Krawtchouk and multivariate Krawtchouk polynomial hypergroups with Markov chains.

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Abstract

Geoff Eagleson’s (1969) characterization of reversible Markov chains with a stationary Binomial \((N, p)\) distribution having Krawtchouk polynomial eigenfunctions relies on the hypergroup property for the Krawtchouk polynomials \(\{Q_k(x; N, p)\}\) that

\[
Q_k(x; N, p)Q_k(y; N, p) = \mathbb{E}_{x,y}[Q_k(Z; N, p)],
\]

where \(Z\) is specific random variable on \(\{0, 1, \ldots, N\}\) whose distribution depends on \(x, y\). The extreme points of the set of Markov chains have eigenvalues \(\{Q_k(z; N, p)\}\) for fixed \(z\) and have an interesting Ehrenfest urn interpretation. Multivariate Krawtchouk polynomials \(\{Q_k(X; N, p, u)\}\) on the \(d\)-dimensional multinomial \((x; N, p)\) distribution are constructed from an orthogonal set of functions \(u = \{u^{(r)}; r = 0, \ldots, d - 1\}\) on \(p\). The polynomials satisfy a hypergroup property if and only if \(u\) satisfies a hypergroup property.
The reproducing kernel polynomials

\[ Q_k(x, y; N, p) = \sum_{|k|=k} Q_k(x; N, p, u) Q_k(y; N, p, u) \]

are invariant under the choice of \( \{ u^{(r)} \} \). They satisfy a duplication formula

\[ Q_k(x, y; N, p) = h_k(p) E_{x,y} \left[ Q_k(Z; N, p) \right] , \]

where the right side polynomials are 1-dimensional Krawtchouk polynomials with \( 1 - p \leq \min_{j \in [d]} p_j \) and \( h_k(p) \) are normalizing constants. This hypergroup property leads to a characterization of reversible Markov chains with multinomial stationary distributions and multivariate Krawtchouk eigenfunctions.
Krawtchouk Polynomials

Orthogonal polynomials on $ \binom{N}{x} p^x q^{N-x}, x = 0, 1, \ldots, N$

Generating function

$$G(z; x) = \sum_{n=0}^{N} P_n(x) \frac{z^n}{n!} = (1 + qz)^x (1 - pz)^{N-x}$$

$$Q_n(x) = \frac{P_n(x)}{P_n(0)}$$

$$Q_n(x) = \sum_{\nu=0}^{N} ( - q/p)^\nu \binom{x}{\nu} \binom{N-x}{n-\nu} \binom{N}{n}$$

$$h_n^{-1} = \mathbb{E}(Q_n(X)^2) = \binom{N}{n}^{-1} (q/p)^n$$

Self-dual relationship $ Q_n(x) = Q_x(n) $
Building Krawtchouck polynomials

Let \( \{\xi_i\}_{i \in [N]} \) be a Bernoulli \((p)\) sequence of iid random variables.

\[
X = \xi_1 + \ldots + \xi_N.
\]

A complete orthogonal set of functions on the distribution of each \( \xi \) is \( \{1, \xi - p\} \), and a complete set on the product distribution of \( \{\xi_i\}_{i \in [N]} \) is the direct product set

\[
\bigotimes_{i=1}^N \{1, \xi_i - p\}
\]

The \( n \)th Krawtchouk polynomial is the \( n \)th symmetric product in the second components of the sets.

\[
P_n(X) = n! \sum_{\sigma \in S_N} (\xi_{\sigma(1)} - p) \cdots (\xi_{\sigma(n)} - p)
\]

\( S_N \) is the symmetric group of permutations of \( 1, 2, \ldots, N \).
Krawtchouck polynomials

\[
Q_n(X) = \binom{N}{n}^{-1} \sum_{\sigma \in S_N} \frac{p - \xi_{\sigma(1)}}{p} \ldots \frac{p - \xi_{\sigma(n)}}{p}
= \sum_{\nu=0}^{N} (-q/p)^\nu \frac{(X)_\nu (N-X)_{n-\nu}}{(N)_n}
\]

\(Q_n(X)\) is the average of the \(n\)th symmetric product in the exchangeable sequence of random variables \(\{(p - \xi_i)/p\}\),
Hypergroup property (Eagleson, 1969)

**Proposition** Take $p \geq 1/2$. For $x, y, z = 0, 1, \ldots, N$

$$K(x, y, z) = \sum_{n=0}^{N} h_n Q_n(x) Q_n(y) Q_n(z) \geq 0$$

and

$$K_{xy}(z) = \binom{N}{z} p^z q^{N-z} K(x, y, z)$$

is a probability distribution in $z$.

