Admissible shift, 
reproducing kernel Hilbert space, 
and abstract Wiener space

The standard Gaussian measure \( \gamma_n \), with density \( \frac{1}{(2\pi)^n} e^{-\frac{1}{2} |x|^2} \), \( x \in \mathbb{R}^n \) with respect to the Lebesgue measure on \( \mathbb{R}^n \), is not translation invariant. Shifted measures are described by

\[
\gamma_n(B + h) = e^{-\frac{1}{2} |h|^2} \int_B e^{-(h, x)} d\gamma_n \tag{1}
\]

where \( B + h = \{ x + h; x \in B \} \), \( B \) Borel set in \( \mathbb{R}^n \) and \( h \in \mathbb{R}^n \). In other words, the shifted measure \( \gamma_n(\cdot + h) \) by an element \( h \in \mathbb{R}^n \) is absolutely continuous with respect to \( \gamma_n \), with density \( e^{-\frac{1}{2} |h|^2 - (h, \cdot)} \).

Let now \( \mu \) be the Wiener measure on the Borel sets of the Banach space \( C([0, 1]) \) of real continuous functions on \([0, 1]\), law of a standard Brownian motion or Wiener process \( W = (W(t))_{t \in [0, 1]} \). It is not entirely clear to give a meaning to the preceding translation formula in this infinite-dimensional context, and in particular to make sense of \( |h|^2 \) and \( (h, \cdot) \). An early result of H. Cameron and W. Martin [7] answers this question in the following form. If (and only if) \( h : [0, 1] \rightarrow \mathbb{R} \) is absolutely continuous on \([0, 1]\), with almost everywhere derivative \( h' \) in \( L^2([0, 1]) \) (for the Lebesgue measure), the shifted measure \( \mu(\cdot + h) \) is absolutely continuous with respect to \( \mu \), with density

\[
\exp \left( -\frac{1}{2} \int_0^1 h'(t)^2 dt - \int_0^1 h'(t)dW(t) \right),
\]

where \( \int_0^1 h'(t)dW(t) \) is understood as a Wiener (-Itô) integral.
This translation formula actually entails some basic features associated to the Wiener measure, namely the so-called Cameron-Martin Hilbert space of absolutely continuous functions on $[0, 1]$ with almost everywhere derivative $h'$ in $L^2([0, 1])$, and the Wiener integral $\int_0^1 h'(t) dW(t)$. These objects are in fact only generated by the covariance function of $W$, $\mathbb{E}(W(s)W(t)) = s \wedge t$, $s, t \in [0, 1]$, and give rise to the specific structure consisting of the space $C([0, 1])$, with its topology, the Cameron-Martin, or reproducing kernel, Hilbert space, and the Wiener measure.

This structure, called abstract Wiener space, may be built for any Gaussian measure (on a Banach space for example), and the text below develops the construction in a rather general setting. While the exposition might appear somewhat abstract, it only relies on some standard functional analysis and is not any longer or difficult than it would be for a specific model like the Wiener space. It covers besides, in a most instructive way, several examples of interest, even finite-dimensional. In addition, it naturally puts forward series representations in orthonormal bases of the reproducing kernel Hilbert space (like the trigonometric or Haar expansions of Brownian motion), a most useful property to transfer, in applications, dimension-free statements from finite to infinite-dimensional Gaussian measures and vectors.

The note is mainly extracted from [12]. Some main expositions on Gaussian measures, vectors, processes, in infinite-dimensional spaces are [16, 4, 11, 13, ?, 8, 10, 5, 15]...

Table of contents

1. Gaussian measure and random vector
2. Wiener space factorization
3. Reproducing kernel Hilbert space
4. Gaussian process
5. Abstract Wiener space
6. Series representation
7. Cameron-Martin translation formula

References
1 Gaussian measure and random vector

It is classical that the Lebesgue measure $\lambda_n$ does not extend to an infinite-dimensional setting. However, Gaussian measures, due in particular to their dimension-free features, may easily be considered in infinite-dimensional spaces. A prototype, and central, example is the Wiener measure, with associated Brownian or Wiener process, on the Banach space $C([0,1])$ of continuous functions on the interval $[0,1]$.

