Region-based theories of space

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2. Algebras of regions, models and representation theory
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Historical excursion

Points

Point-based theories of space
  ▶ Classical Euclidean geometry
  ▶ Topology

Point
  ▶ Simplest spatial entity
  ▶ Without dimension and internal structure
Historical excursion

Topology

\[ T = (X, \mathcal{O}) \]

- \( X \): nonempty set of “points”
- \( \mathcal{O} \): set of subsets of \( X \) such that
  - \( \emptyset \in \mathcal{O} \)
  - \( X \in \mathcal{O} \)
  - if \( u, v \in \mathcal{O} \) then \( u \cap v \in \mathcal{O} \)
  - if \( u_i \in \mathcal{O} \) for each \( i \in I \) then \( \bigcup \{u_i : x \in I\} \in \mathcal{O} \)
Historical excursion

Topology

Let $T = (X, \mathcal{O})$ be a topology

Open

- subset $u$ of $X$ such that $u \in \mathcal{O}$

Closed

- subset $u$ of $X$ such that $X \setminus u \in \mathcal{O}$
Let $T = (X, \mathcal{O})$ be a topology

Interior of $u \subseteq X$: $\text{In}(u)$
- greatest open set contained in $u$

Closure of $u \subseteq X$: $\text{Cl}(u)$
- least closed set containing $u$
Historical excursion
Bodies

Idea: develop an alternative theory of space
▶ Basic notion
  ▶ Solid bodies
▶ Basic relations
  ▶ “One solid is part of another solid”
  ▶ “Two solids overlap”
  ▶ “One solid touches another solid”

Abstract philosophical disciplines
▶ Ontology
  ▶ Theory of “existent”
▶ Mereology
  ▶ Theory of “part-whole” relations
Historical excursion

Mereology

Founders of mereology

▶ Leśnewski (1927–1931)
▶ Tarski (1927)

Pointless approach

▶ De Laguna (1922)
  ▶ “Point, line, and surface, as sets of solids”
▶ Whitehead (1929)
  ▶ “Process and Reality”
  ▶ Pointless geometry
Historical excursion
De Laguna (1922) and Whitehead (1929)

De Laguna (1922)
- Solids
- Ternary relation between solids
  - “x connects y with z”

Whitehead (1929)
- Regions
- Binary relation between regions
  - “x is connected with y”
Historical excursion
Whitehead (1929)

Whitehead (1929)

- Regions
- Binary relation between regions
  \( xCy \): “\( x \) is in contact with \( y \)”
- Mereological relations
  \( \subseteq \): “Part-of”
  \( O \): “Overlap”
  \( C^{\text{ext}} \): “External contact”
  \( \lesssim \): “Tangential inclusion”
  \( \ll \): “Nontangential inclusion”
Historical excursion
Whitehead (1929)

Some formal properties of the contact relation

\((W1)\): \(\forall x \ xCx\)

\((W2)\): \(\forall x \ \forall y \ (xCy \to yCx)\)

\((W3)\): \(\forall x \ \forall y \ (x = y \leftrightarrow \forall z \ (xCz \leftrightarrow yCz))\)

Some other Whitehead’s spatial relations between regions

“Part-of”: \(x \leq y ::= \forall z \ (xCz \to yCz)\)

“Overlap”: \(xOy ::= \exists z \ (z \leq x \land z \leq y)\)

“External contact”: \(xC_{\text{ext}}y ::= xCy \land x\bar{O}y\)

“Tangential inclusion”: \(x \leq^{\circ} y ::= x \leq y \land \exists z \ (zC_{\text{ext}}x \land zC_{\text{ext}}y)\)

“Nontangential inclusion”: \(x \ll y ::= x \leq y \land x \leq^{\circ} y\)
Abstractive set

- Set $\alpha$ of regions such that
  - $\alpha$ is totally ordered by the nontangential inclusion
  - There is no region included in every element of $\alpha$

Covering relation

- An abstractive set $\alpha$ covers an abstractive set $\beta$ if every region in $\alpha$ contains a region in $\beta$

Equivalence relation

- $\alpha \equiv \beta$ provided $\alpha$ covers $\beta$ and $\beta$ covers $\alpha$
Historical excursion
Whitehead (1929)

Geometrical element

- Class of equivalence modulo \( \equiv \)

Incidence relation

- Order relation induced by the covering relation in the set of all geometrical elements

Point

- Minimal (with respect to the incidence relation) geometrical element
Historical excursion
Tarski (1927) and Grzegorczyk (1960)

Tarski (1927)
- Spheres
- Binary relation between spheres
  - “x is part of y”

Grzegorczyk (1960)
- Bodies
- Binary relations between bodies
  - Part-of
  - Separation
Historical excursion
Grzegorczyk (1960)

Grzegorczyk’s pointless geometry \((R, \leq, C)\)

\((G0)\): \((R, \leq)\) is a mereological field, i.e. a complete Boolean algebra with deleted zero element

\((G1)\): \(C\) is a reflexive relation in \(R\)

\((G2)\): \(C\) is a symmetric relation in \(R\)

\((G3)\): \(C\) is monotone with respect to \(\leq\) in the sense that we have: \(x \leq y \rightarrow \forall z \in R (xCz \rightarrow yCz)\)

Relation \(\ll\) of nontangential inclusion

- Defined in the same way as by Whitehead
Historical excursion
Grzegorczyk (1960)

Representative of a point in \((R, \leq, C)\)

- Set \(p\) of regions such that
  1. \(p\) has no minimum and is totally ordered by the relation \(\ll\)
  2. Given two regions \(x\) and \(y\), if we have \(zOx\) and \(zOy\) for every \(z \in p\), then \(xCy\)

Point in \((R, \leq, C)\)

- Filter \(P\) in \(R\) generated by a representative of a point
- \(P\) belongs to a region \(x\) if \(x\) is a member of \(P\)

Notations

\(\Pi\): Set of all points of \((R, \leq, C)\)
\(\pi(x)\): Set of all points of a region \(x\)
Grzegorczyk’s pointless geometry \((R, \leq, C)\)

\((G0)\): \((R, \leq)\) is a mereological field, i.e. a complete Boolean algebra with deleted zero element

\((G1)\): \(C\) is a reflexive relation in \(R\)

\((G2)\): \(C\) is a symmetric relation in \(R\)

\((G3)\): \(C\) is monotone with respect to \(\leq\) in the sense that we have: \(x \leq y \rightarrow \forall z \in R (xCz \rightarrow yCz)\)

Two additional axioms are introduced

\((G4)\): Every region has at least one point

\((G5)\): If \(xCy\), then \(x\) and \(y\) overlap with every member of \(P\) for some point \(P\)
Historical excursion
Grzegorczyk (1960)

Let \( T = (X, \mathcal{O}) \) be a topology

Regular open
- subset \( u \) of \( X \) such that \( \text{In}(\text{Cl}(u)) = u \)

Regular closed
- subset \( u \) of \( X \) such that \( \text{Cl}(\text{In}(u)) = u \)
Historical excursion
Grzegorczyk (1960)

Let $T = (X, \mathcal{O})$ be a topology

The set $RO(T)$ of all regular open subsets of $T$ constitutes a Boolean algebra

$0_T = \emptyset$, $1_T = X$

$-T u = ln(X \setminus u)$

$u \oplus_T v = ln(Cl(u \cup v))$, $u \otimes_T v = u \cap v$

The set $RC(T)$ of all regular closed subsets of $T$ constitutes a Boolean algebra

$0_T = \emptyset$, $1_T = X$

$-T u = Cl(X \setminus u)$

$u \oplus_T v = u \cup v$, $u \otimes_T v = Cl(ln(u \cap v))$
Theorem (Grzegorczyk)

Let $T = (X, \mathcal{O})$ be a Hausdorff topological space and $R = RO(T)$ be the set of all its nonempty regular-open sets. For any $x, y \in R$, we set $xCy$ if and only if $Cl(x) \cap Cl(y) \neq \emptyset$. Then $(R, \subseteq, C)$ satisfies $(G0)$–$(G3)$. If every point of $X$ is the intersection of a decreasing (with respect to $\ll$) family of open sets, then axioms $(G4)$ and $(G5)$ are also satisfied.

