d\mu = 1 \]

\[ \int_{\mathbb{R}^n} f \log f \, d\mu \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\mu \]

\[ d\nu = f \, d\mu \quad \text{probability} \]

\[ \int_{\mathbb{R}^n} f \log f \, d\mu = H(\nu \mid \mu) \quad \text{(relative) entropy} \]

\[ \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\mu = I(\nu \mid \mu) \quad \text{(relative) Fisher information} \]

entropy \quad \[ H(\nu \mid \mu) \leq \frac{1}{2} I(\nu \mid \mu) \quad \text{Fisher information} \]
**PDE description**

$f$ function $\rightarrow$ probability (Lebesgue) density $\rho$

$$\rho_\infty = \frac{e^{-|x|^2/2}}{(2\pi)^{n/2}}$$

$$\int_{\mathbb{R}^n} f \, d\mu = 1, \quad \rho = f \rho_\infty, \quad \int_{\mathbb{R}^n} \rho \, dx = 1$$

**logarithmic Sobolev inequality**

$$\int_{\mathbb{R}^n} \rho \log \left( \frac{\rho}{\rho_\infty} \right) \, dx = H(\rho \mid \rho_\infty) \leq \frac{1}{2} I(\rho \mid \rho_\infty) = 2 \int_{\mathbb{R}^n} \left| \nabla \left( \sqrt{\frac{\rho}{\rho_\infty}} \right) \right|^2 \rho_\infty \, dx$$

another formulation of the Euclidean logarithmic Sobolev inequality
trend to equilibrium

\[ \rho > 0 \text{ smooth, } \int_{\mathbb{R}^n} \rho \, dx = 1 \]

\[ H(\rho | \rho_\infty) \leq \frac{1}{2} I(\rho | \rho_\infty) \]

\[ V(x) = \frac{|x|^2}{2} \]

linear Fokker-Planck equation

\[ \frac{\partial \rho}{\partial t} = \nabla \cdot [\rho \nabla (\log \rho + V)] \]

Boltzmann \( H \)-theorem

\[ \frac{d}{dt} H(\rho_t | \rho_\infty) = -I(\rho_t | \rho_\infty) \]

\[ \rho_t \to \rho_\infty = \frac{e^{-V}}{Z} \]

\[ H(\rho_t | \rho_\infty) \leq e^{-2t} H(\rho_0 | \rho_\infty) \]
history

(Euclidean) logarithmic Sobolev inequality

\[ \int_{\mathbb{R}^n} f^2 \log f^2 \, dx \leq \frac{n}{2} \log \left( \frac{2}{n \pi e} \int_{\mathbb{R}^n} |\nabla f|^2 \, dx \right), \quad \int_{\mathbb{R}^n} f^2 \, dx = 1 \]

(Gaussian) logarithmic Sobolev inequality

\[ \int_{\mathbb{R}^n} f^2 \log f^2 d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu, \quad \int_{\mathbb{R}^n} f^2 d\mu = 1 \]

\[ d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]
logarithmic Sobolev inequalities

various origins

Boltzmann $H$-theorem

mathematical physics

quantum field theory (hypercontractivity)

information theory
L. Gross

Logarithmic Sobolev inequalities

Amer. J. Math. 97, 1061-1083 (1975)
A partially alternate derivation of a result of Nelson

Shannon-Stam entropy power inequality

\[ \frac{2}{n} H(\rho * h) \geq \frac{2}{n} H(\rho) + \frac{2}{n} H(h) \]

\[ H(\rho) = -\int_{\mathbb{R}^n} \rho \log \rho \, dx, \quad \rho > 0, \quad \int_{\mathbb{R}^n} \rho \, dx = 1 \]

\[ h = h_\varepsilon \quad \text{Gaussian kernel,} \quad \varepsilon \to 0 \]

\[ e^{-\frac{2}{n} H(\rho)} \leq \frac{1}{2n\pi e} \int_{\mathbb{R}^n} \frac{\lvert \nabla \rho \rvert^2}{\rho} \, dx \]

\[ \int_{\mathbb{R}^n} \rho \log \rho \, dx \leq \frac{n}{2} \log \left( \frac{1}{2n\pi e} \int_{\mathbb{R}^n} \frac{\lvert \nabla \rho \rvert^2}{\rho} \, dx \right) \]

(\rho \to f^2) \quad \text{(Euclidean) logarithmic Sobolev inequality}

A. Stam (1959)
(Gaussian) logarithmic Sobolev inequality

\[ f : \mathbb{R}^n \to \mathbb{R} \text{ smooth, } \int_{\mathbb{R}^n} f^2 d\mu = 1, \quad d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]

\[ \int_{\mathbb{R}^n} f^2 \log f^2 d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu \]

