Proofs of the
Gaussian isoperimetric inequality
( as of 2017)

The note reviews the known proofs of the isoperimetric inequality for Gaussian measures (as of 2017 – any relevant informations and references on omitted or new further proofs are welcome, and will be incorporated).

Let $\gamma = \gamma_n$ be the standard Gaussian probability measure on the Borel sets of $\mathbb{R}^n$, with density $\varphi_n(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}|x|^2}$, $x \in \mathbb{R}^n$, with respect to the Lebesgue measure. Denote by $\Phi(t) = \int_{-\infty}^t \varphi_1(x)dx$, $t \in \mathbb{R}$, the (continuous, strictly increasing) distribution function in dimension one, and define then the Gaussian isoperimetric profile

$$I(s) = \varphi_1 \circ \Phi^{-1}(s), \quad s \in [0,1].$$

(1)

The function $I$ is symmetric along the vertical line $s = \frac{1}{2}$, and such that $I(0) = I(1) = 0$. It is worthwhile observing that $I(s) \sim s \sqrt{2 \log \left( \frac{1}{s} \right)}$ as $s \to 0$. 
Given \( r > 0 \), \( A_r = \{ x \in \mathbb{R}^n; \inf_{a \in A} |x - a| \leq r \} \) is the (closed) \( r \)-neighborhood of a set \( A \) in \( \mathbb{R}^n \). The (Gaussian) outer Minkowski content of Borel set \( A \) is defined as

\[
\gamma^+(A) = \liminf_{r \to 0} \frac{1}{r} [\gamma(A_r) - \gamma(A)].
\]

**Theorem** [The Gaussian isoperimetric inequality] For any Borel set \( A \) in \( \mathbb{R}^n \),

\[
\gamma^+(A) \geq I(\gamma(A)).
\]  

Equality is achieved on the half-spaces \( H = \{ x \in \mathbb{R}^n; \langle x, u \rangle \leq h \} \) where \( u \) is a unit vector and \( h \in \mathbb{R} \).

The measure of a half-space is computed in dimension one, \( \gamma(H) = \Phi(h) \), and its boundary measure is

\[
\gamma^+(H) = \liminf_{r \to 0} \frac{1}{r} [\Phi(h + r) - \Phi(h)] = \varphi_1(h).
\]

The Gaussian isoperimetric inequality thus expresses equivalently that, if \( H \) is a half-space such that \( \Phi(h) = \gamma(H) = \gamma(A) \), then

\[
\gamma^+(A) \geq \gamma^+(H),
\]  

and half-spaces are the extremal sets of the Gaussian isoperimetric problem.

Integrating along the neighborhoods, (3) is equivalently formulated as

\[
\gamma(A_r) \geq \gamma(H_r), \quad r > 0,
\]  

provided that \( \gamma(A) = (\geq) \gamma(H) \), or

\[
\Phi^{-1}(\gamma(A_r)) \geq \Phi^{-1}(\gamma(A)) + r, \quad r > 0
\]
(since $\gamma(H_r) = \Phi(h + r)$).

Linear (affine) transformations yield the isoperimetric statement for any Gaussian measure. The dimension-free character allows furthermore for an infinite-dimensional formulation on an abstract Wiener space $(E, \mathcal{H}, \mu)$, developed first in [7], as

$$\Phi^{-1}(\mu(A + rK)) \geq \Phi^{-1}(\mu(A)) + r, \quad r \geq 0,$$

where $K$ is the unit ball of the reproducing kernel Hilbert space $\mathcal{H}$ (cf. [21]). (Here $A + rK = \{a + rh; a \in A, h \in K\}$, which, in $\mathbb{R}^n$, amounts to $A_r$ for $K$ the Euclidean unit ball.)

The following sections briefly present the various known proofs of the Gaussian isoperimetric inequality.

## 1 Limit of spherical isoperimetry

In the neighborhood formulation, the isoperimetric inequality for the (normalized) uniform measure $\sigma_N$ on the $N$-sphere $S^N$ in $\mathbb{R}^{N+1}$, due to P. Lévy [22] and E. Schmidt [28], expresses that whenever $A$ is a Borel set in $S^N$, and $B$ a spherical cap (geodesic ball) such that $\sigma_N(A) \geq \sigma_N(B)$, then

$$\sigma_N(A_r) \geq \sigma_N(B_r)$$

for any $r \geq 0$, where $A_r$ is the $r$-neighborhood of $A$ in the geodesic metric.

It is a folklore result, usually quoted as “Poincaré’s lemma”, that the normalized uniform measure on the sphere $\sqrt{N}S^N$, when projected on a $n$-dimensional subspace, converges as $N \to \infty$ to the standard $n$-dimensional Gaussian measure (cf. e.g. [21]). Via this limit, V. Sudakov and B. Tsirel’son [29], and C. Borell [7], independently, put forward the Gaussian isoperimetric inequality from the corresponding one on the sphere, the extremal spherical caps turning into half-spaces.

