Admissible shift, reproducing kernel Hilbert space, and abstract Wiener space

The standard Gaussian measure $\gamma_n$, with density $\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}|x|^2}$, $x \in \mathbb{R}^n$ with respect to the Lebesgue measure on $\mathbb{R}^n$, is not translation invariant. Shifted measures are described by

$$\gamma_n(B + h) = e^{-\frac{1}{2}|h|^2} \int_B e^{-\langle h, x \rangle} d\gamma_n$$  \hspace{1cm} (1)

where $B + h = \{x + h; x \in B\}$, $B$ Borel set in $\mathbb{R}^n$ and $h \in \mathbb{R}^n$. In other words, the shifted measure $\gamma_n(\cdot + h)$ by an element $h \in \mathbb{R}^n$ is absolutely continuous with respect to $\gamma_n$, with density $e^{-\frac{1}{2}|h|^2 - \langle h, \cdot \rangle}$.

Let now $\mu$ be the Wiener measure on the Borel sets of the Banach space $C([0,1])$ of real continuous functions on $[0,1]$, law of a standard Brownian motion or Wiener process $W = (W(t))_{t \in [0,1]}$. It is not entirely clear to give a meaning to the preceding translation formula in this infinite-dimensional context, and in particular to make sense of $|h|^2$ and $\langle h, \cdot \rangle$. An early result of H. Cameron and W. Martin [7] answers this question in the following form. If (and only if) $h : [0,1] \to \mathbb{R}$ is absolutely continuous on $[0,1]$, with almost everywhere derivative $h'$ in $L^2([0,1])$ (for the Lebesgue measure), the shifted measure $\mu(\cdot + h)$ is absolutely continuous with respect to $\mu$, with density

$$\exp \left( -\frac{1}{2} \int_0^1 h'(t)^2 \, dt - \int_0^1 h'(t) dW(t) \right),$$

where $\int_0^1 h'(t) dW(t)$ is understood as a Wiener (Itô) integral.
This translation formula actually entails some basic features associated to the Wiener measure, namely the so-called Cameron-Martin Hilbert space of absolutely continuous functions on $[0, 1]$ with almost everywhere derivative $h'$ in $L^2([0, 1])$, and the Wiener integral $\int_0^1 h'(t)dW(t)$. These objects are in fact only generated by the covariance function of $W$, $\mathbb{E}(W(s)W(t)) = s \land t$, $s, t \in [0, 1]$, and give rise to the specific structure consisting of the space $C([0, 1])$, with its topology, the Cameron-Martin, or reproducing kernel, Hilbert space, and the Wiener measure.

This structure, called *abstract Wiener space*, may be built for any Gaussian measure (on a Banach space for example), and the text below develops the construction in a rather general setting. While the exposition might appear somewhat abstract, it only relies on some standard functional analysis and is not any longer or difficult than it would be for a specific model like the Wiener space. It covers besides, in a most instructive way, several examples of interest, even finite-dimensional. In addition, it naturally puts forward series representations in orthonormal bases of the reproducing kernel Hilbert space (like the trigonometric or Haar expansions of Brownian motion), a most useful property to transfer, in applications, dimension-free statements from finite to infinite-dimensional Gaussian measures and vectors.

The note is mainly extracted from [12]. Some main expositions on Gaussian measures, vectors, processes, in infinite-dimensional spaces are [16, 4, 11, 13, 14, 8, 10, 5, 18, 19, 15]...

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References
1 Gaussian measure and random vector

It is classical that the Lebesgue measure $\lambda_n$ does not extend to an infinite-dimensional setting. However, Gaussian measures, due in particular to their dimension-free features, may easily be considered in infinite-dimensional spaces. A prototype, and central, example is the Wiener measure, with associated Brownian or Wiener process, on the Banach space $C([0,1])$ of continuous functions on the interval $[0,1]$.