A Gaussian measure $\mu$ on a real separable Banach space $E$ equipped with its Borel $\sigma$-algebra $\mathcal{B}$, and with norm $\| \cdot \|$, is a Borel probability measure on $(E, \mathcal{B})$ such that the law of each continuous linear functional on $E$ is Gaussian. Equivalently, a random variable, or vector, $X$ on some probability space $(\Omega, \mathcal{A}, P)$ with values in $(E, \mathcal{B})$ is Gaussian if its law, on the Borel sets of $E$, is Gaussian, that is, for every element $\xi$ of the dual space $E^*$ of $E$, $\langle \xi, X \rangle$ is a real Gaussian variable.

By separability of $\mathcal{B}$, the distribution of $X$ may also be described by the finite-dimensional distributions of the random process $\langle \xi, X \rangle$, $\xi \in E^*$, and therefore by the covariance operator

$$
E(\langle \xi, X \rangle \langle \zeta, X \rangle) = \int_E \langle \xi, x \rangle \langle \zeta, x \rangle d\mu(x), \quad \xi, \zeta \in E^*
$$

(for $\mu$ the law of $X$). As such, all the standard properties of finite-dimensional Gaussian random vectors extend to this infinite-dimensional setting.

The infinite dimensional setting may be extended to locally convex vector spaces [6], but for simplicity, the exposition here is limited to Banach spaces.

Throughout the note, only centered Gaussian measures and vectors are considered, without further notice.

2 Wiener space factorization

Let $\mu$ be a Gaussian measure on $(E, \mathcal{B})$. As $E$ is separable, $\mu$ is a Radon measure in the sense that, for every $B \in \mathcal{B}$,

$$
\mu(B) = \sup \{ \mu(K); K \subset B, K \text{ compact in } E \}.
$$

It is known from the integrability properties of norms of Gaussian random vectors (cf. [1]), that

$$
\sigma = \sup_{\xi \in E^*, \| \xi \| \leq 1} \left( \int_E \langle \xi, x \rangle^2 d\mu(x) \right)^{1/2} < \infty,
$$

(2)
and actually
\[ \int_E \|x\|^p d\mu(x) < \infty \quad \text{for every } p > 0. \]  

(3)

The abstract Wiener space factorization of the Gaussian measure $\mu$ on $(E, \mathcal{B})$ is given by
\[ E^* \xrightarrow{j} L^2(\mu) \xrightarrow{j^*} E, \]
where $j$ is the injection map from $E^*$ into $L^2(\mu) = L^2(E, \mathcal{B}, \mu; \mathbb{R})$ (i.e. $j(\xi) = \langle \xi, \cdot \rangle \in L^2(\mu)$), the dual map $j^*$ of $j$ mapping $L^2(\mu)$ into $E$ (rather than the bi-dual). Indeed, by the integrability property (3), for any element $\varphi$ of $L^2(\mu)$, the integral $\int_E x \varphi(x) d\mu(x)$ is defined, as an element of $E$, in the strong sense since
\[ \int_E \|x\| \varphi(x) d\mu(x) \leq \left( \int_E \|x\|^2 d\mu(x) \right)^{1/2} \left( \int_E |\varphi|^2 d\mu \right)^{1/2} < \infty. \]

Now, for every $\xi \in E^*$,
\[ \langle j(\xi), \varphi \rangle_{L^2(\mu)} = \int_E \langle \xi, x \rangle \varphi(x) d\mu(x) = \langle \xi, \int_E x \varphi(x) d\mu(x) \rangle \]
so that $j^*(\varphi) = \int_E x \varphi(x) d\mu(x) \in E$.

3 Reproducing kernel Hilbert space

The reproducing kernel Hilbert space $\mathcal{H}$ of $\mu$ is defined as the subspace $j^*(L^2(\mu))$ of $E$. By the preceding, its elements are of the form $\int_E x \varphi(x) d\mu(x)$ with $\varphi \in L^2(\mu)$. This description induces a natural scalar product on $\mathcal{H}$ via the covariance of $\mu$ by
\[ \langle j^*(\varphi), j^*(\psi) \rangle_{\mathcal{H}} = \langle \varphi, \psi \rangle_{L^2(\mu)}, \quad \varphi, \psi \in L^2(\mu). \]

Since $j(E^*) = \text{Ker}(j^*)$, $j^*$ restricted to the closure $E_2^*$ of $E^*$ in $L^2(\mu)$ is linear and bijective onto $\mathcal{H}$. For simplicity in the notation, set below for $h \in \mathcal{H}$,
\[ \tilde{h} = (j^*|_{E_2^*})^{-1}(h) \in E_2^* \subset L^2(\mu). \]

Under $\mu$, $\tilde{h}$ is Gaussian with variance $|h|_{\mathcal{H}}^2$.