Theorem (Grzegorczyk)

Suppose that $(R, \leq, C)$ satisfies $(G0)$–$(G5)$. Let $\mathcal{O}$ be the topology in $\Pi$ generated by the set $\{\pi(x): x \in R\}$. Then $\{\pi(x): x \in R\}$ coincides with the set of all nonempty regular-open sets of $(\Pi, \mathcal{O})$ and $\pi$ is an isomorphism.
The notions of a point used by Whitehead and Grzegorczyk are equivalent when

\((G6)\): If \(x \ll y\), then \(x \ll z\) and \(z \ll y\) for some \(z \in R\)

\((G6')\): If \(x \bar{C} y\), then \(x \bar{C} z\) and \(z \star \bar{C} y\) for some \(z \in R\)

Unpleasant feature of Grzegorczyk’s system

- It is not a first-order system
Historical excursion
Clarke (1981, 1985)

Clarke’s system \((R, C)\)

\[(C1): \forall x \ xCx\]

\[(C2): \forall x \ \forall y \ (xCy \rightarrow yCx)\]

\[(C3): \forall x \ \forall y \ (x = y \leftrightarrow \forall z \ (xCz \leftrightarrow yCz))\]

\[(C4): \text{If } A \text{ is a nonempty subset of } R, \text{ then } C(x) = \bigcup\{C(y): y \in A\} \text{ for some } x \in R\]

Points

- Certains subsets \(P\) of \(R\) satisfying some closure conditions

Additional axiom

\[(C5): \text{If } xCy \text{ then } x, y \in P \text{ for some point } P\]
Historical excursion
Clarke (1981, 1985)

The relations of contact and overlap used by Clarke are equivalent

- \( xCy \) if and only if \( xOy \)

Unpleasant feature of Clarke’s system

- It is not a first-order system
Historical excursion
Qualitative spatial reasoning

Interval logic for reasoning about space
- Pointless approach to the theory of space
- Ontological primitives
  - Physical objects
  - Regions
- Primitive relations
  - Part-of
  - Contact

Important systems
- Boolean contact algebras
- Polygonal plane mereotopology
- Region connection calculus


Bibliography


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Contact algebra $\mathcal{B} = (B, 0, 1, \cdot, +, \star, C)$

(DiVa0): $(B, 0, 1, \cdot, +, \star)$ is a Boolean algebra with $\star$ as the Boolean complement

(DiVa1): $xCy$ implies $x \neq 0$ and $y \neq 0$

(DiVa2): $(x + y)Cz$ is equivalent to $xCz$ or $yCz$

(DiVa3): $xC(y + z)$ is equivalent to $xCy$ or $xCz$

(DiVa4): $x \neq 0$ implies $xCx$

(DiVa5): $xCy$ implies $yCx$

Relation $\ll$ of nontangential inclusion

$\triangleright x \ll y ::= x\bar{C}y^*$
Contact algebra of regular-closed sets

Let $P$ be a point-based topology. Let $\mathcal{RC}_P = (\mathcal{RC}_P, 0_P, 1_P, \cdot_P, +_P, \star_P, C_P)$ be the structure defined as follows:

- $(\mathcal{RC}_P, 0_P, 1_P, \cdot_P, +_P, \star_P)$ is the Boolean algebra of the regular-closed sets of $P$,
- $xC_P y$ iff $x \cap y \neq \emptyset$.

Then $\mathcal{RC}_P$ meets constraints $(DiVa1)$–$(DiVa5)$. Remark that

- $x \ll_P y$ iff $x \subseteq \text{In}_P(y)$.
Contact algebra of regular-open sets

Let $P$ be a point-based topology. Let $\mathcal{RO}_P = (\mathcal{RO}_P, 0_P, 1_P, \cdot_P, +_P, \star_P, C_P)$ be the structure defined as follows:

- $(\mathcal{RO}_P, 0_P, 1_P, \cdot_P, +_P, \star_P)$ is the Boolean algebra of the regular-open sets of $P$,
- $xC_Py$ iff $Cl_P(x) \cap Cl_P(y) \neq \emptyset$.

Then $\mathcal{RO}_P$ meets constraints $(DiVa1)$–$(DiVa5)$. Remark that

- $x \ll_P y$ iff $Cl_P(x) \subseteq y$. 
Algebras of regions, models and representation theory
Contact algebras: Dimov and Vakarelov (2006)

Let $\mathcal{B} = (B, 0, 1, \cdot, +, \ast, C)$ be a contact algebra

▶ Mereological relations $\leq$ and $O$ in $\mathcal{B}$

$\leq$: $x \leq y ::= x \cdot y = x$

$O$: $xOy ::= x \cdot y \neq 0$

▶ Mereotopological relation $Con$ in $\mathcal{B}$

$Con$: $Con(x) ::= \forall y \forall z (y \neq 0 \land z \neq 0 \land x = y + z \rightarrow yCz)$
Let $\mathcal{B} = (B, 0, 1, \cdot, +, \ast, C)$ be a contact algebra

- **$RCC$** – 8 basic mereotopological relations in $\mathcal{B}$ between two nonzero regions

  - **$DC$:** $DC(x, y) ::= x\bar{C}y$
  - **$EC$:** $EC(x, y) ::= xCy \land x\bar{O}y$
  - **$PO$:** $PO(x, y) ::= xOy \land x\preceq y \land y\preceq x$
  - **$TPP$:** $TPP(x, y) ::= x \preceq y \land x\ll y \land y\ll x$
  - **$TPPI$:** $TPPI(x, y) ::= y \ll x \land y\ll x \land x\ll y$
  - **$NTPP$:** $NTPP(x, y) ::= x \ll y \land y\ll x$
  - **$NTPPI$:** $NTPPI(x, y) ::= y \ll x \land x\ll y$
  - **$Id$:** $Id(x, y) ::= x \preceq y \land y \preceq x$
Let $\mathcal{B} = (B, 0, 1, \cdot, +, \ast, C)$ be a contact algebra

- $\mathcal{B}$ is connected:
  \[(Con): x \ll x \text{ implies } x = 0 \text{ or } x = 1\]

- $\mathcal{B}$ is extensional:
  \[(Ext): x \neq 1 \text{ implies } x \ll y \text{ for some } y \in B \text{ such that } y \neq 1\]

- $\mathcal{B}$ is normal:
  \[(Nor): x \ll y \text{ implies } x \ll z \text{ and } z \ll y \text{ for some } z \in B\]
Let $\mathcal{B} = (B, 0, 1, \cdot, +, \star, C)$ be a contact algebra

$\blacktriangleright \hspace{0.2cm}$ (Ext) is equivalent to each of the following axioms:

$(\text{Ext}')$: $x \leq y$ iff $xCz$ implies $yCz$ for every $z \in B$

$(\text{Ext}''')$: $x = y$ iff $xCz$ is equivalent to $yCz$ for every $z \in B$

$(\text{Ext}''''')$: $x \neq 0$ implies $y \ll x$ for some $y \in B$ such that $y \neq 0$
Algebras of regions, models and representation theory

Extensions of contact algebras by adding new axioms

A point-based topology $P$ is

**semiregular** iff it has a base of regular-closed sets

**normal** iff every pair of closed disjoint sets can be separated by a pair of open sets

**$\chi$-normal** iff every pair of regular-closed disjoint sets can be separated by a pair of open sets

**extensional** iff $RC_P$ satisfies axiom (Ext)

**weakly regular** iff it is semiregular and for every nonempty open set $x$ there exists a nonempty open set $y$ such that $Cl_P(x) \subseteq y$
A point-based topology $P$ is

connected \textit{iff} it cannot be represented as the sum of two disjoint nonempty open sets

$T_0$ \textit{iff} for every two different points there exists an open set that contains one of them and does not contain the other

$T_1$ \textit{iff} every one-point set is a closed set

Hausdorff \textit{iff} every two different points can be separated by a pair of disjoint open sets

compact \textit{iff} every cover of $P$ composed of open sets contains a finite subset which also covers $P$
Proposition

Let $P$ be a semiregular point-based topology

1. $P$ is weakly regular iff $RC_P$ satisfies axiom $(Ext)$.
2. $P$ is $\chi$-normal iff $RC_P$ satisfies axiom $(Nor)$.
3. $P$ is connected iff $RC_P$ satisfies axiom $(Con)$.
4. If $P$ is Hausdorff and compact then $RC_P$ satisfies axioms $(Ext)$ and $(Nor)$.
5. If $P$ is normal and Hausdorff then $RC_P$ satisfies axiom $(Nor)$. 
Let $P$ be a point-based topology and $u \in P$ be a point.

The set $\Gamma_u = \{x \in RC_P: u \in x\}$ satisfies the following conditions:

- $P \in \Gamma_u$,
- $x \cup y \in \Gamma_u$ iff $x \in \Gamma_u$ or $y \in \Gamma_u$,
- $x \in \Gamma_u$ and $y \in \Gamma_u$ imply $xC_{\Gamma_u}y$.
Let $\mathcal{B} = (B, 0, 1, \cdot, +, *, C)$ be a contact algebra

- A set $\Gamma \subseteq B$ of regions is called a clan iff it satisfies the following conditions:
  - $1 \in \Gamma$,
  - $x + y \in \Gamma$ iff $x \in \Gamma$ or $y \in \Gamma$,
  - $x \in \Gamma$ and $y \in \Gamma$ imply $xCy$.

- A clan is said to be maximal if it is maximal with respect to inclusion

Notations

- $CLANS_\mathcal{B}$: Set of all clans of $\mathcal{B}$
- $MaxCLANS_\mathcal{B}$: Set of all maximal clans of $\mathcal{B}$
Let $\mathcal{B} = (B, 0, 1, \cdot, +, \star, C)$ be a contact algebra

- Every ultrafilter of $B$ is a clan
- If $(\Gamma_i)_i$ is a collection of ultrafilters of $B$ such that $xCy$ for every $x, y \in \bigcup_i \Gamma_i$ then $\bigcup_i \Gamma_i$ is a clan and every clan can be obtained by such a construction
Lemma (Dimov and Vakarelov)

Let $\mathcal{B} = (B, 0, 1, \cdot, +, \star, C)$ be a contact algebra and $x \mapsto h(x) = \{ \Gamma \in CLANS_{\mathcal{B}} : x \in \Gamma \}$ be a function of $B$ in $2^{CLANS_{\mathcal{B}}}$. Then

- $h(x + y) = h(x) \cup h(y)$,
- $h(0) = \emptyset$,
- $h(1) = CLANS_{\mathcal{B}}$,
- $x \leq y$ iff $h(x) \subseteq h(y)$,
- $x = 1$ iff $h(x) = CLANS_{\mathcal{B}}$,
- $xCy$ iff $h(x) \cap h(y) \neq \emptyset$. 