A Gaussian measure $\mu$ on a real separable Banach space $E$ equipped with its Borel $\sigma$-algebra $\mathcal{B}$, and with norm $\|\cdot\|$, is a Borel probability measure on $(E,\mathcal{B})$ such that the law of each continuous linear functional on $E$ is Gaussian. Equivalently, a random variable, or vector, $X$ on some probability space $(\Omega,\mathcal{A},\mathbb{P})$ with values in $(E,\mathcal{B})$ is Gaussian if its law, on the Borel sets of $E$, is Gaussian, that is, for every element $\xi$ of the dual space $E^*$ of $E$, $\langle \xi, X \rangle$ is a real Gaussian variable.

By separability of $\mathcal{B}$, the distribution of $X$ may also be described by the finite-dimensional distributions of the random process $\langle \xi, X \rangle$, $\xi \in E^*$, and therefore by the covariance operator

$$\mathbb{E}(\langle \xi, X \rangle \langle \zeta, X \rangle) = \int_E \langle \xi, x \rangle \langle \zeta, x \rangle d\mu(x), \quad \xi, \zeta \in E^*$$

(for $\mu$ the law of $X$). As such, all the standard properties of finite-dimensional Gaussian random vectors extend to this infinite-dimensional setting.

The infinite dimensional setting may be extended to locally convex vector spaces [6], but for simplicity, the exposition here is limited to Banach spaces.

Throughout the note, only centered Gaussian measures and vectors are considered, without further notice.

2 Wiener space factorization

Let $\mu$ be a Gaussian measure on $(E,\mathcal{B})$. As $E$ is separable, $\mu$ is a Radon measure in the sense that, for every $B \in \mathcal{B}$,

$$\mu(B) = \sup \{ \mu(K); K \subset B, K \text{ compact in } E \}.$$

It is known from the integrability properties of norms of Gaussian random vectors (cf. [1]), that

$$\sigma = \sup_{\xi \in E^*, \|\xi\| \leq 1} \left( \int_E \langle \xi, x \rangle^2 d\mu(x) \right)^{1/2} < \infty,$$

(2)
and actually
\[ \int_E \|x\|^p d\mu(x) < \infty \quad \text{for every } p > 0. \] (3)

The abstract Wiener space factorization of the Gaussian measure \( \mu \) on \((E, \mathcal{B})\) is given by
\[ E^* \xrightarrow{j} L^2(\mu) \xrightarrow{j^*} E, \]
where \( j \) is the injection map from \( E^* \) into \( L^2(\mu) = L^2(E, \mathcal{B}, \mu; \mathbb{R}) \) (i.e. \( j(\xi) = \langle \xi, \cdot \rangle \in L^2(\mu) \)), the dual map \( j^* \) of \( j \) mapping \( L^2(\mu) \) into \( E \) (rather than the bi-dual). Indeed, by the
integrability property (3), for any element \( \varphi \) of \( L^2(\mu) \), the integral \( \int_E x \varphi(x) d\mu(x) \) is defined, as an element of \( E \), in the strong sense since
\[ \int_E \|x\| \varphi(x) |d\mu(x) \leq \left( \int_E \|x\|^2 d\mu(x) \right)^{1/2} \left( \int_E |\varphi|^2 d\mu \right)^{1/2} < \infty. \]
Now, for every \( \xi \in E^* \),
\[ \langle j(\xi), \varphi \rangle_{L^2(\mu)} = \int_E \langle \xi, x \rangle \varphi(x) d\mu(x) = \left\langle \xi, \int_E x \varphi(x) d\mu(x) \right\rangle \]
so that \( j^*(\varphi) = \int_E x \varphi(x) d\mu(x) \in E \).