**at least 15 different proofs**

two-point space (central limit theorem)

hypercontractivity

analytic semigroup theory

geometric convexity

optimal transportation
(Gaussian) logarithmic Sobolev inequality

\[ f : \mathbb{R}^n \to \mathbb{R} \text{ smooth, } \int_{\mathbb{R}^n} f^2 \, d\mu = 1, \quad d\mu(x) = e^{-\frac{|x|^2}{2}} \frac{dx}{(2\pi)^{n/2}} \]

\[ \int_{\mathbb{R}^n} f^2 \log f^2 \, d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu \]

at least 15 different proofs

two-point space (central limit theorem)

hypercontractivity

analytic semigroup theory

geometric convexity

optimal transportation
two-point space

\[ f : \{-1, +1\} \to \mathbb{R}, \quad \int_{\{-1,+1\}} f^2 \, d\nu = 1, \quad \nu(-1) = \nu(+1) = \frac{1}{2} \]

\[ \int_{\{-1,+1\}} f^2 \log f^2 \, d\nu \leq \frac{1}{2} \int_{\{-1,+1\}} |Df|^2 \, d\nu \]

\[ Df = f(+1) - f(-1) \]

\[ f(-1) = \alpha, \quad f(+1) = \beta, \quad \frac{\alpha^2}{2} + \frac{\beta^2}{2} = 1 \]

\[ \alpha^2 \log \alpha^2 + \beta^2 \log \beta^2 \leq (\alpha - \beta)^2 \]

(not so easy) exercise

L. Gross (1975)
\[
\int_{\{-1,+1\}} f^2 \log f^2 \, d\nu \leq \frac{1}{2} \int_{\{-1,+1\}} |Df|^2 \, d\nu
\]

tensorization

\[
\int_{\{-1,+1\}^n} f^2 \log f^2 \, d\nu \otimes^n \leq \frac{1}{2} \int_{\{-1,+1\}^n} \sum_{i=1}^n |D_i f|^2 \, d\nu \otimes^n
\]

central limit theorem

\[
\nu \otimes^n \rightarrow \mu \quad \text{Gaussian measure}
\]

\[
\int_{\mathbb{R}^n} f^2 \log f^2 \, d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu, \quad \int_{\mathbb{R}^n} f^2 \, d\mu = 1
\]
hypercontractivity

two-point space

\( f : \{-1, +1\} \to \mathbb{R}, \quad f(x) = a + bx \)

\( P_t f(x) = a + be^{-t}x, \quad t \geq 0 \)

\( (P_t)_{t \geq 0} \) semigroup of contractions on \( L^p(\nu) \)

\( 1 < p < q < \infty, \quad e^{2t} \geq \frac{q - 1}{p - 1} \)

\[ \|P_t f\|_q \leq \|f\|_p \]

\[ \left(\frac{1}{2} |a + be^{-t}|^q + \frac{1}{2} |a - be^{-t}|^q\right)^{1/q} \leq \left(\frac{1}{2} |a + b|^p + \frac{1}{2} |a - b|^p\right)^{1/p} \]

\textbf{A. Bonami (1970), W. Beckner (1975)}
two-point space $\rightarrow$ Gaussian

$$d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$$

$L = \Delta - x \cdot \nabla$ Ornstein-Uhlenbeck operator (Fokker-Planck)

$\mu$ invariant measure, $P_t = e^{tL}$ semigroup (contractions on $L^p(\mu)$)

hypercontractivity property

$$1 < p < q < \infty, \quad e^{2t} \geq \frac{q - 1}{p - 1}$$

$$\|P_tf\|_q \leq \|f\|_p$$

E. Nelson (1966-73)

quantum field theory
L. Gross (1975)

logarithmic Sobolev inequality

equivalent

hypercontractivity

(general context of Markov operators)

\[ \| P_t f \|_q \leq \| f \|_p \]

\[ q = q(t) = 1 + e^{2t(p-1)}, \quad t \geq 0 \]

\[ \frac{d}{dt} \| P_t f \|_{q(t)} \leq 0 \]

\[ \int_{\mathbb{R}^n} f^2 \log f^2 \, d\mu \leq 2 \int_{\mathbb{R}^n} f(-Lf) \, d\mu = 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu \]
three proofs of the logarithmic Sobolev inequality

analytic: parametrisation by heat kernels

geometric: Brunn-Minkowski inequality

measure theoretic: parametrisation by optimal transport

interface of analysis, probability and geometry
analytic proof (semigroup)