Theorem (Representation of contact algebras)

* Let $\mathcal{B} = (\mathcal{B}, 0, 1, \cdot, +, \ast, C)$ be a contact algebra. Then there exists a semiregular $T_0$ compact space $P$ and a dense $C$-separable embedding $h$ of $\mathcal{B}$ in the contact algebra of regular closed sets $\mathcal{R}_P$. Moreover,

  * $\mathcal{B}$ satisfies $(\text{Con})$ iff $P$ is connected,
  * $\mathcal{B}$ satisfies $(\text{Ext})$ iff $P$ is weakly regular,
  * $\mathcal{B}$ satisfies $(\text{Nor})$ iff $P$ is $\chi$-normal.
Theorem (Representation of extensional contact algebras)

Let $\mathcal{B} = (\mathcal{B}, 0, 1, \cdot, +, \star, C)$ be a contact algebra satisfying axiom $(\text{Ext})$. Then there exists a weakly regular $T_1$ compact space $P$ and a dense $C$-separable embedding $h$ of $\mathcal{B}$ in the contact algebra of regular closed sets $\mathcal{RC}_P$. Moreover,

- $\mathcal{B}$ satisfies $(\text{Con})$ iff $P$ is connected,
- $\mathcal{B}$ satisfies $(\text{Nor})$ iff $P$ is $\chi$-normal.
Theorem (Representation of extensional normal contact algebras)

Let $\mathcal{B} = (B, 0, 1, \cdot, +, \ast, C)$ be a contact algebra satisfying axioms $(Ext)$ and $(Nor)$. Then there exists a Hausdorff compact space $P$ and a dense $C$-separable embedding $h$ of $\mathcal{B}$ in the contact algebra of regular closed sets $\mathcal{RC}_P$. Moreover,

$\mathcal{B}$ satisfies $(Con)$ iff $P$ is connected.


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Mereotopologies
Mereotopology of open sets / Mereotopology of closed sets

Let $T = (X, \mathcal{O})$ be a topology

Mereotopology of open sets over $T$

- Boolean sub-algebra $(D, 0_T, 1_T, -T, \oplus_T, \otimes_T)$ of $(RO(T), 0_T, 1_T, -T, \oplus_T, \otimes_T)$ such that
  - if $U \in X$, $u$ is open in $T$ and $U \in u$ then there is $v \in D$ such that $U \in v$ and $v \subseteq u$

Mereotopology of closed sets over $T$

- Boolean sub-algebra $(D, 0_T, 1_T, -T, \oplus_T, \otimes_T)$ of $(RC(T), 0_T, 1_T, -T, \oplus_T, \otimes_T)$ such that
  - if $U \in X$, $u$ is closed in $T$ and $U \in u$ then there is $v \in D$ such that $U \in v$ and $v \subseteq u$
Mereotopologies
Examples of mereotopologies

Let $n \geq 1$

Semi-algebraic subset of $\mathbb{R}^n$
  - Boolean combination of polynomial inequations

Semi-linear subset (polytope) of $\mathbb{R}^n$
  - Boolean combination of linear inequalities

Algebraic polytope of $\mathbb{R}^n$
  - Boolean combination of polynomial inequations with algebraic coefficients

Rational polytope of $\mathbb{R}^n$
  - Boolean combination of polynomial inequations with rational coefficients
Mereotopologies
Examples of mereotopologies

Let $n \geq 1$

$RO(\mathbb{R}^n) / RC(\mathbb{R}^n)$  
▶ regular open / closed subsets of $\mathbb{R}^n$

$ROS(\mathbb{R}^n) / RCS(\mathbb{R}^n)$  
▶ regular open / closed semi-alg. subsets of $\mathbb{R}^n$

$ROP(\mathbb{R}^n) / RCP(\mathbb{R}^n)$  
▶ regular open / closed polytopes of $\mathbb{R}^n$

$ROP_{alg}(\mathbb{R}^n) / RCP_{alg}(\mathbb{R}^n)$  
▶ regular open / closed alg. polytopes of $\mathbb{R}^n$

$ROP_{rat}(\mathbb{R}^n) / RCP_{rat}(\mathbb{R}^n)$  
▶ regular open / closed rat. polytopes of $\mathbb{R}^n$
Euclidean logics
Syntax

Terms
► $s ::= x \mid 0 \mid 1 \mid -s \mid (s \oplus t) \mid (s \otimes t)$ where $x \in \text{VAR}$

Formulas
► $\phi ::= s \leq t \mid s \geq t \mid C(s, t) \mid c(s) \mid \bot \mid \neg \phi \mid (\phi \lor \psi) \mid \forall x \phi$

Abbreviation
► $s = t ::= s \leq t \land s \geq t$
Euclidean logics

Syntax

Reading

- $s \leq t$: “$s$ is contained in $t$”
- $s \geq t$: “$s$ contains $t$”
- $C(s, t)$: “$s$ is in contact with $t$”
- $c(s)$: “$s$ is connected”

Examples

- $C(x, y) \rightarrow \exists z (C(x, z) \land c(z) \land z \leq y)$
- $c(x) \land c(y) \land c(z) \land c(x \oplus y \oplus z) \rightarrow c(x \oplus y) \lor c(x \oplus z)$
- $c(x) \land c(y) \rightarrow c(x \otimes y) \lor C(-x, -y)$
Euclidean logics

Semantics

Let \( T = (X, \mathcal{O}) \) be a topology and \( M = (D, 0_T, 1_T, -T, \oplus_T, \otimes_T) \) be a mereotopology over \( T \)

Interpretation

\[ \begin{align*}
\text{function } \theta : x & \mapsto \theta(x) \in D \\
n & \mapsto \bar{\theta}(n) \in D
\end{align*} \]

Satisfaction

\[ \begin{align*}
M, \theta \models s & \leq t \text{ iff } \bar{\theta}(s) \text{ is contained } \bar{\theta}(t) \\
M, \theta \models s & \geq t \text{ iff } \bar{\theta}(s) \text{ contains } \bar{\theta}(t) \\
M, \theta \models C(s, t) & \text{ iff } \bar{\theta}(s) \text{ is in contact with } \bar{\theta}(t) \\
M, \theta \models c(s) & \text{ iff } \bar{\theta}(s) \text{ is connected}
\end{align*} \]

Validity

\[ M \models \phi \text{ iff for all interpretations } \theta, \ M, \theta \models \phi \]
Euclidean logics

definitions

Examples

If \( \phi = C(x, y) \rightarrow \exists z(C(x, z) \land c(z) \land z \leq y) \) then

\[ RO(\mathbb{R}^2) \not\models \phi \]

\[ \text{for all finitary mereotopologies } M, M \models \phi \]

If \( \phi = c(x) \land c(y) \land c(z) \land c(x \oplus y \oplus z) \rightarrow c(x \oplus y) \lor c(x \oplus z) \) then

\[ RO(\mathbb{R}^2) \not\models \phi \]

\[ ROS(\mathbb{R}^2) \models \phi \]

If \( \phi = c(x) \land c(y) \rightarrow c(x \otimes y) \lor C(-x, -y) \) then

\[ RO(\mathbb{R}^2) \models \phi \]

\[ RO(\text{torus}) \not\models \phi \]
Let $M = (D, 0_T, 1_T, -T, \oplus_T, \otimes_T)$ be a mereotopology over $\mathbb{IR}^n$

Then

\begin{itemize}
  \item $M \models x \leq y \iff \forall z (C(x, z) \rightarrow C(y, z))$
  \item $M \models x \geq y \iff \forall z (C(y, z) \rightarrow C(x, z))$
\end{itemize}
Expressivity
Definition of $c$ by means of $C$

Let $M = (D, 0_T, 1_T, -T, \oplus_T, \otimes_T)$ be a mereotopology over $\mathbb{R}^n$

If $M$ respects components then

$M \models c(x) \iff$

$\forall x_1 \forall x_2 (x_1 \neq 0 \land x_2 \neq 0 \land x_1 \oplus x_2 = x \land x_1 \otimes x_2 = 0 \rightarrow$

$\exists y_1 \exists y_2 (x_1 \geq y_1 \land x_2 \geq y_2 \land C(y_1, y_2) \land \neg C(y_1 \oplus y_2, -x)))$
Expressivity
Definition of $RCC8$ by means of $C$

- $DC(x, y) ::= \neg C(x, y)$
- $EC(x, y) ::= C(x, y) \land \forall z(x \geq z \land y \geq z \rightarrow z = 0)$
- $PO(x, y) ::= \exists z(x \geq z \land y \geq z \land z \neq 0) \land x \nless y \land x \ngeq y$
- $TPP(x, y) ::= x \leq y \land \exists z(EC(x, z) \land EC(y, z))$
- $NTPP(x, y) ::= x \leq y \land \forall z(\neg EC(x, z) \lor \neg EC(y, z))$
- $TPPI(x, y) ::= x \geq y \land \exists z(EC(x, z) \land EC(y, z))$
- $NTPPI(x, y) ::= x \geq y \land \forall z(\neg EC(x, z) \lor \neg EC(y, z))$
- $EQ(x, y) ::= x \leq y \land x \geq y$
Expressivity
Definition of \textit{halfspace} by means of \textless, \textgreater\ and \textit{convex}