**Duplication formula**

$$Q_n(x) Q_n(y) = \mathbb{E}_{K_{xy}} [Q_n(Z)]$$
Hypergroup property, proof

\[
K(x, y, z) = \sum_{n=0}^{N} h_n Q_n(x) Q_n(y) Q_n(z)
\]

\[
= q^{-N} \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n} Q_x(n) Q_y(n) Q_z(n)
\]

\[
= q^{-N} \mathbb{E}[Q_x(W) Q_y(W) Q_z(W)],
\]

where \(W\) is a Binomial \((N, p)\) random variable.

Now use the symmetric function representation for the polynomials in \(W\).

\[
\mathbb{E}[(p - \xi_i)(p - \xi_j)(p - \xi_k)] \geq 0
\]

The only case that needs checking (recall \(p \geq 1/2\)):

\[
\mathbb{E}[(p - \xi_i)^3] = p(-q)^3 + qp^3 = qp(2p - 1) \geq 0
\]
Lancaster problem

Characterize the convex set of transition functions in terms of \( \{\rho_n\} \).

\[
P(X_t = y \mid X_{t-1} = x) = \binom{N}{y} p^y q^{N-y} \left\{ 1 + \sum_{n=1}^{N} \rho_n h_n Q_n(x) Q_n(y) \right\}
\]

Solution

\[
\rho_n = \mathbb{E}[Q_n(Z)]
\]

for a random variable \( Z \) on \( \{0, 1, \ldots, N\} \).
Generalized Ehrenfest Urn $p, q$

An urn with $X_t$, the number of red balls, having a stationary Binomial $(N, p)$ distribution. $q \leq p$.

Choose $z \in \{0, 1, 2, \ldots, N\}$ balls at random without replacement and independently change the colours of the balls drawn according to the transition matrix

$$P = \begin{bmatrix} 0 & 1 \\ q/p & 1 - q/p \end{bmatrix}$$

$X_t$ is a reversible Markov Chain with Binomial $(N, p)$ stationary distribution. The transition functions of $X_t$ are extreme points in the Lancaster class

$$P(X_t = y \mid X_{t-1} = x) = \binom{N}{y} p^y q^{N-y} \left\{ 1 + \sum_{n=1}^{N} h_n Q_n(z) Q_n(x) Q_n(y) \right\}$$
Multivariate Krawtchouk polynomials.
Orthogonal polynomials on the multinomial distribution.

\[ m(x; N, p) = \binom{N}{x} p_1^{x_1} \cdots p_d^{x_d}, \quad \sum_{i=1}^{d} x_i = N. \]

Let \( Z_1, \ldots, Z_N \) be independent identically distributed random variables indicating outcomes in \( N \) trials each with \( d \) possible outcomes such that

\[ P(Z = k) = p_k, \quad k = 1, \ldots, d. \]

Then

\[ X_i =^D |\{Z_k : Z_k = i, k = 1, \ldots, d\}| \]
Orthogonal basis

Let \( \{ u^{(l)} \}_{l=0}^{d-1} \) be a complete set of orthogonal functions on a probability distribution \( \{ p_j \}_{j=1}^{d} \) with \( u^{(0)} \equiv 1 \) such that for \( k, l = 0, 1, \ldots, d-1 \)

\[
\sum_{j=1}^{d} u_j^{(k)} u_j^{(l)} p_j = a_k \delta_{kl}
\]

A complete orthogonal basis on the \( N \) trials is the direct product

\[
\bigotimes_{i=1}^{N} \{ 1, u_{Z_i}^{(k)} \}_{k=1}^{d-1}
\]

The symmetrized direct product basis is the set of multivariate Krawtchouk polynomials.
Multivariate Krawtchouk polynomials