## 2 Gaussian symmetrization

Classical proofs of the isoperimetric inequality on the sphere use symmetrization techniques (see e.g. [16]). It is the contribution of A. Ehrhard [13] to have introduced a powerful (Steiner) symmetrization procedure specifically attached to the Gaussian framework, with which he provided a direct independent proof of the Gaussian isoperimetric inequality (along the standard symmetrization scheme). Specifically, given a Borel set $A$ in $\mathbb{R}^n$, and $u$ a direction
vector, define the (Gaussian) symmetrized set $A^*$ (in the direction $u$) such that, for any $x \in (\mathbb{R}u)^\perp$, $A^* \cap (x + \mathbb{R}u) = (-\infty, a]$ where $a \in [-\infty, +\infty]$ is given by

$$\Phi(a) = \gamma_1(A \cap (x + \mathbb{R}u)).$$

Then $\gamma(A^*) = \gamma(A)$, and the task is to show that symmetrization decreases the boundary measure $\gamma^+(A^*) \leq \gamma^+(A)$. For infinitely many directions $u$, the resulting symmetrized set is a half-space.

### 3 Kernel rearrangement inequality

For Borel sets $A, B$ in $\mathbb{R}^n$, and $t > 0$, set

$$K_t(A, B) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_A(x)1_B(e^{-t}x + \sqrt{1-e^{-2t}}y)d\gamma(x)d\gamma(y).$$

It has been shown by C. Borell [8], using the Gaussian symmetrization technology of [13, 14], that, whenever $H$ is a half-space with the same Gaussian measure as a Borel set $A$, then

$$K_t(A, A) \leq K_t(H, H).$$

(7)

A heat flow argument of this inequality is provided in [26], extended in a diffusion process picture in [15]. It is shown in [20, 21] that, for any Borel set $A$ and any $t > 0$,

$$\gamma(A) - K_t(A, A) = K_t(A, A^c) \leq \frac{\text{arccos}(e^{-t})}{\sqrt{2\pi}} \gamma^+(A),$$

and that, if $H$ is a half-space,

$$\lim_{t \to 0} \frac{\sqrt{2\pi}}{\text{arccos}(e^{-t})} K_t(H, H^c) = \gamma^+(H).$$

Combined with (7), the latter yields that $\gamma^+(A) \geq \gamma^+(H)$ whenever $\gamma(A) = \gamma(H)$, that is the Gaussian isoperimetric inequality.

### 4 Brunn-Minkowski inequality

In [13], A. Ehrhard discovered, using Gaussian symmetrization, an improved form of the Brunn-Minkowski inequality for Gaussian measures

$$\Phi^{-1}(\gamma(\theta A + (1-\theta)B)) \geq \theta \Phi^{-1}(\gamma(A)) + (1-\theta) \Phi^{-1}(\gamma(B))$$

(8)
for any $\theta \in [0, 1]$ and any convex bodies $A, B$ in $\mathbb{R}^n$. This inequality has been extended to the case of only one convex body in [19], and finally to all Borel sets in [9] by pde methods\(^1\).

The inequality (8) applied to $B$ the Euclidean ball with center the origin and radius $\frac{r}{1-\theta}$ yields (5) as $\theta \to 1$.

5 Limit of a two-point inequality

In [5], S. Bobkov showed that for any smooth function $f : \mathbb{R}^n \to [0, 1],$

\[
\mathcal{I}\left( \int_{\mathbb{R}^n} f \, d\gamma \right) \leq \int_{\mathbb{R}^n} \sqrt{\mathcal{I}(f)^2 + \|
abla f\|^2} \, d\gamma. 
\]  

(9)

Applied to a (smooth) approximation of $f = \chi_A$, this inequality yields (2). This functional form is actually equivalent to (2) when considering the level sets of functions defined on $\mathbb{R}^{n+1}$.

The proof of (9) in [5] is based on the two-point inequality

\[
\mathcal{I}\left( \frac{a+b}{2} \right) \leq \frac{1}{2} \sqrt{\mathcal{I}(a)^2 + \frac{1}{2} |a-b|^2 + \frac{1}{2} \sqrt{\mathcal{I}(b)^2 + \frac{1}{2} |a-b|^2}}, 
\]

for all $a, b \in [0, 1]$, and a tensorization argument and the central limit theorem. The stability by product of the functional inequality (9) is indeed a main feature (being true for $n = 1$, it holds for any dimension $n$).