D. Bakry, M. Emery (1985)

(the?) simplest one (according to L. Gross 2010)

\[ f > 0 \text{ smooth, } \quad d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]

\((P_t)_{t \geq 0}\) heat semigroup, generator \(\Delta\)

\[ P_t f(x) = \int_{\mathbb{R}^n} f(y) e^{-|x-y|^2/4t} \frac{dy}{(4\pi t)^{n/2}} , \quad t > 0, \ x \in \mathbb{R}^n \]

\[ t = \frac{1}{2} \ (x = 0) : \quad P_t \to \mu \]
analytic proof (semigroup)

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\[ f > 0 \text{ smooth, } \quad d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]

\[
\int_{\mathbb{R}^n} f \log f \, d\mu \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|
abla f|^2}{f} \, d\mu, \quad \int_{\mathbb{R}^n} f \, d\mu = 1
\]

\((P_t)_{t \geq 0}\)  heat semigroup, generator  \(\Delta\)

\[
P_tf(x) = \int_{\mathbb{R}^n} f(y) e^{-|x-y|^2/4t} \frac{dy}{(4\pi t)^{n/2}}, \quad t > 0, \ x \in \mathbb{R}^n
\]

\[
t = \frac{1}{2} \quad (x = 0) : \quad P_t \to \mu
\]
analytic proof (semigroup)

D. Bakry, M. Emery (1985)

(the?) simplest one (according to L. Gross 2010)

\[ f > 0 \text{ smooth, } \quad d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]

\[
\int_{\mathbb{R}^n} f \log f \, d\mu - \int_{\mathbb{R}^n} f \, d\mu \log \left( \int_{\mathbb{R}^n} f \, d\mu \right) \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\mu
\]

\[(P_t)_{t \geq 0} \quad \text{heat semigroup, generator } \Delta\]

\[P_t f(x) = \int_{\mathbb{R}^n} f(y) e^{-|x-y|^2/4t} \frac{dy}{(4\pi t)^{n/2}}, \quad t > 0, \; x \in \mathbb{R}^n\]

\[t = \frac{1}{2} \quad (x = 0) : \quad P_t \to \mu\]
$f > 0$ smooth, $t > 0$, at any point

\[
P_t(f \log f) - P_t f \log P_t f = \int_0^t \frac{d}{ds} P_s(P_{t-s}f \log P_{t-s}f) \, ds
\]

\[
\frac{d}{ds} P_s(P_{t-s}f \log P_{t-s}f)
\]

\[
= P_s \left( \Delta(P_{s-t}f \log P_{t-s}f) - \Delta P_{t-s}f \log P_{t-s}f - \Delta P_{t-s}f \right)
\]

\[
= P_s \left( \frac{|\nabla P_{t-s}f|^2}{P_{t-s}f} \right)
\]
\[ f > 0 \text{ smooth, } t > 0, \text{ at any point} \]

\[
P_t(f \log f) - P_t f \log P_t f = \int_0^t \frac{d}{ds} P_s(P_{t-s} f \log P_{t-s} f) \, ds
\]

\[ = \int_0^t P_s \left( \frac{|\nabla P_{t-s} f|^2}{P_{t-s} f} \right) \, ds \]

\[
\nabla P_{uf} = P_u(\nabla f)
\]

\[ |\nabla P_{uf}|^2 \leq \left[ P_u(|\nabla f|) \right]^2 \leq P_u \left( \frac{|\nabla f|^2}{f} \right) P_{uf} \]

\[ u = t - s \]

\[
\frac{|\nabla P_{t-s} f|^2}{P_{t-s} f} \leq P_{t-s} \left( \frac{|\nabla f|^2}{f} \right) \]

\[
P_t(f \log f) - P_t f \log P_t f \leq \int_0^t P_s \left( P_{t-s} \left( \frac{|\nabla f|^2}{f} \right) \right) ds = t \, P_t \left( \frac{|\nabla f|^2}{f} \right) \]
same proof

\[ d\mu = e^{-V} dx \quad \text{probability measure} \]

\[ V : \mathbb{R}^n \to \mathbb{R} \quad \text{smooth} \]

\[ \nabla \nabla V \geq c > 0 \]

\[ \int_{\mathbb{R}^n} f^2 \log f^2 d\mu \leq \frac{2}{c} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu, \quad \int_{\mathbb{R}^n} f^2 d\mu = 1 \]

weighted Riemannian manifold

\[ \text{Ric} + \nabla \nabla V \geq c > 0 \]