3 Reproducing kernel Hilbert space

The reproducing kernel Hilbert space \( \mathcal{H} \) of \( \mu \) is defined as the subspace \( j^*(L^2(\mu)) \) of \( E \). By the preceding, its elements are of the form \( \int_E x \varphi(x) d\mu(x) \) with \( \varphi \in L^2(\mu) \). This description induces a natural scalar product on \( \mathcal{H} \) via the covariance of \( \mu \) by
\[ \langle j^*(\varphi), j^*(\psi) \rangle_{\mathcal{H}} = \langle \varphi, \psi \rangle_{L^2(\mu)}, \quad \varphi, \psi \in L^2(\mu). \]

Since \( j(E^*) \perp = \text{Ker}(j^*) \), \( j^* \) restricted to the closure \( E_2^* \) of \( E^* \) in \( L^2(\mu) \) is linear and bijective onto \( \mathcal{H} \). For simplicity in the notation, set below for \( h \in \mathcal{H} \),
\[ \tilde{h} = (j^*|_{E_2^*})^{-1}(h) \in E_2^* \subset L^2(\mu). \]
Under \( \mu \), \( \tilde{h} \) is Gaussian with variance \( |h|_{\mathcal{H}}^2 \).

Note that \( \sigma \) of (2) is then also \( \sup_{x \in \mathcal{K}} \|x\| \) where \( \mathcal{K} \) is the closed unit ball of \( \mathcal{H} \) for this
Hilbert space scalar product. In particular, for every \( x \) in \( \mathcal{H} \),
\[ \|x\| \leq \sigma |x|_{\mathcal{H}} \]
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where \(|x|_H = \langle x, x \rangle_H^{1/2}\). Moreover, \(\mathcal{K}\) is a compact subset of \(E\). Indeed, if \((\xi_n)_{n \in \mathbb{N}}\) is a sequence in the unit ball of \(E^*\), there is a subsequence \((\xi_{n'})_{n' \in \mathbb{N}}\) which converges weakly to some \(\xi\) in \(E^*\). Now, since the \(\xi_n\)'s are Gaussian under \(\mu\), \(\xi_{n'} \to \xi\) in \(L^2(\mu)\) so that \(j\) is a compact operator. Hence \(j^*\) is also a compact operator, from which the compactness of \(\mathcal{K}\) follows.

The terminology “reproducing kernel” stems from the fact that an element \(\varphi \in L^2(\mu)\) is reproduced, by duality, from the covariance kernel of \(\mu\) as

\[
\int_E \varphi \psi \, d\mu = K(\varphi, \psi)
\]

where \(\psi\) is running through \(L^2(\mu)\). A further illustration of this property in the context of Gaussian processes is provided below.

It is useful to visualize the preceding abstract construction on a number of basic examples.

For \(\gamma_n\) the canonical Gaussian measure on \(\mathbb{R}^n\) (equipped with an arbitrary norm), it is plain that \(\mathcal{H} = \mathbb{R}^n\) with its Euclidean structure, and \(\mathcal{K}\) is the Euclidean (closed) unit ball \(B(0,1)\).

If \(X\) is a Gaussian vector on \(\mathbb{R}^n\) with non-degenerate covariance matrix \(\Sigma = M^\top M\), the unit ball \(\mathcal{K}\) of the reproducing kernel Hilbert space associated to the distribution of \(X\) is the ellipsoid \(M(B(0,1))\).

An infinite dimensional version of \(\gamma_n\) might consist of an infinite sequence \((Y_n)_{n \in \mathbb{N}}\) of independent standard normal random variables (on some probability space \((\Omega, \mathcal{A}, \mathbb{P})\)). This sequence does not belong almost surely to the Hilbert space \(\ell^2\) of square summable sequences, but as soon as \((a_n)_{n \in \mathbb{N}}\) is a (deterministic) sequence in \(\ell^2\), the new Gaussian sequence \((a_n Y_n)_{n \in \mathbb{N}}\) belongs to \(E = \ell^2\), and its law \(\mu\) defines an abstract Wiener space \((E, \mathcal{H}, \mu)\) with reproducing kernel Hilbert space \(\mathcal{H}\) given by the infinite-dimensional ellipsoid consisting of the sequences \((b_n)_{n \in \mathbb{N}}\) such that \((\frac{b_n}{a_n})_{n \in \mathbb{N}}\) belongs to \(\ell^2\) (assuming the \(a_n\)'s different from zero).