A set of orthogonal polynomials on the multinomial distribution $m(x; N, p)$, \{ $Q_n(X; N, p, u)$ \}, with $n = (n_1, \ldots, n_{d-1})$, and $|n| \leq N$, are the coefficients of $w_1^{n_1} \cdots w_{d-1}^{n_{d-1}}$ in the generating function

$$G(X, w; u) = \prod_{k=1}^{N} \left( 1 + \sum_{l=1}^{d-1} w_l u^{(l)}_{Z_k} \right) = \prod_{j=1}^{d} \left( 1 + \sum_{l=1}^{d-1} w_l u^{(l)}_{Z_k} \right)^{X_i}$$

Combinatorial expression

$$Q_n(X; N, p, u) = \sum \prod_{\{A_l\} \in A_1} u^{(1)}_{Z_{k_1}} \cdots \prod_{k_{d-1} \in A_{d-1}} u^{(d-1)}_{Z_{k_{d-1}}}$$

Summation is over all partitions of subsets of \{1, \ldots, N\}, \{A_l\} such that $|A_l| = n_l$, $l = 1, \ldots, d - 1$. 
Hypergeometric form

Define

\[ F_1^{(n)}(-m, -x; -N; z) := \sum_{k_{..} \leq N} \frac{\prod_{i=1}^{n} (-m_i)^{(k_i).} \prod_{j=1}^{n} (-x_j)^{(k,j).}}{\prod_{ij} k_{ij}!(-N)_{(k..)}} \prod_{i,j} z_{ij}^{k_{ij}} \]

where a \( \cdot \) in an index means sum, and the sum is over all \( n \times n \) matrices \((k_{ij})\) with non-negative integer entries with sum of entries at most \( N \).

Proposition Let \( \{u_j^{(l)}\} \) be an orthogonal basis, \( 0 \leq l \leq d - 1, 1 \leq j \leq d \), with \( u_d^{(l)} \equiv 1 \) for all \( l \). Let \( z_{ij} = 1 - u_j^{(i)} \), \( i, j \in [d - 1], \)

\[ Q_m(x, u) = \frac{N!}{\prod_{i=1}^{d} m_i!} F_1^{(d-1)}(-m, -x; -N; z) \]
Indexing

$Q_n(X; N, p, u)$ is a polynomial in

$$S_1 = \sum_{j=1}^{d} u_j^{(1)} X_j, \ldots, S_{d-1} = \sum_{j=1}^{d} u_j^{(d-1)} X_j$$

whose only leading term is proportional to $S_1^{n_1} \cdots S_{d-1}^{n_{d-1}}$.

Orthonormal scaling

$$\mathbb{E} \left[ Q_m(X; N, p, u) Q_n(X; N, p, u) \right] = \delta_{mn} \binom{N}{n} a_1^{n_1} \cdots a_{d-1}^{n_{d-1}}$$

$$= \delta_{mn} h_n^{-1}$$

Orthonormal polynomials

$$Q_n^\circ(X; N, p, u) = \sqrt{h_n} Q_n(X; N, p, u)$$
$r \times s$ Contingency table ($r \leq s$)

$N$ observations are placed independently into a $r \times s$ table with the probability of an observation falling into cell $(i, j)$ being $p_{ij}$.

**Lancaster expansion**

$$p_{ij} = p^r_i p^c_j \left\{ 1 + \sum_{k=1}^{r-1} \rho_k u^{(k)}_i v^{(k)}_j \right\}$$

$u$ and $v$ are orthonormal functions on $p^r$ and $p^c$.

$N_{ij}$ fall into cell $(i, j)$.

**Marginal counts** $X_i = \sum_{j=1}^{c} N_{ij}$, $Y_j = \sum_{i=1}^{r} N_{ij}$

$$P(X = x, Y = y) = m(x; N, p^r)m(y; N, p^c)$$

$$\times \left\{ 1 + \sum_n \rho_1^{n_1} \cdots \rho_{r-1}^{n_{r-1}} \binom{N}{n}^{-1} Q_n(x; N, u, p^r) Q_n(y; N, v, p^c) \right\}$$

Aitken and Gonin (1935), $2 \times 2$ table with Binomial marginals.
Hypergroup properties

Let \( \{u^{(i)}\}_{i=1}^{d-1} \) be orthogonal functions on \( p \) with \( u^{(0)} \equiv 1 \).

Assume that there is a scaling such that \( u^{(l)}_{d} = 1, \ l = 0, \ldots, d - 1 \).

Recall that \( \sum_{j=1}^{d} u^{(r)}_{j} u^{(s)}_{j} p_{j} = a_{r} \delta_{rs} \).