D. Bakry, M. Emery (1985)
geometric (convexity) proof

Brunn-Minkowski-Lusternik inequality

\[ A, B \text{ compact subsets of } \mathbb{R}^n \]

\[ \text{vol}_n(A + B)^{1/n} \geq \text{vol}_n(A)^{1/n} + \text{vol}_n(B)^{1/n} \]

\[ A + B = \{x + y; x \in A, y \in B\} \]

isoperimetric inequality

\[ B = B(0, \varepsilon), \quad \varepsilon \to 0 \]
Brunn-Minkowski: functional form

Prékopa-Leindler (1971) theorem

\[ \theta \in [0, 1], \quad u, v, w \geq 0 \quad \text{on} \quad \mathbb{R}^n \]

if \[ w(\theta x + (1 - \theta) y) \geq u(x)^\theta v(y)^{1-\theta}, \quad x, y \in \mathbb{R}^n \]

then \[ \int_{\mathbb{R}^n} w \, dx \geq \left( \int_{\mathbb{R}^n} u \, dx \right)^\theta \left( \int_{\mathbb{R}^n} v \, dx \right)^{1-\theta} \]

\[ u = \chi_A, \ v = \chi_B \]

(equivalent, dimension free) multiplicative form of Brunn-Minkowski

\[ \text{vol}_n(\theta A + (1 - \theta) B) \geq \text{vol}_n(A)^\theta \text{vol}_n(B)^{1-\theta} \]
Brunn-Minkowski : functional form

Prékopa-Leindler (1971) theorem

\[ \theta \in [0, 1], \quad u, v, w \geq 0 \quad \text{on} \quad \mathbb{R}^n \]

if \[ w(\theta x + (1 - \theta) y) \geq u(x)^\theta v(y)^{1-\theta}, \quad x, y \in \mathbb{R}^n \]
then \[ \int_{\mathbb{R}^n} w \, dx \geq \left( \int_{\mathbb{R}^n} u \, dx \right)^{\theta} \left( \int_{\mathbb{R}^n} v \, dx \right)^{1-\theta} \]

\[ dx \to d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]
Brunn-Minkowski : functional form

Prékopa-Leindler (1971) theorem

\[
dx \rightarrow d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}
\]

\[
f \rightarrow f e^{-|x|^2/2}
\]

\[
\theta \in [0, 1], \quad u, v, w \geq 0 \quad \text{on} \quad \mathbb{R}^n
\]

then

\[
\int_{\mathbb{R}^n} w \, d\mu \geq \left( \int_{\mathbb{R}^n} u \, d\mu \right)^\theta \left( \int_{\mathbb{R}^n} v \, d\mu \right)^{1-\theta}
\]
Brunn-Minkowski: functional form

Prékopa-Leindler (1971) theorem

\[ dx \rightarrow d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]

\[ f \rightarrow f e^{-|x|^2/2} \]

\[ \theta \in [0, 1], \quad u, v, w \geq 0 \quad \text{on} \quad \mathbb{R}^n \]

if \[ w(\theta x + (1 - \theta)y) \geq u(x)^\theta v(y)^{1-\theta}, \quad x, y \in \mathbb{R}^n \]

then \[ \int_{\mathbb{R}^n} w \, d\mu \geq \left( \int_{\mathbb{R}^n} u \, d\mu \right)^\theta \left( \int_{\mathbb{R}^n} v \, d\mu \right)^{1-\theta} \]
Brunn-Minkowski : functional form

Prékopa-Leindler (1971) theorem

\[ dx \rightarrow d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]

\[ f \rightarrow f e^{-|x|^2/2} \]

\[ \theta \in [0, 1], \quad u, v, w \geq 0 \quad \text{on} \quad \mathbb{R}^n \]

if \[ w(\theta x + (1 - \theta) y) \geq u(x)^\theta v(y)^{1-\theta} e^{-\theta(1-\theta)|x-y|^2/2}, \quad x, y \in \mathbb{R}^n \]

then \[ \int_{\mathbb{R}^n} w \, d\mu \geq \left( \int_{\mathbb{R}^n} u \, d\mu \right)^\theta \left( \int_{\mathbb{R}^n} v \, d\mu \right)^{1-\theta} \]
\( f : \mathbb{R}^n \to \mathbb{R} \) bounded, \( \theta \in (0, 1) \)

\[
\begin{align*}
  w(z) &= e^{f(z)} \\
  v(y) &= 1 \\
  u(x) &= e^{g(x)}
\end{align*}
\]

\[
  w(\theta x + (1 - \theta) y) \geq u(x)^\theta v(y)^{1-\theta} e^{-\theta(1-\theta)|x-y|^2/2}, \quad x, y \in \mathbb{R}^n
\]