Let $M = (D, 0_T, 1_T, -T, \oplus_T, \otimes_T)$ be a mereotopology over IR$^n$

Then

$\models M \models \text{halfspace}(x) \iff x \neq 0 \land -x \neq 0 \land \text{convex}(x) \land \text{convex}(-x)$
Expressivity
Definition of \textit{parallel} by means of $\leq$, $\geq$ and \textit{convex}

Let $M = (D, 0_T, 1_T, -T, \oplus_T, \otimes_T)$ be a mereotopology over $\mathbb{IR}^n$

Then

$\blacktriangleright$ $M \models parallel(x, y) \iff$
$halfspace(x) \land halfspace(y) \land x \neq y \land x \neq -y \land$
$(x \otimes y = 0 \lor x \otimes -y = 0 \lor -x \otimes y = 0 \lor -x \otimes -y = 0)$
Expressivity
Definition of $C$ by means of $\leq$, $\geq$ and *convex*

Let $M = (D, 0_T, 1_T, -T, \oplus_T, \otimes_T)$ be a mereotopology over $\mathbb{R}^n$

If $M$ is finitary then

$\vdash M \models C(x, y) \iff$

$\exists x' \exists y' (x \geq x' \land \text{convex}(x') \land y \geq y' \land \text{convex}(y') \land$

$\forall x'' \forall y'' (x' \leq x'' \land y' \leq y'' \land \text{parallel}(x'', y'') \rightarrow$

$x'' \otimes y'' \neq 0)$)
Let $\Sigma$ be a signature and $\mathcal{C}$ be a class of mereotopologies

Let $\text{SAT}(\Sigma, \mathcal{C})$ be the following decision problem

- input: a formula $\phi$ over $\Sigma$
- output: determine whether there exists
  - a mereotopology $M = (D, 0, 1, -, \oplus, \otimes)$ in $\mathcal{C}$
  - an interpretation $\theta: x \mapsto \theta(x) \in D$

such that $M, \theta \models \phi$
With quantification

In the table below, \( n \geq 2 \) and every signature contains 0, 1, \(-\), \(\oplus\) and \(\otimes\)

<table>
<thead>
<tr>
<th></th>
<th>all</th>
<th>( RC(\mathbb{IR}) )</th>
<th>( RC(\mathbb{IR}^n) )</th>
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<tr>
<td>(\leq, \geq)</td>
<td>(NEXPTIME)-h (\text{in EXPSPACE})</td>
<td>(NEXPTIME)-h (\text{in EXPSPACE})</td>
<td>(NEXPTIME)-h (\text{in EXPSPACE})</td>
</tr>
<tr>
<td>(\leq, \geq, c)</td>
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<td>non-elem.</td>
<td>undec.</td>
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<td>(\leq, \geq, C, c)</td>
<td>?</td>
<td>non-elem.</td>
<td>undec.</td>
</tr>
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Decidability/complexity

The SAT problem

Without quantification

In the table below, \( n \geq 3 \) and every signature contains 0, 1, −, \( \oplus \) and \( \otimes \)

<table>
<thead>
<tr>
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<th>( RC(\mathbb{IR}^2) )</th>
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<td>( \text{EXPTIME-}c )</td>
<td>( \text{NPTIME-}c )</td>
<td>( \text{undec.} )</td>
<td>( \text{EXPTIME-h} )</td>
</tr>
<tr>
<td>( \leq, \geq, C )</td>
<td>( \text{NPTIME-}c )</td>
<td>( \text{PSPACE-}c )</td>
<td>( \text{PSPACE-}c )</td>
<td>( \text{PSPACE-}c )</td>
</tr>
<tr>
<td>( \leq, \geq, C, c )</td>
<td>( \text{EXPTIME-}c )</td>
<td>( \text{PSPACE-}c )</td>
<td>( \text{undec.} )</td>
<td>( \text{EXPTIME-h} )</td>
</tr>
</tbody>
</table>


Contents

1. Historical excursion
2. Algebras of regions, models and representation theory
3. First-order mereotopology
4. Qualitative spatial reasoning using constraint calculi
5. Modal logic and mereotopology
Qualitative spatial reasoning using constraint calculi

Constraint-based methods

Examples

► “The room is 5 metres in length and 6 in breadth”
► “The desk should be placed in front of the window”
► “The table should be placed between the sofa and the armchair”
Qualitative spatial reasoning using constraint calculi

Constraint-based methods

A partial method for determining inconsistency of a CSP is the path-consistency method

- A CSP is path-consistent iff for any partial instantiation of any two variables satisfying the constraints between the two variables, it is possible for any third variable to extend the partial instantiation to this third variable satisfying the constraints between the three variables.

A straightforward way to enforce path-consistency on a binary CSP is to strengthen relations by successively applying the following operation until a fixed point is reached:

\[
R_{ij} := R_{ij} \cap (R_{ik} \circ R_{kj})
\]

The resulting CSP is equivalent to the original CSP.
A straightforward way to enforce path-consistency on a binary CSP is to strengthen relations by successively applying the following operation until a fixed point is reached

\[ R_{ij} := R_{ij} \cap (R_{ik} \circ R_{kj}) \]

The resulting CSP is equivalent to the original CSP.

The algorithm sketched has a running time of \( O(n^5) \).

Advanced algorithms can enforce path-consistency in time \( O(n^3) \).
Qualitative spatial reasoning using constraint calculi

Constraint-based methods

Relation algebras based on JEPD relations
- Jointly exhaustive and pairwise disjoint (JEPD) relations are sometimes called atomic, basic or base relations

Examples
- Interval algebra (Allen, 1983)
- Region connection calculus (Randell, Cui and Cohn, 1992)
\( \mathcal{B} \) being a finite set of JEPD binary relations, the consistency problem \( \text{CSPSAT}(S) \) for \( S \subseteq 2^\mathcal{B} \) is defined as follows

**input:** a finite set \( \mathcal{V} \) of variables over a domain \( \mathcal{D} \) and a finite set \( \Theta \) of binary constraints \( R(x_i, x_j) \) where \( R \in S \) and \( x_i, x_j \in \mathcal{V} \)

**output:** is there an instantiation of all variables in \( \mathcal{V} \) with values from \( \mathcal{D} \) such that all constraints in \( \Theta \) are satisfied?
Point algebra (Vilain and Kautz)

- temporal beings: points in dimension 1
- basic relations: $<$, $ld$, $>$
- temporal relations: $\emptyset$, $<$, $ld$, $>$, $\leq$, $\neq$, $\geq$, $\star$
- composition table

<table>
<thead>
<tr>
<th></th>
<th>&lt;</th>
<th>Id</th>
<th>&gt;</th>
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<td>&gt;</td>
<td>&gt;</td>
</tr>
</tbody>
</table>
Qualitative spatial reasoning using constraint calculi

Temporal constraint calculi

Point algebra (Vilain and Kautz)

composition in the point algebra

\[
\begin{array}{ccccccc}
\emptyset & < & ld & > & \leq & \neq & \geq & *\\
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \\
< & \emptyset & < & < & * & < & * & * & * \\
ld & \emptyset & < & ld & > & \leq & \neq & \geq & * \\
> & \emptyset & * & > & > & * & * & > & * \\
\leq & \emptyset & < & \leq & * & \leq & * & * & * \\
\neq & \emptyset & * & \neq & * & * & * & * & * \\
\geq & \emptyset & * & \geq & > & * & * & \geq & * \\
* & \emptyset & * & * & * & * & * & * & * \\
\end{array}
\]
Point network: \((n, C)\)

\begin{itemize}
\item \(n\) is a positive integer
\item \(C: (i, j) \mapsto \{\emptyset, <, ld, >, \leq, \neq, \geq, \star\}\)
\end{itemize}

Given a point network \((n, C)\), by successively applying the following operation until a fixed point is reached

\[
C_{ij} := C_{ij} \cap (C_{ik} \circ C_{kj})
\]

one can compute an equivalent path-consistent point network \((n, C')\) in polynomial time.
Proposition (Vilain and Kautz)

- Let \((n, C)\) be a point network and \((n, C')\) be an equivalent path-consistent point network. The following conditions are equivalent:
  1. \((n, C)\) is consistent.
  2. \((n, C')\) does not contain the \(\emptyset\) relation.

Proposition

- The consistency problem of point networks is in \(P\).
Interval algebra (Allen)

- temporal beings: intervals in dimension 1
- basic relations: $p, m, o, s, d, fi, ld, f, di, si, oi, mi, pi$
- temporal relations: subsets of \{p, m, o, s, d, fi, ld, f, di, si, oi, mi, pi\}
- composition table

|   | $p$ | $m$ | $o$ | ...
|---|-----|-----|-----|-----
| $p$ | $p$ | $p$ | $p$ | ...
| $m$ | $p$ | $p$ | $p$ | ...
| $o$ | $p$ | $p$ | $p, m, o$ | ...
| ... | ... | ... | ... | ...
Interval network: \((n, C)\)
- \(n\) is a positive integer
- \(C: (i, j) \mapsto \) subsets of \(\{p, m, o, s, d, fi, ld, f, di, si, oi, mi, pi\}\)

Given an interval network \((n, C)\), by successively applying the following operation until a fixed point is reached

- \(C_{ij} := C_{ij} \cap (C_{ik} \circ C_{kj})\)

one can compute an equivalent path-consistent interval network \((n, C')\) in polynomial time
Qualitative spatial reasoning using constraint calculi
Temporal constraint calculi

There exists interval networks \((n, C)\) such that

- \((n, C)\) is inconsistent
- \((n, C)\) is path-consistent

Proposition (Vilain and Kautz)

- The consistency problem of interval network is \(NP\)-hard.