Another illustrative, infinite-dimensional, example is the classical Wiener space associated with Brownian motion, say on \([0, 1]\) and with real values for simplicity (cf. [2]). Let thus \(E\) be the Banach space \(C([0, 1])\) of all real continuous functions \(x\) on \([0, 1]\) equipped with the uniform norm (the Wiener space), and let \(\mu\) be the distribution of a standard Brownian motion, or Wiener process, \(W = (W(t))_{t \in [0,1]}\) starting at the origin (the Wiener measure). The dual space of \(C([0, 1])\) is the space of signed measures on \([0, 1]\), and if \(m\) and \(m'\) are finitely supported measures on \([0, 1]\), \(m = \sum_i c_i \delta_{t_i}, c_i \in \mathbb{R}, t_i \in [0, 1]\), \(m' = \sum_j c'_j \delta_{t'_j}, c'_j \in \mathbb{R}\),
by definition of the covariance of Brownian motion. It follows that the element $h = j^* j(m) = \int_E x^t_m x d\mu(x)$ of $H$ is the map $h : t \in [0, 1] \mapsto \sum_i c_i (t_i \wedge t)$. This map is absolutely continuous, with almost everywhere derivative $h'$ satisfying

\[
\int_0^1 h'(t)^2 dt = \int_0^1 \left| \sum_i c_i 1_{[0,t_i]} \right|^2 dt
\]

\[
= \int_0^1 \sum_{i,j} c_i c_j 1_{[0,t_i]} 1_{[0,t_j]} dt
\]

\[
= \sum_{i,j} c_i c_j (t_i \wedge t_j) = \int_E (m,x)^2 d\mu(x) = |h|^2_{H_2}.
\]

By a standard extension, the reproducing kernel Hilbert space $H$ associated to the Wiener measure $\mu$ on $E$ may then be identified with the Cameron-Martin Hilbert space \([7]\) of the absolutely continuous elements $h$ of $C([0,1])$ such that $\int_0^1 h'(t)^2 dt < \infty$. Moreover, if $h \in H$,

\[
\tilde{h} = (j^*_{H_2})^{-1}(h) = \int_0^1 h'(t) dW(t)
\]

as a Wiener (-Itô) integral, defining a Gaussian random variable with mean zero and variance $\int_0^1 h'(t)^2 dt$.

While the Wiener space $C([0,1])$ is equipped here with the uniform topology, other choices are possible. Let $F$ be a separable Banach space such that the Wiener process $W$ belongs almost surely to $F$. Using probabilistic notation, the previous abstract Wiener space theory indicates that if $\varphi$ is a real valued random variable, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}(\varphi^2) < \infty$, then $h = \mathbb{E}(W \varphi) \in F$. Since $\mathbb{P}(W \in F \cap C([0,1])) = 1$, it immediately follows that the Cameron-Martin Hilbert space may be identified with a subset of $F$, and is also the reproducing kernel Hilbert space of the Wiener measure on $F$. Examples of subspaces $F$ include the Lebesgue spaces $L^p([0,1])$, $1 \leq p < \infty$, or the Hölder spaces with exponent $\alpha$, $0 < \alpha < \frac{1}{2}$, given by

\[
\|x\|_\alpha = \sup_{0 \leq s \neq t \leq 1} \frac{|x(s) - x(t)|}{|s-t|^\alpha}, \quad x \in C([0,1]).
\]
4 Gaussian process

The construction of the reproducing kernel Hilbert space \( H \) of the law of a Gaussian random vector with values in a Banach space may be, at least formally, extended to the setting of Gaussian processes. By definition, a Gaussian process \( X = (X_t)_{t \in T} \), on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\), indexed by a parameter set \( T \), is a random process such that any finite-dimensional vector \((X_{t_1}, \ldots, X_{t_n})\), \( t_1, \ldots, t_n \in T \), is a Gaussian vector in \( \mathbb{R}^n \). The finite-dimensional distributions of the process \( X = (X_t) \), on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\), indexed by a parameter set \( T \), are therefore fully determined by the covariance function \( \Sigma(s, t) = \mathbb{E}(X_s X_t) \), \( s, t \in T \). As for the Brownian motion, the associated reproducing kernel Hilbert space \( H \) is the span of the functions \( s \mapsto \Sigma(s, t) \), \( t \in T \), with scalar product
\[
\langle h, k \rangle_H = \sum_{i,j} c_i d_j \Sigma(s_i, t_j)
\]
whenever \( h = \sum_i c_i \Sigma(s_i, \cdot) \), for a finite collection of \( c_i \in \mathbb{R} \), \( s_i \in T \), and similarly \( k = \sum_j d_j \Sigma(\cdot, t_j) \), and
\[
\mathbb{E}\left( \left| \sum_i c_i X_{s_i} \right|^2 \right) = \langle h, h \rangle_H.
\]