\[
  f(\theta x + (1 - \theta) y) \geq \theta g(x) - \frac{\theta(1-\theta)}{2} |x - y|^2
\]
\begin{align*}
  f(\theta x + (1 - \theta)y) & \geq \theta g(x) - \frac{\theta(1-\theta)}{2} |x - y|^2 \\
  g(x) & = \frac{1}{\theta} Q_{(1-\theta)/\theta} f(x) \\
  Q_t f(x) & = \inf_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2t} |x - y|^2 \right\}, \quad t > 0, \ x \in \mathbb{R}^n
\end{align*}

infimum-convolution with quadratic cost
Prékopa-Leindler theorem

\[ w(z) = e^{f(z)} \]

\[ v(y) = 1 \]

\[ u(x) = e^{\frac{1}{\theta} Q_{(1-\theta)/\theta} f(x)} \]

\[ w(\theta x + (1 - \theta)y) \geq u(x)^{\theta} v(y)^{1-\theta} e^{-\theta(1-\theta)|x-y|^2/2}, \quad x, y \in \mathbb{R}^n \]

\[ \int_{\mathbb{R}^n} e^f \, d\mu \geq \left( \int_{\mathbb{R}^n} e^{\frac{1}{\theta} Q_{(1-\theta)/\theta} f} \, d\mu \right)^\theta \]

\[ \frac{1}{\theta} = 1 + t \]

\[ \int_{\mathbb{R}^n} e^f \, d\mu \geq \left( \int_{\mathbb{R}^n} e^{(1+t) Q_t f} \, d\mu \right)^{1/(1+t)}, \quad t > 0 \]
\[
\| e^{Q_t f} \|_{1+t} \leq \| e^f \|_1
\]

\[
Q_t f(x) = \inf_{y \in \mathbb{R}^n} \{ f(y) + \frac{1}{2t} |x - y|^2 \}, \quad t > 0, \ x \in \mathbb{R}^n
\]

**Hopf-Lax representation of Hamilton-Jacobi solutions**

\[
\partial_t Q_t f \big|_{t=0} = -\frac{1}{2} |\nabla f|^2
\]

differentiate at \( t = 0 \)

\[
\int_{\mathbb{R}^n} f e^f d\mu \leq \frac{1}{2} \int_{\mathbb{R}^n} e^f |\nabla f|^2 d\mu, \quad \int_{\mathbb{R}^n} e^f d\mu = 1
\]

\( f \to \log f^2 \) \quad logarithmic Sobolev inequality
same proof

\[ d\mu = e^{-V} \, dx \quad \text{probability measure} \]

\[ V : \mathbb{R}^n \to \mathbb{R} \quad \text{smooth} \]

\[ \nabla \nabla V \geq c > 0 \]

\[ \| e^{Q_t f} \|_{1+t} \leq \| e^f \|_1 \]

\[ \int_{\mathbb{R}^n} f^2 \log f^2 \, d\mu \leq \frac{2}{c} \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu, \quad \int_{\mathbb{R}^n} f^2 \, d\mu = 1 \]

weighted Riemannian manifold

\[ \text{Ric} + \nabla \nabla V \geq c > 0 \]

\[ \| e^{Q_t f} \|_{1+t} \leq \| e^f \|_1 \]

analogue of hypercontractivity

equivalent to logarithmic Sobolev inequality

\[ a \to 0 \]

\[ \int_{\mathbb{R}^n} e^{Q_t f} d\mu \leq e^{\int_{\mathbb{R}^n} f d\mu} \]

\[ f : \mathbb{R}^n \to \mathbb{R} \quad \text{measurable bounded} \]

\[ Q_t f(x) = \inf_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2t} |x - y|^2 \right\}, \quad t > 0, \ x \in \mathbb{R}^n \]
\[ \| e^{Q_t f} \|_{a+t} \leq \| e^f \|_a, \quad a > 0 \]

analogue of hypercontractivity

equivalent to logarithmic Sobolev inequality

\[ a \to 0 \]

\[ \int_{\mathbb{R}^n} e^{Q_t f} \, d\mu \leq e^{\int_{\mathbb{R}^n} f \, d\mu} \]

\( f : \mathbb{R}^n \to \mathbb{R} \) measurable bounded

\[ Q_t f(x) = \inf_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2t} |x - y|^2 \right\}, \quad t > 0, \ x \in \mathbb{R}^n \]
\[
\int_{\mathbb{R}^n} e^{Q_t f} \, d\mu \leq e^{\int_{\mathbb{R}^n} f \, d\mu}
\]

dual form of \textit{transportation cost inequality}

\textbf{Wasserstein distance} \quad W_2(\nu, \mu)^2 \leq 2 H(\nu \mid \mu) \quad \text{relative entropy}

\[H(\nu \mid \mu) = \int_{\mathbb{R}^n} \log \frac{d\nu}{d\mu} \, d\nu, \quad \nu \ll \mu\]

