Proofs of the
Gaussian isoperimetric inequality
(as of 2017)

The note reviews the known proofs of the isoperimetric inequality for Gaussian measures (as of 2017 – any relevant informations and references on omitted or new further proofs are welcome, and will be incorporated).

Let \( \gamma = \gamma_n \) be the standard Gaussian probability measure on the Borel sets of \( \mathbb{R}^n \), with density \( \varphi_n(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}||x||^2}, \ x \in \mathbb{R}^n, \) with respect to the Lebesgue measure. Denote by \( \Phi(t) = \int_{-\infty}^{t} \varphi_1(x)dx, \ t \in \mathbb{R}, \) the (continuous, strictly increasing) distribution function in dimension one, and define then the Gaussian isoperimetric profile

\[
I(s) = \varphi_1 \circ \Phi^{-1}(s), \ s \in [0, 1].
\] (1)

The function \( I \) is symmetric along the vertical line \( s = \frac{1}{2} \), and such that \( I(0) = I(1) = 0. \) It is worthwhile observing that \( I(s) \sim s \sqrt{2 \log \left( \frac{1}{s} \right)} \) as \( s \to 0. \)
Given $r > 0$, $A_r = \{ x \in \mathbb{R}^n; \inf_{a \in A} |x - a| < r \}$ is the $r$-neighborhood of a set $A$ in $\mathbb{R}^n$. The (Gaussian) outer Minkowski content of Borel set $A$ is defined as

$$\gamma^+(A) = \lim_{r \to 0} \inf \frac{1}{r} \left[ \gamma(A_r) - \gamma(A) \right].$$

**Theorem** [The Gaussian isoperimetric inequality] For any Borel set $A$ in $\mathbb{R}^n$,

$$\gamma^+(A) \geq \mathcal{I}(\gamma(A)). \quad (2)$$

Equality is achieved on the half-spaces $H = \{ x \in \mathbb{R}^n; \langle x, u \rangle \leq h \}$ where $u$ is a unit vector and $h \in \mathbb{R}$.

The measure of a half-space is computed in dimension one, $\gamma(H) = \Phi(h)$, and its boundary measure is

$$\gamma^+(H) = \lim_{r \to 0} \inf \frac{1}{r} \left[ \Phi(h + r) - \Phi(h) \right] = \varphi_1(h).$$

The Gaussian isoperimetric inequality thus expresses equivalently that, if $H$ is a half-space such that $\Phi(h) = \gamma(H) = \gamma(A)$, then

$$\gamma^+(A) \geq \gamma^+(H), \quad (3)$$

and half-spaces are the extremal sets of the Gaussian isoperimetric problem.

Integrating along the neighborhoods, (3) is equivalently formulated as

$$\gamma(A_r) \geq \gamma(H_r), \quad r > 0, \quad (4)$$

provided that $\gamma(A) = (\geq) \gamma(H)$, or

$$\Phi^{-1}(\gamma(A_r)) \geq \Phi^{-1}(\gamma(A)) + r, \quad r > 0 \quad (5)$$
(since $γ(H_r) = Φ(h + r)$).

Linear (affine) transformations yield the isoperimetric statement for any Gaussian measure. The dimension-free character allows furthermore for an infinite-dimensional formulation on an abstract Wiener space $(E, H, μ)$ as, for example,

$$Φ^{-1}(μ(A + rK)) \geq Φ^{-1}(μ(A)) + r, \quad r > 0,$$

where $K$ is the unit ball of the reproducing kernel Hilbert space $H$ (cf. [7, 21]).

The following sections briefly present the various known proofs of the Gaussian isoperimetric inequality.

1 Limit of spherical isoperimetry

In the neighborhood formulation, the isoperimetric inequality for the (normalized) uniform measure $σ_N$ on the $N$-sphere $S^N$ in $\mathbb{R}^{N+1}$, due to P. Lévy [22] and E. Schmidt [28], expresses that whenever $A$ is a Borel set in $S^N$, and $B$ a spherical cap (geodesic ball) such that $σ_N(A) = (≥) σ_N(B)$, then

$$σ_N(A_r) ≥ σ_N(B_r)$$

for any $r > 0$, where $A_r$ is the $r$-neighborhood of $A$ in the geodesic metric.

It is a folklore result, usually quoted as “Poincaré’s lemma”, that the normalized uniform measure on the sphere $\sqrt{N}S^N$, when projected on a $n$-dimensional subspace, converges as $N \to \infty$ to the standard $n$-dimensional Gaussian measure (cf. e.g. [21]). Via this limit, V. Sudakov and B. Tsirel’son [29], and C. Borell [7], independently, put forward the Gaussian isoperimetric inequality from the corresponding one on the sphere, the extremal spherical caps turning into half-spaces.