Note that $\sigma$ of (2) is then also $\sup_{x \in K} \|x\|$ where $K$ is the closed unit ball of $\mathcal{H}$ for this Hilbert space scalar product. In particular, for every $x$ in $\mathcal{H}$,
\[ \|x\| \leq \sigma |x|_{\mathcal{H}} \]
where \( |x|_H = \langle x, x \rangle_H^{1/2} \). Moreover, \( K \) is a compact subset of \( E \). Indeed, if \((\xi_n)_{n \in \mathbb{N}}\) is a sequence in the unit ball of \( E^* \), there is a subsequence \((\xi_{n'})_{n' \in \mathbb{N}}\) which converges weakly to some \( \xi \) in \( E^* \). Now, since the \( \xi_n \)'s are Gaussian under \( \mu \), \( \xi_{n'} \to \xi \) in \( L^2(\mu) \) so that \( j \) is a compact operator. Hence \( j^* \) is also a compact operator, from which the compactness of \( K \) follows.

The terminology “reproducing kernel” stems from the fact that an element \( \varphi \in L^2(\mu) \) is reproduced, by duality, from the covariance kernel of \( \mu \) as

\[
\int_E \varphi \psi \, d\mu = K(\varphi, \psi)
\]

where \( \psi \) is running through \( L^2(\mu) \). A further illustration of this property in the context of Gaussian processes is provided below.

It is useful to visualize the preceding abstract construction on a number of basic examples. For \( \gamma_n \) the canonical Gaussian measure on \( \mathbb{R}^n \) (equipped with an arbitrary norm), it is plain that \( H = \mathbb{R}^n \) with its Euclidean structure, and \( K \) is the Euclidean (closed) unit ball \( B(0, 1) \).

If \( X \) is a Gaussian vector on \( \mathbb{R}^n \) with non-degenerate covariance matrix \( \Sigma = M^\top M \), the unit ball \( K \) of the reproducing kernel Hilbert space associated to the distribution of \( X \) is the ellipsoid \( M(B(0, 1)) \).

An infinite dimensional version of \( \gamma_n \) might consist of an infinite sequence \( (Y_n)_{n \in \mathbb{N}} \) of independent standard normal random variables (on some probability space \((\Omega, \mathcal{A}, \mathbb{P})\)). This sequence does not belong almost surely to the Hilbert space \( \ell^2 \) of square summable sequences, but as soon as \((a_n)_{n \in \mathbb{N}}\) is a (deterministic) sequence in \( \ell^2 \), the new Gaussian sequence \((a_n Y_n)_{n \in \mathbb{N}}\) belongs to \( E = \ell^2 \), and its law \( \mu \) defines an abstract Wiener space \((E, H, \mu)\) with reproducing kernel Hilbert space \( H \) given by the infinite-dimensional ellipsoid consisting of the sequences \((b_n)_{n \in \mathbb{N}}\) such that \((\frac{b_n}{a_n})_{n \in \mathbb{N}}\) belongs to \( \ell^2 \) (assuming the \( a_n \)'s different from zero).