\text{relative entropy}

\[W_2(\nu, \mu)^2 = \inf_{\nu \leftarrow \pi \to \mu} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x - y|^2 \, d\pi(x, y)\]

\textbf{Kantorovitch-Rubinstein-Wasserstein distance}

\[W_2(\nu, \mu)^2 = \sup \left\{ \int_{\mathbb{R}^n} Q_1 f \, d\nu - \int_{\mathbb{R}^n} f \, d\mu \right\}\]
\[ W_2(\nu, \mu)^2 \leq 2 H(\nu \mid \mu), \quad \nu \ll \mu \]

\[ d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]

M. Talagrand (1996)


consequence of logarithmic Sobolev inequality

general \mu

optimal transportation framework

\[ \| e^{Q_t f} \|_{a+t} \leq \| e^f \|_a, \quad a > 0 \]

\[ a \to 0 \]
parametrisation proof by optimal transportation

$\mu, \nu$ probability measures on $\mathbb{R}^n$ smooth densities

$T : \mu \rightarrow \nu$

optimal: $W_2(\mu, \nu)^2 = \int_{\mathbb{R}^n} |x - T(x)|^2 d\mu(x)$

$T = \nabla \phi, \phi$ convex

Y. Brenier, S. T. Rachev - L. Rüschendorf (1990)

manifold case R. McCann (1995)
transportation proof of the logarithmic Sobolev inequality

\[ d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \]

\[ f > 0, \quad \int_{\mathbb{R}^n} f \, d\mu = 1, \quad d\nu = f \, d\mu \]

Brenier map: \[ T : f \mu \rightarrow \mu \]

\[ \int_{\mathbb{R}^n} b \circ T f \, d\mu = \int_{\mathbb{R}^n} b \, d\mu \]

\[ T = \nabla \phi = x + \nabla \psi, \quad \phi \text{ convex} \]

Monge-Ampère equation

\[ f(x) e^{-|x|^2/2} = e^{-|T(x)|^2/2 \det(\text{Id} + \nabla\nabla \psi(x))} \]
\[ f(x) e^{-|x|^2/2} = e^{-|T(x)|^2/2 \det (\text{Id} + \nabla \nabla \psi(x))} \]

\[ \log f = \frac{1}{2} \left[ |x|^2 - |T|^2 \right] + \log \det (\text{Id} + \nabla \nabla \psi) \]

integrate with respect to \( f \, d\mu \)

\[ \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|
abla f|^2}{f} \, d\mu \]

logarithmic Sobolev inequality
\[ f(x) e^{-|x|^2/2} = e^{-|T(x)|^2/2} \det(\Id + \nabla \nabla \psi(x)) \]

\[ \log f = \frac{1}{2} \left[ |x|^2 - |T|^2 \right] + \log \det (\Id + \nabla \nabla \psi) \]

\[ = -x \cdot \nabla \psi - \frac{1}{2} |\nabla \psi|^2 + \log \det (\Id + \nabla \nabla \psi) \]

integrate with respect to \( f \, d\mu \)

\[ \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|
abla f|^2}{f} \, d\mu \]

logarithmic Sobolev inequality
\[ f(x) e^{-|x|^2/2} = e^{-|T(x)|^2/2 \det(\text{Id} + \nabla \nabla \psi(x))} \]

\[ \log f = \frac{1}{2} \left[ |x|^2 - |T|^2 \right] + \log \det \left( \text{Id} + \nabla \nabla \psi \right) \]

\[ \leq -x \cdot \nabla \psi - \frac{1}{2} |\nabla \psi|^2 + \Delta \psi \]

integrate with respect to \( f \, d\mu \)

\[ \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|
abla f|^2}{f} \, d\mu \]

logarithmic Sobolev inequality
\begin{align*}
f(x) \, e^{-|x|^2/2} &= e^{-|\mathcal{T}(x)|^2/2} \det(\operatorname{Id} + \nabla \nabla \psi(x)) \\
\log f &= \frac{1}{2} \left[ |x|^2 - |\mathcal{T}|^2 \right] + \log \det (\operatorname{Id} + \nabla \nabla \psi) \\
\int f \, d\mu &\leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|
abla f|^2}{f} \, d\mu \end{align*}

integrate with respect to $f \, d\mu$

logarithmic Sobolev inequality
\[ f(x) e^{-|x|^2/2} = e^{-|T(x)|^2/2} \det(\text{Id} + \nabla \nabla \psi(x)) \]