6 Heat flow monotonicity

A direct heat flow proof of Bobkov’s inequality (9) has been presented in [1]. Let

\[
p_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{1}{4t}|x|^2}, \quad t > 0, \quad x \in \mathbb{R}^n,
\]

be the standard heat kernel, fundamental solution of the heat equation $\partial_t p_t = \Delta p_t$. The convolution semigroup $P_t f(x) = f * p_t(x), \ t > 0,$ solves $\partial_t P_t f = \Delta P_t f$ with initial data $f$.

At $t = \frac{1}{2}$, $p_t$ is just the standard Gaussian density so that $P_{t} f(0) = \int_{\mathbb{R}^n} f \, d\gamma$ (while $P_0 f = f$). In order to verify (9), it suffices therefore to show that, for a smooth function $f : \mathbb{R}^n \to [0, 1], \ (at \ any \ point),$.

\[
P_s\left( \sqrt{\mathcal{I}(P_{\frac{1}{2}-s} f)^2 + 2s|\nabla P_{\frac{1}{2}-s} f|^2} \right), \quad s \in [0, \frac{1}{2}]
\]

is increasing, which is simply achieved taking its derivative (cf. [1]). A martingale proof along the same line, which includes extensions to path (Wiener) spaces, is provided in [4, 11].

\(^1\)New recent proofs include [30, 18, 27].
7 Geometric measure theory

A proof of the Gaussian isoperimetric inequality relying on geometric measure theory is presented in the note by F. Morgan [24], with the suitable version of the Heinze-Karcher inequality on weighted manifolds. This inequality provides an upper bound on the volume of a one-sided neighborhood of a hypersurface in terms of its mean curvature and the Ricci curvature of the ambient manifold. In Gauss space, it yields

\[
\frac{\gamma(A)}{\gamma^+(S)} \leq \frac{\gamma(H)}{\gamma^+(H)}
\]

where \(S\) is a minimizing hypersurface enclosing a set \(A\) with \(\gamma(A) = \gamma(H)\). See also E. Milman [23], relying on regularity of isoperimetric minimizers, both in the interior and on the boundary, as emphasized in the early work by M. Gromov [17].

8 Deficit

A stronger version of the isoperimetric inequality examines lower bounds on the deficit

\[
\gamma^+(A) - \gamma(H^+)
\]

in terms of a functional measuring the proximity of a half-space \(H = H_u = \{x \in \mathbb{R}^n; \langle x, u \rangle \leq h\}\) such as \(\gamma(H_u) = \gamma(A)\), with the Borel set \(A\). First steps in this investigation involved a geometric analysis with the Ehrhard symmetrization [12], and a study of the deficit in the kernel rearrangement inequality (7) [25, 26, 15]. A variational method is developed by M. Barchiesi, A. Brancolini and V. Julin [3] providing sharp bounds on the deficit. These authors introduce a technique which is based on an analysis of the first and the second variation conditions of solutions to a suitable minimization problem, providing a direct proof of the sharp deficit bound

\[
\gamma^+(A) - \gamma(H^+) \geq c(\gamma(A)) \sqrt{\inf_{u \in S^{n-1}} \gamma(A \Delta H_u)}
\]

(where \(c(\gamma(A)) > 0\) only depends on the measure of \(A\)).

9 Extension to strongly log-concave measures

The Gauss space and measure is a model example (of positive curvature and infinite dimension in the language of [2]) to which other examples may be compared. A most natural and famous instance is the case of a probability measure \(d\mu = e^{-V}dx\) on \(\mathbb{R}^n\) whose potential \(V : \mathbb{R}^n \to \mathbb{R}\) is more convex than the quadratic potential, that is \(V(x) - \frac{c}{2}|x|^2\), \(x \in \mathbb{R}^n\), is
convex for some $c > 0$. A main result in this setting is that the isoperimetric profile $I_\mu$ of $\mu$ is bounded from below by the Gaussian one. That is, if

$$I_\mu(s) = \inf \{ \mu^+(A) ; \mu(A) = s \}, \quad s \in [0,1],$$

where the infimum is running over all Borel sets $A$ in $\mathbb{R}^n$ (and with a definition of $\mu^+(A)$ similar to $\gamma^+(A)$), then

$$I_\mu \geq \sqrt{c} I.$$

(10)

The property (10) has been established in [1] by the heat flow monotonicity method (Section 6). A proof using needle decomposition has been proposed in [6]. A celebrated contraction principle in optimal transport by L. Caffarelli [10], expressing that $\mu$ is the $\frac{1}{\sqrt{c}}$-Lipschitz image of $\gamma$, produces a neat and direct proof of (10) (although not saying anything on the Gaussian case itself). The geometric measure theory approach outlined in Section 7 covers the framework of weighted Riemannian manifolds with (generalized) curvature bounded from below by a positive constant, also covered by the heat flow argument (cf. [1, 2]).

References