The hypergroup property, if it holds, is that for \( j, k, l = 1, \ldots d \)

\[
\sum_{i=0}^{d-1} a_{i}^{-1} u^{(i)}_{j} u^{(i)}_{k} u^{(i)}_{l} \geq 0
\]

The Lancaster problem

\[
p_{k}\{1 + \sum_{i=1}^{d-1} \rho_{i} a_{i}^{-1} u^{(i)}_{j} u^{(i)}_{k}\} \geq 0
\]

is solved by \( \rho_{i} = \mathbb{E}[u_{\xi}^{(i)}] \) with \( \xi \) a random variable on \( 1, \ldots, d \).
Irwin-Lancaster basis

Orthogonal basis, \( u^{(0)} \equiv 1 \) and

\[
\begin{aligned}
u_k^{(j)} &= \begin{cases} 
0, & k < j \\
-(1 - |p_j|)/p_j, & k = j \\
1, & j < k \leq d 
\end{cases}
\end{aligned}
\]

Note that \( u_d^{(l)} = 1, \ l = 1, \ldots, d - 1 \)

**Definition** \( \mathbf{p} \) is strongly monotone if

\[
p_d \leq p_{d-1}, \ p_d + p_{d-1} \leq p_{d-2}, \ldots, \ p_d + \cdots + p_2 \leq p_1
\]

**Proposition** Let \( \mathbf{p} \) be strongly monotone, then \( \mathbf{u} \) satisfies the hypergroup property.
Hypergroup Property for Krawtchouk polynomials

Let \( \{ u^{(i)} \} \) be an orthogonal basis such that \( u_d^{(i)} = 1, \ i = 0, \ldots, d \). Then scale (with \( e_d = (\delta_{id}) \))

\[
Q_n^\diamond(x, u) = \binom{N}{n}^{-1} Q_n(x, u) \quad \text{so} \quad Q_n^\diamond(Ne_d, u) = 1.
\]

Hypergroup property

\[
\sum_{n:|n|\geq0} h_n Q_n(x, u) Q_n(y, u) Q_n^\diamond(z, u) \geq 0
\]

**Proposition** The hypergroup property for the multivariate Krawtchouk polynomials holds if and only if the hypergroup property for \( u \) holds; that is

\[
s(j, k, l) \geq 0 \quad \text{for all } j, k, l \in \{1, 2, \ldots, d\}
\]
Irwin-Lancaster Krawtchouk polynomials

Proposition

\[ Q_n(x; N, p, u) = c_n \prod_{j=1}^{d-1} \left( -N + |x_{j-1}| + |n^{j+1}| \right)_{n_j} \]

\[ \times P_{n_j}(x_j; N - |x_{j-1}| - |n^{j+1}|) \]

where \( c_n \) are constants and \( \{ P_n(x; N, p) \} \) are 1-dimensional Krawtchouk polynomials.

Notation: \( n^{j+1} = (n_{j+1}, \ldots, n_d) \), \( x_{j-1} = (x_1, \ldots, x_{j-1}) \).

This conditional Binomial construction is used by Xu (2013).

\( Q_n(x; N, p, u) \) satisfies a hypergroup property if \( p \) is strongly monotone.
Markov chains, Lancaster problem

Transition functions

\[
P(X_t = y \mid X_{t-1} = x) = m(y; N, p) \left\{ 1 + \sum_{n; |n| \geq 1} \rho_n h_n Q_n(x, u) Q_n(y, u) \right\}
\]

Assume that

\[s(j, k, l) \geq 0\] for all \(j, k, l \in \{1, 2, \ldots, d\}\)

The right side of (*) is \(\geq 0\) if and only if

\[\rho_n = \mathbb{E}[Q_n^\circ(Z; u)]\]

where \(Z\) is a \(d\)-dimensional non-negative integer random variable with \(|Z| = N\).