Another illustrative, infinite-dimensional, example is the classical Wiener space associated with Brownian motion, say on \([0, 1]\) and with real values for simplicity (cf. [2]). Let thus \( E \) be the Banach space \( C([0, 1]) \) of all real continuous functions \( x \) on \([0, 1]\) equipped with the uniform norm (the Wiener space), and let \( \mu \) be the distribution of a standard Brownian motion, or Wiener process, \( W = (W(t))_{t \in [0, 1]} \) starting at the origin (the Wiener measure). The dual space of \( C([0, 1]) \) is the space of signed measures on \([0, 1]\), and if \( m \) and \( m' \) are finitely supported measures on \([0, 1]\), \( m = \sum_i c_i \delta_{t_i}, \; c_i, t_i \in \mathbb{R}, \; t_i \in [0, 1], \; m' = \sum_j c'_j \delta_{t'_j}, \; c'_j \in \mathbb{R}, \)
\[ t' \in [0, 1], \]

\[ \int_E \langle m, x \rangle \langle m', x \rangle d\mu(x) = \mathbb{E}(\langle m, W \rangle \langle m', W \rangle) \]
\[ = \sum_{i, j} c_i c'_j \mathbb{E}(W(t_i)W(t'_j)) \]
\[ = \sum_{i, j} c_i c'_j (t_i \wedge t'_j) \]

by definition of the covariance of Brownian motion. It follows that the element \( h = j^* j(m) = \int_E x \langle m, x \rangle d\mu(x) \) of \( \mathcal{H} \) is the map \( h : t \in [0, 1] \mapsto \sum_i c_i (t_i \wedge t) \). This map is absolutely continuous, with almost everywhere derivative \( h' \) satisfying
\[
\int_0^1 h'(t)^2 dt = \int_0^1 \left| \sum_i c_i \mathbb{1}_{[0, t_i]} \right|^2 dt
\]
\[ = \int_0^1 \sum_{i, j} c_i c_j \mathbb{1}_{[0, t_i]} \mathbb{1}_{[0, t_j]} dt
\]
\[ = \sum_{i, j} c_i c_j (t_i \wedge t_j) = \int_E \langle m, x \rangle^2 d\mu(x) = |h|_{\mathcal{H}}^2. \]

By a standard extension, the reproducing kernel Hilbert space \( \mathcal{H} \) associated to the Wiener measure \( \mu \) on \( E \) may then be identified with the Cameron-Martin Hilbert space [7] of the absolutely continuous elements \( h \) of \( C([0, 1]) \) such that \( \int_0^1 h'(t)^2 dt < \infty \). Moreover, if \( h \in \mathcal{H} \),
\[
\tilde{h} = (j^*_{|E_2^1})^{-1}(h) = \int_0^1 h'(t) dW(t)
\]
as a Wiener (-Itô) integral, defining a Gaussian random variable with mean zero and variance \( \int_0^1 h'(t)^2 dt \).

While the Wiener space \( C([0, 1]) \) is equipped here with the uniform topology, other choices are possible. Let \( F \) be a separable Banach space such that the Wiener process \( W \) belongs almost surely to \( F \). Using probabilistic notation, the previous abstract Wiener space theory indicates that if \( \varphi \) is a real valued random variable, on a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \), with \( \mathbb{E}(\varphi^2) < \infty \), then \( h = \mathbb{E}(W \varphi) \in F \). Since \( \mathbb{P}(W \in F \cap C([0, 1])) = 1 \), it immediately follows that the Cameron-Martin Hilbert space may be identified with a subset of \( F \), and is also the reproducing kernel Hilbert space of the Wiener measure on \( F \). Examples of subspaces \( F \) include the Lebesgue spaces \( L^p([0, 1]) \), \( 1 \leq p < \infty \), or the Hölder spaces with exponent \( \alpha \), \( 0 < \alpha < \frac{1}{2} \), given by
\[
\|x\|_\alpha = \sup_{0 \leq s \neq t \leq 1} \frac{|x(s) - x(t)|}{|s - t|^{\alpha}}, \quad x \in C([0, 1]).
\]
4 Gaussian process

The construction of the reproducing kernel Hilbert space $\mathcal{H}$ of the law of a Gaussian random vector with values in a Banach space may be, at least formally, extended to the setting of Gaussian processes. By definition, a Gaussian process $X = (X_t)_{t \in T}$, on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, indexed by a parameter set $T$, is a random process such that any finite-dimensional vector $(X_{t_1}, \ldots, X_{t_n})$, $t_1, \ldots, t_n \in T$, is a Gaussian vector in $\mathbb{R}^n$. The finite-dimensional distributions of the process $X = (X_t)_{t \in T}$ are therefore fully determined by the covariance function $\Sigma(s, t) = \mathbb{E}(X_s X_t)$, $s, t \in T$. As for the Brownian motion, the associated reproducing kernel Hilbert space $\mathcal{H}$ is the span of the functions $s \mapsto \Sigma(s, \cdot)$, $t \in T$, with scalar product $\langle h, k \rangle_{\mathcal{H}} = \sum_{i,j} c_i d_j \Sigma(s_i, t_j)$ whenever $h = \sum_i c_i \Sigma(s_i, \cdot)$, for a finite collection of $c_i \in \mathbb{R}$, $s_i \in T$, and similarly $k = \sum_j d_j \Sigma(\cdot, t_j)$, and