2 Gaussian symmetrization

Classical proofs of the isoperimetric inequality on the sphere use symmetrization techniques (see e.g. [16]). It is the contribution of A. Ehrhard [13] to have introduced a powerful (Steiner) symmetrization procedure specifically attached to the Gaussian framework, with which he provided a direct independent proof of the Gaussian isoperimetric inequality (along the standard symmetrization scheme). Specifically, given a Borel set $A$ in $\mathbb{R}^n$, and $u$ a direction vector, define the (Gaussian) symmetrized set $A^*$ (in the direction $u$) such that, for any $x \in (\mathbb{R}u)^\perp$, $A^* \cap (x + \mathbb{R}u) = (-\infty, a]$ where $a \in [-\infty, +\infty]$ is given by

$$Φ(a) = γ_1(A \cap (x + \mathbb{R}u)).$$
Then $\gamma(A^*) = \gamma(A)$, and the task is to show that symmetrization decreases the boundary measure $\gamma^+(A^*) \leq \gamma^+(A)$. For infinitely many directions $u$, the resulting symmetrized set is a half-space.

3 Kernel rearrangement inequality

For Borel sets $A, B$ in $\mathbb{R}^n$, and $t > 0$, set

$$K_t(A, B) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_A(x) 1_B(e^{-t}x + \sqrt{1 - e^{-2t}} y) d\gamma(x) d\gamma(y).$$

It has been shown by C. Borell [8], using the Gaussian symmetrization technology of [13, 14], that, whenever $H$ is a half-space with the same Gaussian measure as a Borel set $A$, then

$$K_t(A, A) \leq K_t(H, H). \tag{7}$$

A heat flow argument of this inequality is provided in [26], extended in a diffusion process picture in [15]. It is shown in [20, 21] that, for any Borel set $A$ and any $t > 0$,

$$\gamma(A) - K_t(A, A) = K_t(A, A^c) \leq \frac{\arccos(e^{-t})}{\sqrt{2\pi}} \gamma^+(A),$$

and that, if $H$ is a half-space,

$$\lim_{t \to 0} \frac{\sqrt{2\pi}}{\arccos(e^{-t})} K_t(H, H^c) = \gamma^+(H).$$

Combined with (7), the latter yields that $\gamma^+(A) \geq \gamma^+(H)$ whenever $\gamma(A) = \gamma(H)$, that is the Gaussian isoperimetric inequality.

4 Brunn-Minkowski inequality

In [13], A. Ehrhard discovered, using Gaussian symmetrization, an improved form of the Brunn-Minkowski inequality for Gaussian measures

$$\Phi^{-1}(\gamma(\theta A + (1 - \theta) B)) \geq \theta \Phi^{-1}(\gamma(A)) + (1 - \theta) \Phi^{-1}(\gamma(B)) \tag{8}$$

for any $\theta \in [0, 1]$ and any convex bodies $A, B$ in $\mathbb{R}^n$. This inequality has been extended to the case of only one convex body in [19], and finally to all Borel sets in [9] by pde methods$^1$.

The inequality (8) applied to $B$ the Euclidean ball with center the origin and radius $\frac{r}{1-\theta}$ yields (5) as $\theta \to 1$.

$^1$New recent proofs include [30, 18, 27].
5 Limit of a two-point inequality

In [5], S. Bobkov showed that for any smooth function \( f : \mathbb{R}^n \to [0, 1] \),

\[
\mathcal{I}\left( \int_{\mathbb{R}^n} f \, d\gamma \right) \leq \int_{\mathbb{R}^n} \sqrt{\mathcal{I}(f)^2 + |\nabla f|^2} \, d\gamma. \tag{9}
\]

Applied to a (smooth) approximation of \( f = 1_A \), this inequality yields (2).

The proof of (9) in [5] is based on the two-point inequality

\[
\mathcal{I}\left( \frac{a+b}{2} \right) \leq \frac{1}{2} \sqrt{\mathcal{I}(a)^2 + \frac{1}{2}|a-b|^2} + \frac{1}{2} \sqrt{\mathcal{I}(b)^2 + \frac{1}{2}|a-b|^2}
\]

for all \( a, b \in [0, 1] \), and a tensorization argument and the central limit theorem. The stability by product of the functional inequality (9) is indeed a main feature (being true for \( n = 1 \), it holds for any dimension \( n \)).