Extreme points \(\rho_n = Q_n^\circ(z; u)\) for fixed \(z\).
Extreme point Markov chains

Transition functions

\[
P(X_t = y \mid X_{t-1} = x) = m(y; N, p)\left\{1 + \sum_{n;|n|\geq1} Q_n^\circ(z; u)h_n Q_n(x; u) Q_n(y; u)\right\}
\]

An urn contains \(N\) balls of \(d\) colours. Choose \(z_1, z_2, \ldots, z_d\) balls without replacement, then in group \(l\), with \(z_l\) balls, independently change colours according to a transition probability matrix

\[
p_{l:j,k} = p_k s(j, k, l) = p_k \left\{1 + \sum_{i=1}^{d-1} u_l^{(i)} a_i^{-1} u_j^{(i)} u_k^{(i)} \right\}
\]

Many other interesting composition Markov chains are in the Lancaster class with Krawtchouk polynomials as eigenfunctions, Zhou and Lange (2009).
Reproducing kernel polynomials

\[ Q_k(x, y; N, p) = \sum_{|k|=k} h_k Q_k(x; N, p, u) Q_k(y; N, p, u) \]

are invariant under the choice of basis \( u \).

**Duplication formula**

\[ Q_k(x, y; N, p) = h_k(p) \mathbb{E}_{x,y}[Q_k(Z; N, p)], \]

where the right side polynomials are 1-dimensional Krawtchouk polynomials with \( 1 - p \leq \min_{j \in [d]} p_j \) and \( h_k(p) \) are normalizing constants.
Poisson Kernel and Contingency tables

In a $d \times d$ contingency table suppose

$$p_{ij} = p_i p_j \left\{ 1 + \sum_{r=1}^{d-1} \rho_r u_i^{(r)} u_j^{(r)} \right\}$$

Then the eigenvalues in a Lancaster expansion of the joint distribution of the marginals in a contingency table with $N$ entries are

$$\rho_k = \rho_1^{k_1} \cdots \rho_{d-1}^{k_{d-1}}$$

If $\rho_1 = \rho_2 = \cdots = \rho_{d-1} = \rho$ then $\rho_k = \rho^{|k|}$ and

$$p_{ij} = p_i p_j \left\{ 1 - \rho + \delta_{ij} \rho p_i^{-1} \right\}$$

which is $\geq 0$ if

$$-\frac{1}{\min_i p_i} - 1 \leq \rho \leq 1$$
Poisson Kernel and Contingency tables

\[ P(X = x, Y = y) \]

\[ = m(x; N, p)m(y; N, p)\{1 + \sum_{k=1}^{d-1} \rho^k Q_k(x, y; N, p)\} \]

which is \( \geq 0 \) if

\[ -\frac{1}{\min_i \rho_i} - 1 \leq \rho \leq 1 \]

A direct probability calculation leads to an explicit expression for the Poisson kernel and reproducing kernel polynomials.

\[ P(X = x, Y = y) = m(x; N, p)m(y; N, p) \]

\[ \times \sum_{|z| \leq N} \binom{N}{z, x - z, y - z} \prod_{i=1}^{d}(1 - \rho + \rho p_i^{-1})^{z_i}(1 - \rho)^{|x| - |z| + |y| - |z|} \]
Markov Chains, reproducing kernel polynomials

Transition functions

\[ P(X_t = y \mid X_{t-1} = x) = m(y; N, p) \left\{ 1 + \sum_{k=1}^{d-1} \rho_k Q_k(x, y; N, p) \right\} \]

The right side is \( \geq 0 \) if and only if

\[ \rho_k = \mathbb{E}[Q_k(Z; N, p)] \]

where \( Z \) is a random variable on \( 0, \ldots, N \) and \( 1 - p \leq \min_{j \in [d]} p_j \).

Extreme points

\[ \rho_k = Q_k(z; N, p), \ z \in \{0, 1, \ldots, N\} \]
An urn has $N$ balls of $d$ colours labelled 1, 2, ..., $d$. $X_t; t = 0, 1, \ldots$ counts the number of balls of the $d$ colours at times $t$.

In a transition choose $z$ balls without replacement from the urn to change colour independently with transition probabilities

$$
\begin{cases}
  i \to j & \frac{p_j}{p}, \ j \neq i \\
  i \to i & \frac{(p_i - q)}{p}, \ i = j
\end{cases}
$$

where $q = 1 - p \leq \min_{j \in [d]} p_j$.

Then

$$
\rho_k = Q_k(z; N, p), \ z \in \{0, 1, \ldots, N\}
$$
Multivariate Poisson-Charlier polynomials

Poisson-Charlier polynomials

\[\sum_{n=0}^{\infty} C_n(x; \mu) \frac{z^n}{n!} = e^z \left(1 - \frac{z}{\mu}\right)^x\]

\(X = (X_1, \ldots, X_d)\) are independent Poisson random variables with means \(\mu = (\mu_1, \ldots, \mu_d)\).