$$\mathbb{E}\left(\left|\sum_i c_i X_{s_i}\right|^2\right) = \langle h, h \rangle_{\mathcal{H}}.$$  

5 Abstract Wiener space

In the preceding context of a Gaussian measure $\mu$ on a Banach space $E$ with reproducing kernel Hilbert space $\mathcal{H}$, the triple $(E, \mathcal{H}, \mu)$ is called, following L. Gross [9], an abstract Wiener space.

A dual point of view, starting from a given Hilbert space, more commonly used by analysts on Wiener spaces, may be emphasized (cf. [11] for further details). Let $\mathcal{H}$ be a real separable Hilbert space with norm $|\cdot|_{\mathcal{H}}$ and let $e_1, e_2, \ldots$ be an orthonormal basis of $\mathcal{H}$. Define a simple additive measure $\nu$ on the cylinder sets in $\mathcal{H}$ by

$$\nu(x \in \mathcal{H}; (\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle) \in B) = \gamma_n(B)$$

for all Borel sets $B$ in $\mathbb{R}^n$. Let $\| \cdot \|$ be a measurable semi-norm on $\mathcal{H}$, and denote by $E$ the completion of $\mathcal{H}$ with respect to $\| \cdot \|$. Then $(E, \| \cdot \|)$ is a real separable Banach space. If $\xi \in E^*$, consider $\xi_{\mathcal{H}} : \mathcal{H} \to \mathbb{R}$ that is identified with an element $h$ in $\mathcal{H} = \mathcal{H}^*$ (in the preceding language, $h = j^* j(\xi)$). Let then $\mu$ be the ($\sigma$-additive) extension of $\nu$ on the Borel sets of $E$. In particular, the distribution of $\xi \in E^*$ under $\mu$ is Gaussian with mean zero and variance $|h|^2_{\mathcal{H}}$. Therefore, $\mu$ is a Gaussian Radon measure on $E$ with reproducing kernel Hilbert space $\mathcal{H}$, and $(E, \mathcal{H}, \mu)$ is an abstract Wiener space. With respect to this
approach, the abstract Wiener space construction of the preceding sections focuses more on the Gaussian measure.

6 Series representation

The next property is a useful series representation of Gaussian random vectors which can efficiently be used to transfer (dimension-free) properties from finite-dimensional to infinite-dimensional Gaussian measures. The Cameron-Martin translation formula (see the next section) may for example be approached in this way. Another illustration is the extension of the isoperimetric inequality to infinite-dimensional Gaussian measures (cf. [3]).

The result puts besides forward the fundamental Gaussian measurable structure consisting of the canonical Gaussian product measure on $\mathbb{R}^N$ with reproducing kernel Hilbert space $\ell^2$.

**Theorem 1.** Let $(E, \mathcal{H}, \mu)$ a Wiener triple, $(e_k)_{k \geq 1}$ an orthonormal basis of $\mathcal{H}$, and $(g_k)_{k \geq 1}$ a sequence of independent real standard normal variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then the series $X = \sum_{k=1}^{\infty} g_k e_k$ converges in $E$ almost surely and in every $L^p$, and is distributed according to $\mu$.

In the example of the Wiener measure on the space $E = C([0, 1])$ of continuous functions on $[0, 1]$, any orthonormal basis $(h_k)_{k \geq 1}$ of $L^2([0, 1])$ for the Lebesgue measure, gives rise to a Schauder basis

$$e_k(t) = \int_0^t h_k(s)ds, \quad t \in [0, 1], \; k \geq 1,$$

of $E = C([0, 1])$ to which the preceding Theorem 1 applies. Now, in this concrete example, specific bases $(h_k)_{k \geq 1}$ are of interest, such as the trigonometric or Haar bases. Each of them actually provides a simple approach to continuity of the Brownian paths (cf. [2]).