6 Heat flow monotonicity

A direct heat flow proof of Bobkov’s inequality (9) has been presented in [1]. Let

\[
p_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{1}{4}\pi |x|^2}, \quad t > 0, \quad x \in \mathbb{R}^n,
\]

be the standard heat kernel, fundamental solution of the heat equation \( \partial_t p_t = \Delta p_t \). The convolution semigroup \( P_t f(x) = f * p_t(x) \), \( t > 0 \), solves \( \partial_t P_t f = \Delta P_t f \) with initial data \( f \).

At \( t = \frac{1}{2} \), \( p_t \) is just the standard Gaussian density so that \( P_{\frac{1}{2}} f(0) = \int_{\mathbb{R}^n} f \, d\gamma \) (while \( P_0 f = f \)). In order to verify (9), it suffices therefore to show that, for a smooth function \( f : \mathbb{R}^n \to [0, 1] \), (at any point),

\[
P_s\left( \sqrt{\mathcal{I}(P_{\frac{1}{2}-s}f)^2 + 2s|\nabla P_{\frac{1}{2}-s}f|^2} \right), \quad s \in [0, \frac{1}{2}],
\]

is increasing, which is simply achieved taking its derivative (cf. [1]). A martingale proof along the same line, which includes extensions to path (Wiener) spaces, is provided in [4, 11].

7 Geometric measure theory

A proof of the Gaussian isoperimetric inequality relying on geometric measure theory is presented in the note by F. Morgan [24], with the suitable version of the Heinze-Karcher
inequality on weighted manifolds. This inequality provides an upper bound on the volume of a one-sided neighborhood of a hypersurface in terms of its mean curvature and the Ricci curvature of the ambient manifold. In Gauss space, it yields

$$\frac{\gamma(A)}{\gamma^+(S)} \leq \frac{\gamma(H)}{\gamma^+(H)}$$

where $S$ is a minimizing hypersurface enclosing a set $A$ with $\gamma(A) = \gamma(H)$. See also E. Milman [23], relying on regularity of isoperimetric minimizers, both in the interior and on the boundary, as emphasized in the early work by M. Gromov [17].

8 Deficit

A stronger version of the isoperimetric inequality examines lower bounds on the deficit

$$\gamma^+(A) - \gamma(H^+)$$

in terms of a functional measuring the proximity of a half-space $H = H_u = \{x \in \mathbb{R}^n; \langle x, u \rangle \leq h\}$ such as $\gamma(H_u) = \gamma(A)$, with the Borel set $A$. First steps in this investigation involved a geometric analysis with the Ehrhard symmetrization [12], and a study of the deficit in the kernel rearrangement inequality (7) [25, 26, 15]. A variational method is developed by M. Barchiesi, A. Brancolini and V. Julin [3] providing sharp bounds on the deficit. These authors introduce a technique which is based on an analysis of the first and the second variation conditions of solutions to a suitable minimization problem, providing a direct proof of the sharp deficit bound

$$\gamma^+(A) - \gamma(H^+) \geq c(\gamma(A)) \sqrt{\inf_{u \in S^{n-1}} \gamma(A \Delta H_u)}$$

(where $c(\gamma(A)) > 0$ only depends on the measure of $A$).

9 Extension to strongly log-concave measures

The Gauss space and measure is a model example (of positive curvature and infinite dimension in the language of [2]) to which other examples may be compared. A most natural and famous instance is the case of a probability measure $d\mu = e^{-V}dx$ on $\mathbb{R}^n$ whose potential $V: \mathbb{R}^n \to \mathbb{R}$ is more convex than the quadratic potential, that is $V(x) - \frac{c}{2}|x|^2$, $x \in \mathbb{R}^n$, is convex for some $c > 0$. A main result in this setting is that the isoperimetric profile $\mathcal{I}_\mu$ of $\mu$ is bounded from below by the Gaussian one. That is, if

$$\mathcal{I}_\mu(s) = \inf \{\mu^+(A); \mu(A) = s\}, \quad s \in [0, 1],$$

where the infimum is running over all Borel sets $A$ in $\mathbb{R}^n$ (and with a definition of $\mu^+(A)$ similar to $\gamma^+(A)$), then

$$I_{\mu} \geq \sqrt{c} I.$$  \hspace{1cm} (10)

The property (10) has been established in [1] by the heat flow monotonicity method (Section 6). A proof using needle decomposition has been proposed in [6]. A celebrated contraction principle in optimal transport by L. Caffarelli [10], expressing that $\mu$ is the $\frac{1}{\sqrt{c}}$-Lipschitz image of $\gamma$, produces a neat and direct proof of (10) (although not saying anything on the Gaussian case itself). The geometric measure theory approach outlined in Section 7 covers the framework of weighted Riemannian manifolds with (generalized) curvature bounded from below by a positive constant, also covered by the heat flow argument (cf. [1, 2]).

References