Proof: Reduction of 3SAT.
Many interval relations can be encoded in the point algebra

- \( x \{p, m, o\} y \) can be encoded as \( x^- < x^+, y^- < y^+, x^- < y^-, x^+ < y^+ \)
- \( x \{d\} y \) can be encoded as \( x^- < x^+, y^- < y^+, x^- > y^-, x^+ < y^+ \)

Proposition

- The consistency problem of pointizable interval networks is in \( P \).
Qualitative spatial reasoning using constraint calculi
Temporal constraint calculi

Convex interval relations
- Definition
- Properties

Proposition
- Let \((n, C)\) be a convex interval network and \((n, C')\) be an equivalent path-consistent interval network. The following conditions are equivalent:
  1. \((n, C)\) is consistent.
  2. \((n, C')\) does not contain the \(\emptyset\) relation.

Proposition
- The consistency problem of convex interval networks is in \(P\).
Preconvex interval relations (Ligozat)

- Definition
- Properties

Proposition (Ligozat)

- Let \((n, C)\) be a preconvex interval network and \((n, C')\) be an equivalent path-consistent interval network. The following conditions are equivalent:
  1. \((n, C)\) is consistent.
  2. \((n, C')\) does not contain the \(\emptyset\) relation.

Proposition (Ligozat)

- The consistency problem of preconvex interval networks is in \(P\).
ORD-Horn interval relations (Nebel and Bürckert)

- Definition
- Properties

Proposition (Nebel and Bürckert)

Let \((n, C)\) be a ORD-Horn interval network and \((n, C')\) be an equivalent path-consistent interval network. The following conditions are equivalent:

1. \((n, C)\) is consistent.
2. \((n, C')\) does not contain the \(\emptyset\) relation.

Proposition (Nebel and Bürckert)

The consistency problem of ORD-Horn interval networks is in \(P\).
Proposition
  ▶ A relation in Allen’s algebra is preconvex iff it is ORD-Horn.

Proposition (Ligozat)
  ▶ The preconvex subclass is the unique greatest tractable subclass that contains all basic relations.

Proposition (Nebel and Bürckert)
  ▶ The ORD-Horn subclass is the unique greatest tractable subclass that contains all basic relations.
Let us consider the following set of JEPD binary relations between nonempty regular closed regions in a topological space

\[ DC: \quad DC(x, y) ::= x \text{ and } y \text{ are disconnected} \]
\[ EC: \quad EC(x, y) ::= x \text{ and } y \text{ are externally connected} \]
\[ PO: \quad PO(x, y) ::= x \text{ and } y \text{ partially overlaps} \]
\[ TPP: \quad TPP(x, y) ::= x \text{ is a tangential proper part of } y \]
\[ TPPI: \quad TPPI(x, y) ::= y \text{ is a tangential proper part of } x \]
\[ NTPP: \quad NTPP(x, y) ::= x \text{ is a nontangential proper part of } y \]
\[ NTPPI: \quad NTPPI(x, y) ::= y \text{ is a nontangential proper part of } x \]
\[ Id: \quad Id(x, y) ::= x \text{ and } y \text{ are equal} \]
RCC8 (Randell, Cui and Cohn)

- spatial beings: nonempty regular closed regions in a topological space
- basic relations: $DC$, $EC$, $PO$, $TPP$, $TPPI$, $NTPP$, $NTPPI$, $Id$
- spatial relations: subsets of $\{DC, EC, PO, TPP, TPPI, NTPP, NTPPI, Id\}$
Qualitative spatial reasoning using constraint calculi
Spatial constraint calculi

Composition table

<table>
<thead>
<tr>
<th></th>
<th>DC</th>
<th></th>
<th>EC</th>
<th></th>
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</thead>
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<tr>
<td>DC</td>
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<td>{DC, EC, PO, TPP, NTPP}</td>
<td></td>
</tr>
<tr>
<td>EC</td>
<td>{DC, EC, PO, TPPI, NTPPI}</td>
<td></td>
<td>*</td>
<td></td>
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<td></td>
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</tbody>
</table>
Qualitative spatial reasoning using constraint calculi

Spatial constraint calculi

\textit{RCC8 network: } \((n, C)\)

\begin{itemize}
  \item \(n\) is a positive integer
  \item \(C: (i, j) \mapsto \text{subsets of}\{DC, EC, PO, TPP, TPPI, NTPP, NTPPI, Id\}\)
\end{itemize}

Given an \textit{RCC8 network} \((n, C)\), by successively applying the following operation until a fixed point is reached

\begin{itemize}
  \item \(C_{ij} := C_{ij} \cap (C_{ik} \circ C_{kj})\)
\end{itemize}

one can compute an equivalent path-consistent \textit{RCC8 network} \((n, C')\) in polynomial time
There exists $\textit{RCC8}$ networks $(n, C)$ such that

- $(n, C)$ is inconsistent
- $(n, C)$ is path-consistent

Proposition (Renz and Nebel)

- The consistency problem of $\textit{RCC8}$ networks is $\textit{NP}$-hard.

Proof: Reduction of $\text{NOT-ALL-EQUAL-3SAT}$.
Proposition (Bennett, Nutt)

- The consistency problem of RCC8 networks is in $PSPACE$.

Proof: Arbitrarily topological set constraints can be translated into multimodal formulas that have an $S4$-operator $I$ and an $S5$-operator $\Box$

- $\pi(DC(x,y)) ::= \Box \neg(x \land y)$
- $\pi(EC(x,y)) ::= \Box \neg(lx \land ly) \land \Diamond(x \land y)$
- $\pi(PO(x,y)) ::= \Diamond(lx \land ly) \land \Diamond(x \land \neg y) \land \Diamond(\neg x \land y)$
- $\pi(TPP(x,y)) ::= \Box(\neg x \lor y) \land \Diamond(x \land \neg ly) \land \Diamond(\neg x \land y)$
- $\pi(NTPP(x,y)) ::= \Box(\neg x \lor ly) \land \Diamond(\neg x \land y)$
- $\pi(Id(x,y)) ::= \Box(\neg x \lor y) \land \Box(x \lor \neg y)$
Proposition (Renz and Nebel)

- The consistency problem of $RCC8$ networks is in $NP$.

Proof: The $RCC8$ network $(n, C)$ is consistent iff its translation

\[ \pi(C_{ij}(x_i, x_j)) : 1 \leq i < j \leq n \]

\[ \Box(x_i \leftrightarrow \neg l \neg l x_i) : 1 \leq i \leq n \]

\[ \Diamond x_i : 1 \leq i \leq n \]

into multimodal formulas that have an $S4$-operator $l$ and an
$S5$-operator $\Box$ is satisfied in a polynomial model of depth 1.
ORD-Horn RCC8 relations (Renz and Nebel)

- Definition
- Properties

Proposition (Renz and Nebel)

- Let \((n, C)\) be a ORD-Horn RCC8 network and \((n, C')\) be an equivalent path-consistent RCC8 network. The following conditions are equivalent:
  1. \((n, C)\) is consistent.
  2. \((n, C')\) does not contain the \(\emptyset\) relation.

Proposition (Renz and Nebel)

- The consistency problem of ORD-Horn RCC8 networks is in \(P\).
Combining topological and size information (Gerevini and Renz)

- There are 8 JEPD binary relations between nonempty regular closed regions in a topological space: $DC$, $EC$, $PO$, $TPP$, $TPPI$, $NTPP$, $NTPPI$, $Id_{\text{topo}}$

- Assuming that our regions are measurable subsets in $\mathbb{R}^n$, there are 3 JEPD binary relations between their size: $<$, $Id_{\text{size}}$, $>$
Combining topological and size information

Example of a constraint network combining topological ($\Theta$) and size ($\Sigma$) information

$\Theta$: $x_0 \{ PO, TPPI, Id_{topo} \} x_1$, $x_0 \{ TPP, Id_{topo} \} x_2$, $x_1 \{ TPP, Id_{topo} \} x_2$, $x_3 \{ TPPI, Id_{topo} \} x_4$

$\Sigma$: $\text{size}(x_0) < \text{size}(x_2)$, $\text{size}(x_1) \geq \text{size}(x_3)$, $\text{size}(x_2) \leq \text{size}(x_4)$
Combining topological and size information

- The topological relations and the relative size relations are not independent

\[
\begin{align*}
DC & \leftrightarrow <, \text{Id}_{size}, > \\
EC & \leftrightarrow <, \text{Id}_{size}, > \\
PO & \leftrightarrow <, \text{Id}_{size}, > \\
TPP & \leftrightarrow < \\
TPPI & \leftrightarrow > \\
NTPP & \leftrightarrow < \\
NTPPI & \leftrightarrow > \\
\text{Id}_{topo} & \leftrightarrow \text{Id}_{size}
\end{align*}
\]
Combining topological and size information

- The topological relations and the relative size relations are not independent

  \[
  < \leftrightarrow DC, EC, PO, TPP, NTPP \\
  \text{id}_{size} \leftrightarrow DC, EC, PO, \text{id}_{topo} \\
  > \leftrightarrow DC, EC, PO, TPPI, NTPPI
  \]
Combining topological and size information

Example of a constraint network combining topological ($\Theta$) and size ($\Sigma$) information

\[ \Theta: x_0 \{ PO, TPPI, Id_{topo} \} x_1, x_0 \{ TPP, Id_{topo} \} x_2, \]
\[ x_1 \{ TPP, Id_{topo} \} x_2, x_3 \{ TPPI, Id_{topo} \} x_4 \]
\[ \Sigma: size(x_0) < size(x_2), size(x_1) \geq size(x_3), \]
\[ size(x_2) \leq size(x_4) \]

