Hydrodynamic limit and numerical solutions in a system of self-propelled particles

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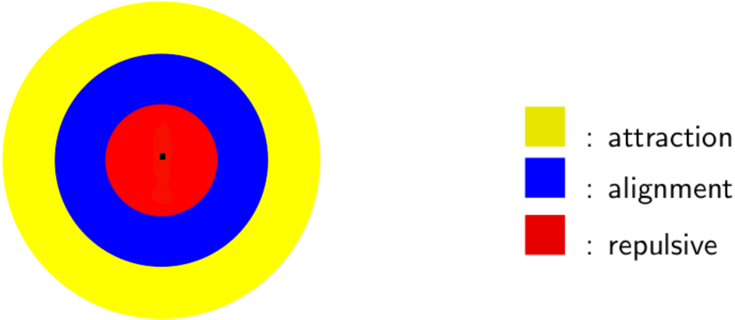
Joint work with: P.Degond, G.Dimarco, N.Wang

The classical model with 3 zones



Ref: Aoki (1982), Reynolds (1986), Couzin(2002) ...

The classical model with 3 zones

Ref: Vicsek (1995), Chaté et al. (2008a),Bertin et al.(2009) Ginelli et al.(2010) ...

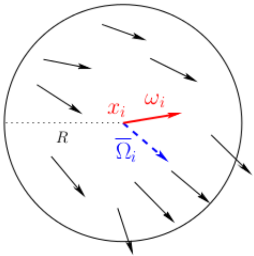
The Vicsek model

Discrete system

$$\frac{X_i^{n+1} - X_i^n}{n\triangle t} = \omega_i^n \quad (1)$$

$$\omega_i^{n+1} = \bar{\Omega}_i^n + \varepsilon \quad (2)$$

with $\bar{\Omega}_i^n = \frac{\sum_{X_j - X_i < R} \omega_j^n}{|\sum_{X_j - X_i < R} \omega_j^n|}$, ε is the noise



The continuous model (P.Degond, S.Motsch 08)

$$\frac{dX_i(t)}{dt} = \omega_i \quad (3)$$

$$d\omega_i(t) = (Id - \omega_i \otimes \omega_i)(\nu\bar{\Omega}_i dt + \sqrt{2D}dB_t) \quad (4)$$

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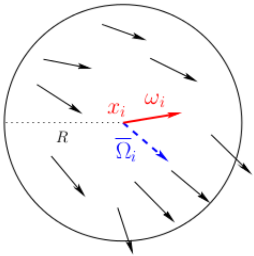

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The individual based model

$$\frac{dX_i(t)}{dt} = v_i \quad (5)$$
$$v_i(t) = v_0\omega_i + \mu F_i \quad (6)$$
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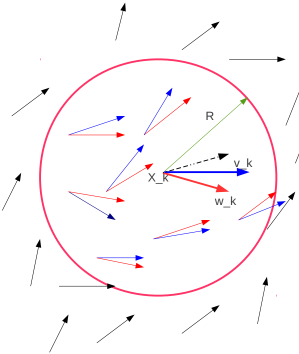

- $\bar{\Omega}_i = \frac{\sum K(|X_i - X_j|)\omega_j}{|\sum K(|X_i - X_j|)\omega_j|}$
- $F_i = -\frac{1}{N}\sum_{j=1}^{N} \nabla\phi(X_i - X_j)$ The repulsive force acts on the i-th particle

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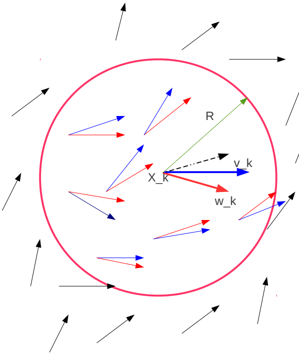

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The kinetic equation

Mean field limit

The density of particles $f(x, \omega, t)$ satisfies:

$$\partial_t f + \nabla_x \cdot (vf) + \nabla_\omega \cdot (Gf) + \nabla_\omega \cdot (Hf) - D\triangle_\omega f = 0 \quad (8)$$

where

$$v = v_0\omega - \mu \int \nabla\phi(|x-y|)f(y, \omega, t)d\omega \quad (9)$$

$$G(x, \omega, t) = \nu(Id - \omega \otimes \omega)\bar{\Omega}_f(x, t) \quad (10)$$

$$\bar{\Omega}_f(x, t) = \frac{j_f}{|j_f|} \quad (11)$$

$$j_f(x, t) = \int K(|x-y|)f(y, \omega, t)dyd\omega \quad (12)$$

$$H(x, \omega, t) = \alpha(Id - \omega \otimes \omega)v \quad (13)$$

Scaling

We define the regime we are interested in:

- The interaction regimes of kernels K, ϕ are small.
- The diffusion and social terms are large.
- The alignment term prevails over the repulsion term
- The other terms are kept order of 1.