5 Abstract Wiener space

In the preceding context of a Gaussian measure \( \mu \) on a Banach space \( E \) with reproducing kernel Hilbert space \( H \), the triple
\[
(E, H, \mu)
\]
is called, following L. Gross [9], an abstract Wiener space.

A dual point of view, starting from a given Hilbert space, more commonly used by analysts on Wiener spaces, may be emphasized (cf. [11] for further details). Let \( H \) be a real separable Hilbert space with norm \( \| \cdot \|_H \) and let \( e_1, e_2, \ldots \) be an orthonormal basis of \( H \). Define a simple additive measure \( \nu \) on the cylinder sets in \( H \) by
\[
\nu(x \in H; (\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle) \in B) = \gamma_n(B)
\]
for all Borel sets \( B \) in \( \mathbb{R}^n \). Let \( \| \cdot \| \) be a measurable semi-norm on \( H \), and denote by \( E \) the completion of \( H \) with respect to \( \| \cdot \| \). Then \((E, \| \cdot \|)\) is a real separable Banach space. If \( \xi \in E^* \), consider \( \xi|_H : H \to \mathbb{R} \) that is identified with an element \( h \) in \( H = H^* \) (in the preceding language, \( h = j^* j(\xi) \)). Let then \( \mu \) be the (\( \sigma \)-additive) extension of \( \nu \) on the Borel sets of \( E \). In particular, the distribution of \( \xi \in E^* \) under \( \mu \) is Gaussian with mean zero and variance \( |h|^2_H \). Therefore, \( \mu \) is a Gaussian Radon measure on \( E \) with reproducing kernel Hilbert space \( H \), and \((E, H, \mu)\) is an abstract Wiener space. With respect to this
approach, the abstract Wiener space construction of the preceding sections focuses more on the Gaussian measure.

6 Series representation

The next property is a useful series representation of Gaussian random vectors which can efficiently be used to transfer (dimension-free) properties from finite-dimensional to infinite-dimensional Gaussian measures. The Cameron-Martin translation formula (see the next section) may for example be approached in this way. Another illustration is the extension of the isoperimetric inequality to infinite-dimensional Gaussian measures (cf. [3]).

The result puts besides forward the fundamental Gaussian measurable structure consisting of the canonical Gaussian product measure on \( \mathbb{R}^N \) with reproducing kernel Hilbert space \( \ell^2 \).

**Theorem 1.** Let \((E, \mathcal{H}, \mu)\) a Wiener triple, \((e_k)_{k \geq 1}\) an orthonormal basis of \( \mathcal{H} \), and \((g_k)_{k \geq 1}\) a sequence of independent real standard normal variables on some probability space \((\Omega, \mathcal{A}, \mathbb{P})\). Then the series \( X = \sum_{k=1}^{\infty} g_k e_k \) converges in \( E \) almost surely and in every \( L^p \), and is distributed according to \( \mu \).

In the example of the Wiener measure on the space \( E = C([0, 1]) \) of continuous functions on \([0, 1]\), any orthonormal basis \((h_k)_{k \geq 1}\) of \( L^2([0, 1]) \) for the Lebesgue measure, gives rise to a Schauder basis

\[
e_k(t) = \int_0^t h_k(s) \, ds, \quad t \in [0, 1], \; k \geq 1,
\]

of \( E = C([0, 1]) \) to which the preceding Theorem 1 applies. Now, in this concrete example, specific bases \((h_k)_{k \geq 1}\) are of interest, such as the trigonometric or Haar bases. Each of them actually provides a simple approach to continuity of the Brownian paths (cf. [2]).