Remark that $\Theta$ and $\Sigma$ are independently consistent, but their union is not consistent
Combining topological and size information

- The following propagation algorithm does not always detect inconsistencies of a constraint network combining a ORD-Horn topological network \((n, T)\) and a size network \((n, S)\)
  - successively apply the following operations until a fixed point is reached
    1. enforce path-consistency to \((n, T)\);
    2. enforce path-consistency to \((n, S)\);
    3. extend \((n, T)\) with the topological constraints entailed by the size constraints in \((n, S)\);
    4. extend \((n, S)\) with the size constraints entailed by the topological constraints in \((n, T)\);
Combining topological and size information

- The following propagation algorithm always detects inconsistencies of a constraint network combining a ORD-Horn topological network \((n, T)\) and a size network \((n, S)\)
  
  - successively apply the following operations until a fixed point is reached

1. \(T_{ij} := (T_{ij} \cap \text{rel}_\text{topo}(S_{ij})) \cap ((T_{ik} \cap \text{rel}_\text{topo}(S_{ik})) \circ (T_{kj} \cap \text{rel}_\text{topo}(S_{kj})))\)
2. \(S_{ij} := (S_{ij} \cap \text{rel}_\text{size}(T_{ij})) \cap ((S_{ik} \cap \text{rel}_\text{size}(T_{ik})) \circ (S_{kj} \cap \text{rel}_\text{size}(T_{kj})))\)
Proposition (Gerevini and Renz)

- Let \((n, T)\) be a ORD-Horn topological network and \((n, S)\) be a size network. Let \((n, T')\) and \((n, S')\) be the networks obtained by means of the previous propagation algorithm. The following conditions are equivalent:
  1. the constraint network combining \((n, T)\) and \((n, S)\) is consistent.
  2. \((n, T')\) does not contain the \(\emptyset_{\text{topo}}\) relation and \((n, S')\) does not contain the \(\emptyset_{\text{size}}\) relation.

Proposition (Gerevini and Renz)

- The consistency problem of constraint networks combining ORD-Horn topological networks and size networks is in \(P\).
Qualitative spatial reasoning using constraint calculi
Variants: rectangle algebra

Rectangle algebra (Condotta)
► spatial beings: isothetic rectangles in dimension 2

Since isothetic rectangles in dimension 2
► can be seen as pairs of intervals
then basic relations between them
► can be seen as pairs of basic relations between intervals
and spatial relations between them
► can be seen as sets of pairs of basic relations between intervals
Qualitative spatial reasoning using constraint calculi

Variants: rectangle algebra

Rectangle algebra

▶ spatial beings: isothetic rectangles in dimension 2
▶ basic relations: pairs of the form \((A, B)\) where 
  \(A, B \in \{p, m, o, s, d, fi, ld, f, di, si, oi, mi, pi\}\)
▶ spatial relations: sets of pairs of the form \((A, B)\) where 
  \(A, B \in \{p, m, o, s, d, fi, ld, f, di, si, oi, mi, pi\}\)
▶ Composition table
  ▶ \((A_1, B_1) \circ_{rec} (A_2, B_2) = (A_1 \circ_{int} A_2) \times (B_1 \circ_{int} B_2)\)
  ▶ Example: \((o, p) \circ_{rec} (s, mi) = \{o\} \times \{p, m, o, s, d\}\)
Qualitative spatial reasoning using constraint calculi

Variants: rectangle algebra

Rectangle network: \((n, C)\)

- \(n\) is a positive integer
- \(C: (i, j) \mapsto \) sets of pairs of the form \((A, B)\) where \(A, B \in \{p, m, o, s, d, fi, ld, f, di, si, oi, mi, pi\}\)

Given a rectangle network \((n, C)\), by successively applying the following operation until a fixed point is reached

\[ C_{ij} := C_{ij} \cap (C_{ik} \circ C_{kj}) \]

one can compute an equivalent path-consistent rectangle network \((n, C')\) in polynomial time
There exists rectangle networks \((n, C)\) such that

- \((n, C)\) is inconsistent
- \((n, C)\) is path-consistent

Proposition

- The consistency problem of rectangle networks is \(NP\)-hard.

Proposition

- The consistency problem of rectangle networks is in \(NP\).
Qualitative spatial reasoning using constraint calculi
Variants: rectangle algebra

Strongly preconvex relations
  ▶ Definition
  ▶ Properties

Proposition (Condotta)
  ▶ Let \((n, C)\) be a strongly preconvex rectangle network and \((n, C')\) be an equivalent path-consistent rectangle network. The following conditions are equivalent:
    1. \((n, C)\) is consistent.
    2. \((n, C')\) does not contain the \(\emptyset\) relation.

Proposition (Condotta)
  ▶ The consistency problem of strongly preconvex rectangle networks is in \(P\).
Let us consider the following set of JEPD binary relations between nonempty subsets of a nonempty set

**DR:**  $DR(x, y) ::= x$ and $y$ are separated

**PO:**  $PO(x, y) ::= x$ and $y$ partially overlaps

**PP:**  $PP(x, y) ::= x$ is a proper part of $y$

**PPI:**  $PPI(x, y) ::= y$ is a proper part of $x$

**Id:**  $Id(x, y) ::= x$ and $y$ are equal
Qualitative spatial reasoning using constraint calculi

Variants: $RCC5$

$RCC5$

- spatial beings: nonempty subsets of a nonempty set
- basic relations: $DR$, $PO$, $PP$, $PPI$, $Id$
- spatial relations: subsets of $\{DR, PO, PP, PPI, Id\}$

Composition table

<table>
<thead>
<tr>
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<th>$DR$</th>
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<td>${DR, PO, PP}$</td>
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<tr>
<td>$PO$</td>
<td>${DR, PO, PPI}$</td>
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Egg yolk representation (Cohn and Gotts)

- How many JEPD relations are there between non crisp regions?

If $x^-$ and $x^+$ are regions such that

- $x^-$ is a nontangential proper part of $x^+$

then one can see $(x^-, x^+)$ as a vague region such that

- $x^-$ denotes the set of all points that necessarily belong to the vague region
- $x^+$ denotes the set of all points that possibly belong to the region
Qualitative spatial reasoning using constraint calculi

Variants: egg yolk representation

Egg yolk representation

- spatial beings: pairs \((x^-, x^+)\) of regions such that \(x^-\) is a nontangential proper part of \(x^+\)

- basic relations: quadruples \(\begin{pmatrix} A^{--} & A^{-+} \\ A^{+-} & A^{++} \end{pmatrix}\) where

  \[ A^{--}, A^{-+}, A^{+-}, A^{++} \in \{ DC, EC, PO, TPP, TPPI, NTPP, NTPPI, Id \} \]

- spatial relations: sets of quadruples \(\begin{pmatrix} A^{--} & A^{-+} \\ A^{+-} & A^{++} \end{pmatrix}\) where

  \[ A^{--}, A^{-+}, A^{+-}, A^{++} \in \{ DC, EC, PO, TPP, TPPI, NTPP, NTPPI, Id \} \]


Cohn, A., Bennett, B., Gooday, J., Gotts, N.: *Qualitative spatial representation and reasoning with the region connection calculus.* GeoInformatica **1** (1997) 275–316.


1. Historical excursion
2. Algebras of regions, models and representation theory
3. First-order mereotopology
4. Qualitative spatial reasoning using constraint calculi
5. Modal logic and mereotopology
Modal logic and mereotopology
Modal logics for mereotopological relations

Mereotopological interpretation of modal logic: syntax

- $\phi ::= p \mid \bot \mid \neg \phi \mid (\phi \lor \psi) \mid [r]\phi$
- where $r \in \{dc, ec, po, tpp, tppi, ntpp, ntppi\}$

- Abbreviations
  - Standard definitions for the remaining Boolean operations
  - $\langle r \rangle \phi ::= \neg [r] \neg \phi$ where $r \in \{dc, ec, po, tpp, tppi, ntpp, ntppi\}$

- Intuitive meaning
  - The interpretation of $[r]\phi$ is $\phi$ holds in every region $r$-related to the current one
  - The interpretation of $\langle r \rangle \phi$ is $\phi$ holds in some region $r$-related to the current one
Modal logic and mereotopology
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Mereotopological interpretation of modal logic: frames

- Concrete frames are structures of the form $\mathcal{F} = (W, R)$ where
  - $W$ is a nonempty set of nonempty regular closed regions in a topological space $(X, \mathcal{O})$
  - $R: r \in \{dc, ec, po, tpp, tppi, ntpp, ntppi\} \mapsto R(r) \subseteq W \times W$
    is a function such that
      - $x R(dc) y$ iff $x \cap y = \emptyset$
      - $x R(ec) y$ iff $x \cap y \neq \emptyset$ and $ln(x) \cap ln(y) = \emptyset$
      - $x R(po) y$ iff $ln(x) \cap ln(y) \neq \emptyset$, $x \not\subseteq y$ and $x \not\supseteq y$
      - $x R(tpp) y$ iff $x \subseteq y$, $x \not\subseteq ln(y)$ and $x \not\supseteq y$
      - $x R(tppi) y$ iff $x \supseteq y$, $ln(x) \not\supseteq y$ and $x \not\subseteq y$
      - $x R(ntpp) y$ iff $x \subseteq ln(y)$ and $x \not\supseteq y$
      - $x R(ntppi) y$ iff $ln(x) \supseteq y$ and $x \not\subseteq y$
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Mereotopological interpretation of modal logic: frames