Theorem 1 actually entails a somewhat more precise statement. Since $\mu$ is a Radon measure, the space $L^2(\mu)$ is separable and the closure $E_2^*$ of $E^*$ in $L^2(\mu)$ consists of Gaussian random variables on the probability space $(E, \mathcal{B}, \mu)$. Let $(g_k)_{k \geq 1}$ denote an orthonormal basis of $E_2^*$, and set $e_k = j^*(g_k), \; k \geq 1$. Then $(e_k)_{k \geq 1}$ defines a complete orthonormal system in $\mathcal{H}$, and $(g_k)_{k \geq 1}$ is a sequence on $(E, \mathcal{B}, \mu)$ of independent standard Gaussian random variables.

A proof of Theorem 1 may, for example, be obtained from a vector valued-martingale convergence theorem (although a direct approach in many specific situations is often easier to apprehend). Here are some details. Recall that $\int_E \|x\|^p d\mu(x) < \infty$ for every $p > 0$. Denote by $\mathcal{B}_n$ the $\sigma$-algebra generated by $g_1, \ldots, g_n$. It is easily seen that the conditional expectation of the identity map on $(E, \mu)$ with respect to $\mathcal{B}_n$ is equal to $X_n = \sum_{k=1}^{n} g_k e_k$. 

8
By the vector-valued martingale convergence theorem, see [17], the series \( X = \sum_{k=1}^{\infty} g_k e_k \) converges almost surely and in any \( L^p \) -space. Since moreover \( e_k = \int_{E} x \varphi_x d\mu \), \( k \geq 1 \), where \( (\varphi_x)_{k \geq 1} \) is an orthonormal basis of \( L^2(\mu) \) (by the reproducing kernel property),

\[
\mathbb{E}(\langle \xi, X \rangle^2) = \sum_{k=1}^{\infty} \langle \xi, e_k \rangle^2 = \sum_{k=1}^{\infty} \left( \int_{E} \langle \xi, x \rangle \varphi_x d\mu \right)^2 = \int_{E} \langle \xi, x \rangle^2 d\mu(x)
\]

for every \( \xi \) in \( E^* \), so that \( X \) has law \( \mu \), and the last claim follows.

As a consequence of this series representation, it may be deduced that the closure \( \overline{\mathcal{H}} \) of \( \mathcal{H} \) in \( E \) coincides with the support of \( \mu \) (for the topology given by the norm on \( E \)), a property that shows the coherence of the abstract Wiener space construction.

### 7 Cameron-Martin translation formula

After the preceding somewhat lengthy developments, this last section addresses the translation formula for infinite-dimensional Gaussian measures. Actually, the series representation in an orthonormal basis of the reproducing kernel Hilbert space may be used to access the Cameron-Martin translation formula discussed in the introduction from its finite-dimensional version (cf. e.g. [5, 7]).

**Theorem 2** (The Cameron Martin formula). On an abstract Wiener space \((E, \mathcal{H}, \mu)\), for any \( h \) in \( \mathcal{H} \), the shifted probability measure \( \mu(\cdot + h) \) is absolutely continuous with respect to \( \mu \), with density given by the formula

\[
\mu(B + h) = e^{-\frac{1}{2}h^2} \int_B e^{-\tilde{h}} d\mu
\]

for every Borel set \( B \) in \( E \), where it is recalled that \( \tilde{h} = (j^*|_{E^2})^{-1}(h) \).

As developed first in [7], it takes an explicit form on the standard Wiener space. Namely, for \( h \in \mathcal{H} \), \( \tilde{h} = (j^*|_{E^2})^{-1}(h) = \int_0^1 h'(t) dW(t) \), so that if \( \mu \) is the Wiener measure on \( E = C([0,1]) \), the shifted measure \( \mu(\cdot + h) \) has density

\[
\exp \left( -\frac{1}{2} \int_0^1 h'(t)^2 dt - \int_0^1 h'(t) dW(t) \right)
\]

with respect to \( \mu \).

### References