\[ \log f = \frac{1}{2} \left[ |x|^2 - |T|^2 \right] + \log \det (\text{Id} + \nabla \nabla \psi) \]

\[ \leq L\psi - \frac{1}{2} |\nabla \psi|^2 \]

integrate with respect to \( f \, d\mu \)

\[ \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\mu \]

logarithmic Sobolev inequality
\[ f(x) e^{-|x|^2/2} = e^{-|T(x)|^2/2} \det \left( \Id + \nabla \nabla \psi(x) \right) \]

\[ \log f = \frac{1}{2} \left[ |x|^2 - |T|^2 \right] + \log \det \left( \Id + \nabla \nabla \psi \right) \]

\[ \leq L \psi - \frac{1}{2} |\nabla \psi|^2 \]

integrate with respect to \( f \, d\mu \)

\[ \int_{\mathbb{R}^n} f \log f \, d\mu \leq \int_{\mathbb{R}^n} L \psi f \, d\mu - \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \psi|^2 f \, d\mu \]

\[ \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\mu \]

logarithmic Sobolev inequality
\[ f(x) e^{-|x|^2/2} = e^{-|T(x)|^2/2} \det (\text{Id} + \nabla \nabla \psi(x)) \]

\[ \log f = \frac{1}{2} \left[ |x|^2 - |T|^2 \right] + \log \det (\text{Id} + \nabla \nabla \psi) \]

\[ \leq L \psi - \frac{1}{2} |\nabla \psi|^2 \]

integrate with respect to \( f \, d\mu \)

\[ \int_{\mathbb{R}^n} f \log f \, d\mu \leq - \int_{\mathbb{R}^n} \nabla \psi \cdot \nabla f \, d\mu - \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \psi|^2 f \, d\mu \]

\[ \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\mu \]

logarithmic Sobolev inequality
same proof

\[ d\mu = e^{-V} \, dx \quad \text{probability measure} \]

\[ V : \mathbb{R}^n \to \mathbb{R} \quad \text{smooth} \]

\[ \nabla \nabla \nabla V \geq c > 0 \]

\[ \int_{\mathbb{R}^n} f^2 \log f^2 \, d\mu \leq \frac{2}{c} \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu, \quad \int_{\mathbb{R}^n} f^2 \, d\mu = 1 \]

weighted Riemannian manifold

\[ \text{Ric} + \nabla \nabla V \geq c > 0 \]

D. Cordero-Erausquin (2002)
general parametrisation

\[ T_\theta = (1 - \theta) \text{Id} + \theta T, \quad \theta \in [0, 1] \]

\( (T_0 \mu = \mu, \quad T_1 \mu = T \mu = \nu) \)

\[ T_\theta : \mu \rightarrow f_\theta \, d\mu \]

\[ d\mu = e^{-V} \, dx \]

Monge-Ampère equation

\[ e^{-V} = f_\theta \circ T_\theta \, e^{-V \circ T_\theta} \det((1 - \theta) \text{Id} + \theta \nabla \nabla \phi) \]

\[ \nabla \nabla \phi \quad \text{symmetric positive definite} \]

non-smooth analysis, PDE methods
optimal parametrisation and entropy

J. Lott - C. Villani, K.-Th. Sturm (2006-10)

Ricci curvature lower bounds in metric measure space

Riemannian geometry of \((\mathcal{P}_2, W_2)\)

\((\mathcal{P}_2, W_2)\) probability measures (second moment)

$\mu_0, \mu_1$ probability measures on $\mathbb{R}^n$

$T : \mu_0 \to \mu_1$ optimal

$T_\theta = (1 - \theta) \text{Id} + \theta \, T$, $\theta \in [0, 1]$ geodesic in $(\mathcal{P}_2, W_2)$

reference measure $d\mu = e^{-V} \, dx$ on $\mathbb{R}^n$, $\nabla \nabla V \geq c$, $c \in \mathbb{R}$

$c$-convexity property of entropy along geodesic $\mu_\theta = T_\theta(\mu_0) = f_\theta \mu$

$H$ relative entropy, $W_2$ Wasserstein distance

R. McCann (1995) displacement convexity
\( \mu_0, \mu_1 \) probability measures on \( \mathbb{R}^n \)

\[ T : \mu_0 \rightarrow \mu_1 \text{ optimal} \]

\[ T_\theta = (1 - \theta) \text{Id} + \theta T, \quad \theta \in [0, 1] \text{ geodesic in } (\mathcal{P}_2, \mathcal{W}_2) \]

reference measure \( d\mu = e^{-V} \, dx \) on \( \mathbb{R}^n \), \( \nabla \nabla V \geq c \), \( c \in \mathbb{R} \)

c-convexity property of entropy along geodesic \( \mu_\theta = T_\theta(\mu_0) = f_\theta \mu \)