- Examples of concrete frames
  1. the set $P_{reg}$ of all nonempty regular closed regions in a point-based topological space $P$
  2. the set $\mathbb{R}_{reg}^n$ of all nonempty regular closed regions in $\mathbb{R}^n$
  3. the set $\mathbb{R}_{con}^n$ of all nonempty convex regular closed regions in $\mathbb{R}^n$
  4. the set $\mathbb{R}_{rec}^n$ of all nonempty rectangle regular closed regions in $\mathbb{R}^n$, i.e. regions of the form $\prod\{C_i : 1 \leq i \leq n\}$ where $C_1, \ldots, C_n$ are non-singleton closed intervals in $\mathbb{R}$
Mereotopological interpretation of modal logic: frames

- Abstract frames are structures of the form $\mathcal{F} = (W, R)$ where
  - $W$ is a nonempty set of regions
  - $R: r \in \{dc, ec, po, tpp, tppi, ntpp, ntppi\} \mapsto R(r) \subseteq W \times W$ is a function such that
    - $R(dc), R(ec), R(po), R(tpp), R(tppi), R(ntpp)$ and $R(ntppi)$ are JEPD binary relations on $W$
    - $R(dc), R(ec)$ and $R(po)$ are symmetrical
    - $R(tppi)$ is the converse of $R(tpp)$ and $R(ntppi)$ is the converse of $R(ntpp)$
    - for any entry $q_1, \ldots, q_k$ in row $r_1$ and column $r_2$ of the composition table for RCC8, $R(r_1) \circ R(r_2) \subseteq R(q_1) \cup \ldots \cup R(q_k)$
Modal logic and mereotopology
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Proposition (Lutz and Wolter)
1. Every concrete frame is an abstract frame.
2. Every abstract frame is isomorphic to a concrete frame.
Mereotopological interpretation of modal logic: semantics

- Let $\mathcal{M} = (W, R, V)$ be a concrete/abstract frame equipped with a valuation function $V: p \mapsto V(p) \subseteq W$
- Truth of a modal formula $\phi$ at a region $x$ in $W$ is defined as follows:
  - $\mathcal{M}, x \models p$ iff $x \in V(p)$
  - $\mathcal{M}, x \not\models \bot$
  - $\mathcal{M}, x \models \neg\phi$ iff $\mathcal{M}, x \not\models \phi$
  - $\mathcal{M}, x \models \phi \lor \psi$ iff $\mathcal{M}, x \models \phi$ or $\mathcal{M}, x \models \psi$
  - $\mathcal{M}, x \models [r]\phi$ iff for all $y \in W$, if $x R(r) y$ then $\mathcal{M}, y \models \phi$
- As a result,
  - $\mathcal{M}, x \models \langle r \rangle \phi$ iff there exists $y \in W$ such that $x R(r) y$ and $\mathcal{M}, y \models \phi$
Modal logic and mereotopology
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Mereotopological interpretation of modal logic: simple observations

**Difference modality:** \([d] \phi ::= [dc] \phi \land \ldots \land [ntppi] \phi\)

**Universal modality:** \([u] \phi ::= \phi \land [dc] \phi \land \ldots \land [ntppi] \phi\)

**Nominal:** \(nom(\phi) ::= \langle u \rangle (\phi \land [d] \neg \phi)\)

**Proper part:**
- \([pp] \phi ::= [tpp] \phi \land [ntpp] \phi\)
- \([ppi] \phi ::= [tppi] \phi \land [ntppi] \phi\)
Mereotopological interpretation of modal logic: examples

- $[u](harborcity \leftrightarrow (city \land \langle ppi \rangle harbor))$
- $[u](harbor \rightarrow (\langle ec \rangle river \lor \langle ec \rangle sea))$
- $[u](Dresden \rightarrow harborcity)$
- $[u](Elbe \rightarrow river)$
- $[u](Dresden \rightarrow \neg sea \land [ec]\neg sea \land [po]\neg sea \land [tpp]\neg sea \land [tppi]\neg sea \land [ntpp]\neg sea \land [ntppi]\neg sea)$
- $[u](Dresden \rightarrow \langle po \rangle Elbe \land (river \rightarrow Elbe) \land [ec](river \rightarrow Elbe) \land [po](river \rightarrow Elbe) \land [tpp](river \rightarrow Elbe) \land [tppi](river \rightarrow Elbe) \land [ntpp](river \rightarrow Elbe) \land [ntppi](river \rightarrow Elbe))$
Some modal logics for mereotopological relations

- $L_{RCC8}(\top)$: the logic of the class of all concrete frames of the form $P_{\text{reg}}$
- $L_{RCC8}(\mathbb{R}^n_{\text{reg}})$: the logic of the concrete frame consisting in all nonempty regular closed regions in $\mathbb{R}^n$
- $L_{RCC8}(\mathbb{R}^n_{\text{con}})$: the logic of the concrete frame consisting in all nonempty convex regular closed regions in $\mathbb{R}^n$
- $L_{RCC8}(\mathbb{R}^n_{\text{rec}})$: the logic of the concrete frame consisting in all nonempty rectangle regular closed regions in $\mathbb{R}^n$
- $L_{RCC8}(RS)$: the logic of the class of all abstract frames
Relationship between some modal logics for mereotopological relations

- $L_{RCC8}(RS) \not\subseteq L_{RCC8}(TOP), L_{RCC8}(IR_{reg}^n), L_{RCC8}(IR_{con}^n), L_{RCC8}(IR_{rec}^n)$

  Proof: $\text{nom}(p) \land \text{nom}(q) \land \langle u \rangle(p \land \langle dc \rangle q) \rightarrow \langle u \rangle(\langle ppi \rangle p \land \langle ppi \rangle q)$ is valid in $TOP, IR_{reg}^n,$ $IR_{con}^n$ and $IR_{rec}^n$ but not in $RS$

- $L_{RCC8}(TOP) \not\subseteq L_{RCC8}(IR_{reg}^n)$

  Proof: $\langle ppi \rangle \top$ is valid in $IR_{reg}^n$ but not in $TOP$
Modal logic and mereotopology
Modal logics for mereotopological relations

Proposition (Lutz and Wolter)
- $L_{RCC8}(TOP)$, $L_{RCC8}(IR_{reg}^n)$, $L_{RCC8}(IR_{con}^n)$, $L_{RCC8}(IR_{rec}^n)$ and $L_{RCC8}(RS)$ are undecidable.

Proposition (Lutz and Wolter)
1. $L_{RCC8}(TOP)$, $L_{RCC8}(IR_{reg}^n)$ and $L_{RCC8}(IR_{con}^n)$ are not recursively enumerable.
2. $L_{RCC8}(RS)$ is recursively enumerable.
One of the main open problems formulated by Lutz and Wolter was to find

- decidable modal logics based on a reasonable set of mereotopological relations

Nenov and Vakarelov present such a logic based on the following mereotopological relations

- overlap $O$ and underlap $\hat{O}$
- part-of $\leq$ and its converse $\geq$
- contact $C$ and dual contact $\hat{C}$
- interior part-of $\ll$ and its converse $\gg$
Contact algebra $\mathcal{B} = (B, 0, 1, \cdot, +, *, C)$

- $(B, 0, 1, \cdot, +, *)$ is a Boolean algebra with $*$ as the Boolean complement
- $xCy$ implies $x \neq 0$ and $y \neq 0$
- $(x + y)Cz$ is equivalent to $xCz$ or $yCz$
- $xC(y + z)$ is equivalent to $xCy$ or $xCz$
- $x \neq 0$ implies $xCx$
- $xCy$ implies $yCx$

The complement of $C$ is denoted by $\bar{C}$
Modal logic and mereotopology
The first-order logic of mereotopological structures

Relations $O$ of overlap and $\hat{O}$ of underlap

- $x O y ::= x \cdot y \neq 0$ and $x \hat{O} y ::= x + y \neq 1$

Relations $\leq$ of part-of and its converse $\geq$

- $x \leq y ::= x \cdot y^* = 0$ and $x \geq y ::= x^* \cdot y = 0$

Relation $\hat{C}$ of dual contact

- $x \hat{C} y ::= x^* \ C \ y^*$

Relations $\ll$ of interior part-of and its converse $\gg$

- $x \ll y ::= x \bar{C} y^*$ and $x \gg y ::= x^* \bar{C} y$
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Example of a contact algebra

- If $P$ is a point-based topology and $\mathcal{RC}_P = (\mathcal{RC}_P, 0_P, 1_P, \cdot_P, +_P, *_P, C_P)$ is the structure defined as follows:
  - $(\mathcal{RC}_P, 0_P, 1_P, \cdot_P, +_P, *_P)$ is the Boolean algebra of the regular-closed sets of $P$
  - $x C_P y$ iff $x \cap y \neq \emptyset$

  then $\mathcal{RC}_P$ is a contact algebra such that

  - $x O_P y$ iff $ln_P(x \cap y) \neq \emptyset$ and $x \hat{O}_P y$ iff $x \cup y \neq P$
  - $x \leq_P y$ iff $x \subseteq y$ and $x \geq_P y$ iff $x \supseteq y$
  - $x \hat{C}_P y$ iff $ln_P(x) \cup ln_P(y) \neq P$
  - $x \ll_P y$ iff $x \subseteq ln_P(y)$ and $x \gg_P y$ iff $ln_P(x) \supseteq y$
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Example of a contact algebra