Construction

\[G^\circ(X, z) = \exp \left\{ \sum_{i=1}^{d} z_i p_i \right\} \prod_{i=1}^{d} \left(1 - \frac{z_i}{\mu}\right)^{x_i}\]

where \(\mu = \sum_{i=1}^{d} \mu_i\), \(p_i = \mu_i / \mu\).

\[\mathbb{E}[G^\circ(X, z)G^\circ(X, z')] = \exp \left\{ \sum_{i=1}^{d} z_i z'_i p_i / \mu \right\}\]
Let $\{u^{(j)}\}_{j=0}^{d-1}$ be an orthogonal basis on $p$, $(u^{(0)} \equiv 1)$ with

$$\sum_{i=1}^{d} u_i^{(j)} u_i^{(k)} p_i = \delta_{jk} a_j$$

Let $z_i := w_0 - \sum_{j=1}^{d-1} u_i^{(j)} w_j$, then

$$\sum_{i=1}^{d} z_i z_i' p_i = \sum_{i=1}^{d} w_j w_j' a_j$$

$$G(x, w, u) = G^\circ(x, z(w))$$

$$= e^{w_0} \prod_{i=1}^{d} \left(1 - \mu^{-1} w_0 + \mu^{-1} \sum_{j=1}^{d-1} u_i^{(j)} w_j\right)^{x_i}$$

generates multivariate Poisson-Charlier polynomials as coefficients of $w_0^{n_0} \ldots w_d^{n_d}$. 
Multivariate Poisson-Charlier polynomials

Explicit form

\[ C_n(x; \mu, u) = (n_0)!^{-1} C_{n_0}(|x| - n_1; \mu) \mu^{-|n_1|} Q_{n_1}(x; |x|, p, u) \]

where \( n_1 = (n_1, \ldots, n_{d-1}) \).

\[ h_n(\mu, u) = \mathbb{E}[C_n(X; \mu, u)^2] = \mu^{-|n|} \prod_{j=0}^{d-1} \frac{a_j^{n_j}}{n_j!} \]

Natural construction: \(|X|\) is Poisson \((\mu)\) and the conditional distribution of \(X\) given \(|X| = |x|\) is multinomial \(m(x; |x|, p)\).
Poisson Table

Let \((X_{ij}), \ i = 1, \ldots, r, \ j = 1, \ldots, c, \ r \leq c\), be an \(r \times c\) array of independent Poisson random variables with means \(\mu = (\mu_{ij})\).

Let \(p_{ij} = \mu_{ij}/|\mu|\). Suppose there is a Lancaster expansion for \(i = 1, \ldots r, \ j = 1, \ldots c\),

\[ p_{ij} = p_i \cdot p_j \left\{ 1 + \sum_{k=1}^{r-1} \rho_k u_i^{(k)} v_j^{(k)} \right\} \]

where \(\{u^{(k)}\}_{k=0}^{r-1}\) and \(\{v^{(k)}\}_{k=0}^{c-1}\) are orthonormal functions on \(\{p_i.\}\) and \(\{p. j\}\) respectively.

Denote the marginals of the table by \(X = (X_i.)_{i=1}^r\), \(Y = (Y. j)_{j=1}^c\).

Then

\[ P\left( X = x, Y = y \right) = P\left( X = x \right) P\left( Y = y \right) \]

\[ \times \left\{ 1 + \sum_{n; |n| \geq 1} \rho_1^{n_1} \cdots \rho_r^{n_{r-1}} h_n(\mu, u) C_n(\mu; \mu_X, u) C_{n^*}(y; \mu_Y, v) \right\}, \]

where \(n = (n_0, n_1, \ldots, n_{r-1})\), \(n^* = (n_0, n_1, \ldots, n_{r-1}, 0, \ldots, 0)\).
1-dimensional Poisson-Charlier polynomials

Lancaster problem

\[ e^{-\mu \mu^x} e^{-\mu \mu^y} \left\{ 1 + \sum_{n=1}^{\infty} \rho_n (n!)^{-1} \mu^n C_n(x; \mu) C_n(y; \mu) \right\} \geq 0 \]

if and only if \( \rho_n = \mathbb{E}[\xi^n] \), where \( \xi \) is a random variable on \([0, 1]\).