Theorem 1 actually entails a somewhat more precise statement. Since \( \mu \) is a Radon measure, the space \( L^2(\mu) \) is separable and the closure \( E_2^* \) of \( E^* \) in \( L^2(\mu) \) consists of Gaussian random variables on the probability space \((E, \mathcal{B}, \mu)\). Let \((g_k)_{k \geq 1}\) denote an orthonormal basis of \( E_2^* \), and set \( e_k = j^*(g_k) \), \( k \geq 1 \). Then \((e_k)_{k \geq 1}\) defines a complete orthonormal system in \( \mathcal{H} \), and \((g_k)_{k \geq 1}\) is a sequence on \((E, \mathcal{B}, \mu)\) of independent standard Gaussian random variables.

A proof of Theorem 1 may, for example, be obtained from a vector valued-martingale convergence theorem (although a direct approach in many specific situations is often easier to apprehend). Here are some details. Recall that \( \int_E \|x\|^p d\mu(x) < \infty \) for every \( p > 0 \). Denote by \( \mathcal{B}_n \) the \( \sigma \)-algebra generated by \( g_1, \ldots, g_n \). It is easily seen that the conditional expectation of the identity map on \((E, \mu)\) with respect to \( \mathcal{B}_n \) is equal to \( X_n = \sum_{k=1}^{n} g_k e_k \).
By the vector-valued martingale convergence theorem, see [17], the series $X = \sum_{k=1}^{\infty} g_\kappa e_\kappa$ converges almost surely and in any $L^p$-space. Since moreover $e_\kappa = \int_E x \varphi_\kappa d\mu$, $k \geq 1$, where $(\varphi_\kappa)_{\kappa \geq 1}$ is an orthonormal basis of $L^2(\mu)$ (by the reproducing kernel property),

$$E(\langle \xi, X \rangle^2) = \sum_{k=1}^{\infty} \langle \xi, e_\kappa \rangle^2 = \sum_{k=1}^{\infty} \left( \int_E \langle \xi, x \rangle \varphi_\kappa d\mu \right)^2 = \int_E \langle \xi, x \rangle^2 d\mu(x)$$

for every $\xi$ in $E^*$, so that $X$ has law $\mu$, and the last claim follows.

As a consequence of this series representation, it may be deduced that the closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ in $E$ coincides with the support of $\mu$ (for the topology given by the norm on $E$), a property that shows the coherence of the abstract Wiener space construction.

## 7 Cameron-Martin translation formula

After the preceding somewhat lengthy developments, this last section addresses the translation formula for infinite-dimensional Gaussian measures. Actually, the series representation in an orthonormal basis of the reproducing kernel Hilbert space may be used to access the Cameron-Martin translation formula discussed in the introduction from its finite-dimensional version (cf. e.g. [5, 14]).

**Theorem 2** (The Cameron Martin formula). On an abstract Wiener space $(E, \mathcal{H}, \mu)$, for any $h$ in $\mathcal{H}$, the shifted probability measure $\mu(\cdot + h)$ is absolutely continuous with respect to $\mu$, with density given by the formula

$$\mu(B + h) = e^{-\frac{1}{2}h_\mathcal{H}^2} \int_B e^{-\tilde{h}} d\mu$$

for every Borel set $B$ in $E$, where it is recalled that $\tilde{h} = (j^*|_{E^*_2})^{-1}(h)$.

As developed first in [7], it takes an explicit form on the standard Wiener space. Namely, for $h \in \mathcal{H}$, $\tilde{h} = (j^*|_{E^*_2})^{-1}(h) = \int_0^1 h'(t)dW(t)$, so that if $\mu$ is the Wiener measure on $E = C([0, 1])$, the shifted measure $\mu(\cdot + h)$ has density

$$\exp \left( -\frac{1}{2} \int_0^1 h'(t)^2 dt - \int_0^1 h'(t)dW(t) \right)$$

with respect to $\mu$.

## References


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