\( H \) relative entropy, \( \mathcal{W}_2 \) Wasserstein distance

R. McCann (1995) displacement convexity
\( \mu_0, \mu_1 \) probability measures on \( \mathbb{R}^n \)

\[
T : \mu_0 \to \mu_1 \quad \text{optimal}
\]

\[
T_\theta = (1 - \theta) \text{Id} + \theta T, \quad \theta \in [0, 1] \quad \text{geodesic in } (\mathcal{P}_2, W_2)
\]

reference measure \( d\mu = e^{-V} dx \) on \( \mathbb{R}^n \), \( \nabla \nabla V \geq c \), \( c \in \mathbb{R} \)

\( c \)-convexity property of entropy along geodesic \( \mu_\theta = T_\theta(\mu_0) = f_\theta \mu \)

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reference measure $d\mu = e^{-V} dx$ on $\mathbb{R}^n$, $\nabla\nabla V \geq c$, $c \in \mathbb{R}$

$c$-convexity property of entropy along geodesic $\mu_\theta = T_\theta(\mu_0) = f_\theta \mu$

$c = 0$ $H(\mu_\theta | \mu) \leq (1 - \theta)H(\mu_0 | \mu) + \theta H(\mu_1 | \mu)$

$H$ relative entropy, $\mathcal{W}_2$ Wasserstein distance

R. McCann (1995) displacement convexity
\( \mu_0, \mu_1 \) probability measures on \( \mathbb{R}^n \)

\[ T : \mu_0 \to \mu_1 \text{ optimal} \]

\[ T_\theta = (1 - \theta) \text{Id} + \theta T, \ \theta \in [0, 1] \text{ geodesic in } (\mathcal{P}_2, \mathcal{W}_2) \]

reference measure \( d\mu = e^{-V} dx \) on \( \mathbb{R}^n \), \( \nabla\nabla V \geq c, \ c \in \mathbb{R} \)

\( c \)-convexity property of entropy along geodesic \( \mu_\theta = T_\theta(\mu_0) = f_\theta \mu \)

\[ H(\mu_\theta | \mu) \leq (1 - \theta)H(\mu_0 | \mu) + \theta H(\mu_1 | \mu) - c \theta(1 - \theta)\mathcal{W}_2(\mu_0, \mu_1)^2 \]

\( H \) relative entropy, \( \mathcal{W}_2 \) Wasserstein distance

R. McCann (1995) displacement convexity
c-convexity property of entropy along geodesic \( \mu_\theta = T_\theta(\mu_0) \)

\[
H(\mu_\theta \mid \mu) \leq (1 - \theta)H(\mu_0 \mid \mu) + \theta H(\mu_1 \mid \mu) - c \theta(1 - \theta)W_2(\mu_0, \mu_1)^2
\]

characterizes \( \nabla \nabla V \geq c \)

reference measure \( d\mu = e^{-V} dx \)

extends to weighted manifolds

characterizes \( \text{Ric} + \nabla \nabla V \geq c \)

M. von Renesse, K.-Th. Sturm (2005)
notion of Ricci curvature bound

in a metric measure space (length space) \((X, d, \mu)\)

\((\mu_\theta)_{\theta \in [0,1]}\) geodesic in \((\mathcal{P}_2(X), W_2)\) connecting \(\mu_0, \mu_1\)

definition of lower bound on curvature

postulate that entropy is \(c\)-convex along one geodesic \((\mu_\theta)_{\theta \in [0,1]}\)

\[
H(\mu_\theta | \mu) \leq (1 - \theta)H(\mu_0 | \mu) + \theta H(\mu_1 | \mu) - c \theta(1 - \theta)W_2(\mu_0, \mu_1)^2
\]

\(H\) relative entropy, \(W_2\) Wasserstein distance
J. Lott - C. Villani, K.-Th. Sturm (2006-09)

definition of lower bound on curvature

in metric measure space

\[ H(\mu_\theta | \mu) \leq (1 - \theta)H(\mu_0 | \mu) + \theta H(\mu_1 | \mu) - c \theta(1 - \theta)W_2(\mu_0, \mu_1)^2 \]

◊ generalizes Ricci curvature in Riemannian manifolds

◊ allows for geometric and functional inequalities

◊ main result: stability by Gromov-Hausdorff limit

◊ analysis on singular spaces (limits of Riemannian manifolds)
mass transportation method