Random elements in common with random means

The joint probability generating function of \((X, Y)\) is

\[ \mathbb{E}\left[ s^X t^Y \right] = \mathbb{E}\left[ \exp \left\{ \mu(1-\xi)(s-1) + \mu(1-\xi)(t-1) + \mu \xi (st - 1) \right\} \right] \]
Lancaster problem for multivariate Poisson-Charlier

**Proposition** Let \( \{u^{(j)}\}_{j=0}^{d-1} \) be an orthogonal basis on \( p = \mu/\mu \) such that \( s(j, k, l) \geq 0, j, k, l = 1, \ldots, d. \)

Then

\[
\prod_{i=1}^{d} \left[ e^{-\mu_i} \frac{\mu_i^{x_i}}{x_i!} e^{-\mu_i} \frac{\mu_i^{y_i}}{y_i!} \right] \left\{ 1 + \sum_{|n| \geq 1} \rho_n h_n(\mu, u) C_n(x; \mu, u) C_n(y; \mu, u) \right\} \geq 0
\]

if and only if

\[
\rho_n = \mathbb{E} \left[ |\xi|^{n_0} \prod_{j=1}^{d-1} \left( \sum_{i=1}^{d-1} u_i^{(j)} \xi_i \right)^{n_j} \right],
\]

where \( \xi \) is a random vector in the unit simplex.
Probability generating function

$$
\mathbb{E} \left[ \prod_{i,j=1}^{d} s_i^{X_i} t_j^{Y_j} \right] = \mathbb{E} \left[ \exp \left\{ (1 - |\xi|) \sum_{i=1}^{d} \mu_i (s_i - 1 + t_i - 1) \right\} 
\times \exp \left\{ \mu \sum_{l=1}^{d} \xi_l \sum_{j,k=0}^{d} p_j p_k s(j, k, l) (s_j t_k - 1) \right\} \right]
$$

Note that

$$p_j p_k s(j, k, l) = p_j p_k \left\{ 1 + \sum_{r=1}^{d-1} u_i^{(r)} a_r^{-1} u_j^{(r)} u_k^{(r)} \right\}$$

is an extreme point Lancaster distribution.

$$X_j \overset{\mathcal{D}}{=} Z_j + \sum_{k=1}^{d} Z_{jk}, \quad Y_k \overset{\mathcal{D}}{=} Z'_k + \sum_{j=1}^{d} Z_{jk}$$

where the $Z$'s are Poisson with random means.
Multivariate Meixner polynomials

Similar construction to the multivariate Charlier polynomials. Orthogonal on

\[ P(x; \alpha, \rho, p) = \frac{\Gamma(\alpha + |x|)}{\Gamma(\alpha)} \frac{\rho^{|x|}}{(1 + \rho)^{\alpha + |x|}} \left( \frac{|x|}{x} \right) \prod_{i=1}^{d} p_{i}^{x_{i}} \]

\( x_1, \ldots x_d = 0, 1, \ldots \) with parameters \( \alpha, \rho > 0. \)

\[ P(x; \mu, p) = \prod_{i=1}^{d} e^{-\mu p_{i}} \frac{(\mu p_{i})^{x_{i}}}{x_{i}!} \]

Poisson-Gamma mixture, \( \mu \sim \text{Gamma} (\alpha, \rho). \)

\[ P(x; \alpha, \rho, p) = \mathbb{E}_{\mu} \left[ P(x; \mu, p) \right] \]
Multivariate Meixner polynomials

Generating function

$$(1 - w_0)^{-(x-\alpha)} \prod_{i=1}^{d} \left( 1 - w_0 \frac{1 + \rho}{\rho} + \sum_{j=1}^{d-1} u_i^{(j)} \right)^{x_i}$$

Mixture

$$\left( \frac{\rho}{1 + \rho} \right)^{n_0} M_n(x; \alpha, \rho, p) = \mathbb{E}_{\mu} \left[ P(x; \mu, p) \mu^{n_1} C_n(x; \mu, u) \right]$$

Explicit form

$$\frac{\Gamma(\alpha + n_0)}{\Gamma(\alpha) n_0!} M_{n_0}(|x| - |n_1|; \alpha, \rho/(1 + \rho)) Q_{n_1}(x; |x|, p, u)$$
References