- If \( G = (N, E) \) is a (reflexive and symmetric) graph then the structure \( S(G) = (2^N, \emptyset, N, \cap, \cup, \setminus, C_G) \) defined as follows:
  - \((2^N, \emptyset, N, \cap, \cup, \setminus)\) is the Boolean algebra of the subsets of \( N \)
  - \( x \ C_G y \) iff \( E(x) \cap E(y) \neq \emptyset \)

is a contact algebra such that

- \( x \ O_G y \) iff \( x \cap y \neq \emptyset \) and \( x \ \hat{O}_G y \) iff \( x \cup y \neq N \)
- \( x \ \leq_N y \) iff \( x \subseteq y \) and \( x \ \geq_N y \) iff \( x \supseteq y \)
- \( x \ \hat{C}_N y \) iff \( E(N \setminus x) \cap E(N \setminus y) \neq \emptyset \)
- \( x \ \ll_G y \) iff \( E(x) \cap E(N \setminus y) = \emptyset \) and \( x \ \gg_P y \) iff \( E(N \setminus x) \cap E(y) = \emptyset \)
Lemma

- The relations $O$, $\hat{O}$, $\leq$, $\geq$, $C$, $\hat{C}$, $\ll$ and $\gg$ satisfy the following first-order conditions:

1. $x \leq x$,
2. if $x \leq y$ and $y \leq x$ then $x = y$,
3. if $x \leq y$ and $y \leq z$ then $x \leq z$,
4. if $x O y$ then $y O x$,
5. if $x O y$ then $x O x$,
6. if $x \bar{O} x$ then $x \leq y$,
7. if $x O y$ and $y \leq z$ then $x O z$,
8. $x O x$ or $x \hat{O} x$,
9. if $x \bar{O} y$ and $x \hat{O} z$ then $y \leq z$,
10. if $x C y$ then $y C x$,
11. if $x O y$ then $x C y$,
12. if $x C y$ then $x O x$,
13. ...
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Lemma

- The relations $O$, $\hat{O}$, $\leq$, $\geq$, $C$, $\hat{C}$, $\ll$ and $\gg$ satisfy the following first-order conditions:
  1. $x \geq x$,
  2. if $x \geq y$ and $y \geq x$ then $x = y$,
  3. if $x \geq y$ and $y \geq z$ then $x \geq z$,
  4. if $x \hat{O} y$ then $y \hat{O} x$,
  5. if $x \hat{O} y$ then $x \hat{O} x$,
  6. if $x \hat{O} x$ then $x \geq y$,
  7. if $x \hat{O} y$ and $y \geq z$ then $x \hat{O} z$,
  8. $x \hat{O} x$ or $x O x$,
  9. if $x \hat{O} y$ and $x \bar{O} z$ then $y \geq z$,
  10. if $x \hat{C} y$ then $y \hat{C} x$,
  11. if $x \hat{O} y$ then $x \hat{C} y$,
  12. if $x \hat{C} y$ then $x \hat{O} x$,
  13. ...
A relational structure $\mathcal{W} = (W, O, \hat{O}, \leq, \geq, C, \hat{C}, \ll, \gg)$ is

- a mereotopological structure iff it satisfies the first-order conditions of the two above lemmas
- a standard (full standard) mereotopological structure iff there exists a contact algebra $\mathcal{B} = (B, 0, 1, \cdot, +, *, C)$ such that

  - $\mathcal{W} \subseteq B$ ($\mathcal{W} = B$)
  - for all $x, y \in B$, $x O y$ iff $x \cdot y \neq 0$, $x \hat{O} y$ iff $x + y \neq 1$, $x \leq y$ iff $x \cdot y^* = 0$, $x \geq y$ iff $x^* \cdot y = 0$, $x \hat{C} y$ iff $x^* C y^*$, $x \ll y$ iff $x \bar{C} y^*$ and $x \gg y$ iff $x^* \bar{C} y$
A relational structure $\mathcal{W} = (\mathcal{W}, O, \hat{O}, \leq, \geq, C, \hat{C}, \ll, \gg)$ is

- a mereotopological structure iff it satisfies the first-order conditions of the two above lemmas

- a completely standard (full completely standard) mereotopological structure iff there exists a contact algebra $\mathcal{B} = (B, 0, 1, \cdot, +, *, C)$ of regular closed subsets of some topological space such that
  - $\mathcal{W} \subseteq B (\mathcal{W} = B)$
  - for all $x, y \in B$, $x O y$ iff $x \cdot y \neq 0$, $x \hat{O} y$ iff $x + y \neq 1$, $x \leq y$ iff $x \cdot y^* = 0$, $x \geq y$ iff $x^* \cdot y = 0$, $x \hat{C} y$ iff $x^* C y^*$, $x \ll y$ iff $x \hat{C} y^*$ and $x \gg y$ iff $x^* \hat{C} y$
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The first-order logic of mereotopological structures

Proposition (Dimov and Vakarelov)

> Every contact algebra $\mathcal{B} = (B, 0, 1, \cdot, +, \star, C)$ can be isomorphically embedded into the contact algebra $\mathcal{R}C_P = (RCP, 0_P, 1_P, \cdot_P, +_P, \star_P, C_P)$ over some point-based topology $P$.

Corollary

> A mereotopological structure is standard iff it is completely standard.

Proposition (Nenov and Vakarelov)

> Every mereotopological structure is completely standard.
Modal logic and mereotopology
A modal logic for mereotopological structures

A polymodal logic based on mereotopological structures: 
**MTML**

- \( \phi ::= p \mid \bot \mid \neg \phi \mid (\phi \lor \psi) \mid [r]\phi \mid [U]\phi \)
  - where \( r \in \{O, \hat{O}, \leq, \geq, C, \hat{C}, \ll, \gg\} \)

- **Abbreviations**
  - Standard definitions for the remaining Boolean operations
  - \( \langle r \rangle \phi ::= \neg [r] \neg \phi \) where \( r \in \{O, \hat{O}, \leq, \geq, C, \hat{C}, \ll, \gg\} \)
  - \( \langle U \rangle \phi ::= \neg [U] \neg \phi \)

- **Intuitive meaning**
  - The interpretation of \([r]\phi\) is \(\phi\) holds in every region \(r\)-related to the current one
  - The interpretation of \(\langle r \rangle \phi\) is \(\phi\) holds in some region \(r\)-related to the current one
  - The interpretation of \([U]\phi\) is \(\phi\) holds in every region
  - The interpretation of \(\langle U \rangle \phi\) is \(\phi\) holds in some region
Modal logic and mereotopology
A modal logic for mereotopological structures

In the two above lemmas, the following first-order conditions are not modally definable

- if $x \leq y$ and $y \leq x$ then $x = y$
- if $x \geq y$ and $y \geq x$ then $x = y$

A generalized mereotopological structure is a generalization of the notion of a mereotopological structure by dropping these conditions and by adding the following conditions:

- if $x \leq y$ and $y \bar{O} y$ then $x = y$
- if $x \geq y$ and $y \bar{O} y$ then $x = y$
- if $x \leq y$, $y \bar{O} z$ and $x \bar{O} z$ then $x = y$
- if $x \geq y$, $y \bar{O} z$ and $x \bar{O} z$ then $x = y$
Modal logic and mereotopology
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Proposition

- Every mereotopological structure is a generalized mereotopological structure.

Proposition (Nenov and Vakarelov)

- Every generalized mereotopological structure is a $p$-morphie image of a mereotopological structure.
Modal logic and mereotopology
A modal logic for mereotopological structures

Axiomatization: Let \( MTML \) be the least normal logic in or language based on

- Sahlqvist axiom schemes corresponding to the first-order conditions defining generalized mereotopological structures
- the following axiom schemes:
  - \([U]\phi \rightarrow \phi\)
  - \(\phi \rightarrow [U]\langle U \rangle \phi\)
  - \([U]\phi \rightarrow [U][U]\phi\)
  - \([U]\phi \rightarrow [O]\phi \land [\hat{O}]\phi \land [\ll]\phi \land [\gg]\phi \land [C]\phi \land [\hat{C}]\phi \land [\lll]\phi \land [\ggg]\phi\)
Proposition (Nenov and Vakarelov): The following conditions are equivalent for any modal formula $\phi$:

1. $\phi$ is a theorem of $MTML$.
2. $\phi$ is valid in the class of all generalized mereotopological structures.
3. $\phi$ is valid in the class of all mereotopological structures.
4. $\phi$ is valid in the class of all standard and completely standard mereotopological structures.
Proposition (Nenov and Vakarelov)

\( \text{MTML does not possess the finite model property with respect to its standard semantics.} \)

Proof: The Grzegorczyk formula \([\leq](\leq(p \rightarrow [\leq]p) \rightarrow p) \rightarrow p\)

\( \rightarrow \) is true in all finite mereotopological structures

\( \rightarrow \) is falsified in the generalized mereotopological structures

\( \overline{W} = (W, O, \hat{O}, \leq, \geq, C, \hat{C}, \ll, \gg) \) defined by

\( W = \{a, b\} \)

\( O, \hat{O}, \leq, \geq, C, \hat{C}, \ll \text{ and } \gg \) coincides with \( W \times W \)
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Proposition (Nenov and Vakarelov)
- $MTML$ admits filtration with respect to its nonstandard semantics and hence is decidable.
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Bibliography


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