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Finally, the function $f^\varepsilon(x,\omega,t)$ satisfies

$$\varepsilon(\partial_t f^\varepsilon + \nabla_x \cdot (vf^\varepsilon) + \nabla_\omega \cdot (Hf^\varepsilon) + \nabla_\omega \cdot (L^\varepsilon f^\varepsilon)) = Q(f^\varepsilon) \quad (14)$$

with

$$v(x,\omega,t) = v_0\omega - \mu\Phi\nabla\left(\int f^\varepsilon d\omega\right) \quad (15)$$

$$G^\varepsilon(x,\omega,t) = \nu(Id - \omega\otimes\omega)\frac{J_f^\varepsilon(x,t)}{|J_f^\varepsilon(x,t)|} \quad (16)$$

$$L_{f^\varepsilon}^\varepsilon = k_0\nu(Id - \omega\otimes\omega) l_{f^\varepsilon}^\varepsilon(x,t) \quad (17)$$

$$l_{f^\varepsilon}^\varepsilon(x,t) = (Id - \Omega\otimes\Omega)\frac{\triangle(J_f^\varepsilon(x,t))}{|J_f^\varepsilon(x,t)|} \quad (18)$$

$$J_f^\varepsilon(x,t) = \int f^\varepsilon \omega d\omega \quad (19)$$

The kinetic equation

The Properties of $Q(f)$

$$Q(f) = -\nabla \cdot (Gf) + D\triangle f \tag{20}$$

- The equilibrium of $Q(f)$

$$M_{\Omega}(\omega) = Cexp(\frac{\omega \cdot \Omega}{d}) \tag{21}$$

with $d = D/\nu$, for Ω is an arbitrary direction.

The macroscopic model

The density $\rho(x,t)$ and the average direction $\Omega(x,t)$ satisfy the following equations

$$\partial_t \rho + \nabla_x \cdot (\rho u_1) = 0 \tag{22}$$

$$\partial_t \Omega + \rho(u_2 \cdot \nabla)\Omega + \delta(Id - \Omega \otimes \Omega)\nabla_x \rho = \gamma(Id - \Omega \otimes \Omega)\triangle(\rho\Omega) \tag{23}$$

with $u_1 = c_1 v_0 \Omega - \mu\Phi\nabla\rho$, $u_2 = c_2 v_0 \Omega - \mu\Phi\nabla\rho$, $\delta = v_0 d + \alpha\mu\Phi\rho(2d + c_2)$, $\gamma = (2d + c_2)k_0$

Numerical solutions

We want to numerically solve the macroscopic model

$$\partial_t \rho + \nabla_x \cdot (\rho u_1) = 0$$

$$\partial_t \Omega + \rho(u_2 \cdot \nabla)\Omega + \delta(Id - \Omega \otimes \Omega)\nabla_x \rho$$

$$= \gamma(Id - \Omega \otimes \Omega)\triangle(\rho\Omega)$$

$$|\Omega| = 1$$

- The model is non- conservative
- has a geometric constraint

$\Longrightarrow$ We replace the geometric constraint by a relaxation operator. (Ref: S.Motsch, L.Navoret)

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$$\partial_t(\rho\Omega) + \nabla \cdot (\rho u_2 \otimes \Omega) + \delta \nabla \rho - \gamma \triangle(\rho\Omega) = \frac{\rho}{\eta}(1 - |\Omega|^2)|\Omega|^2$$

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Splitting method

The idea is to split the relaxation model into two parts. Firstly, we consider this part

$$\partial_t \rho + \nabla \cdot (\rho u_1) = 0 \tag{24}$$

$$\partial_t(\rho\Omega) + \nabla \cdot (\rho u_2 \otimes \Omega) + \delta \nabla \rho - \gamma \triangle(\rho\Omega) = 0 \tag{25}$$

and the relaxation part

$$\partial_t \rho = 0$$

$$\partial_t(\rho\Omega) = \frac{\rho}{\eta}(1 - |\Omega|^2)|\Omega|^2. \tag{26}$$

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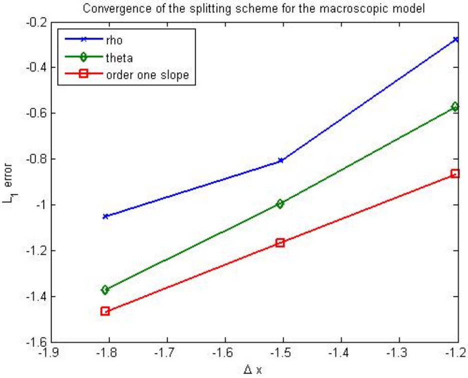
and the relaxation part

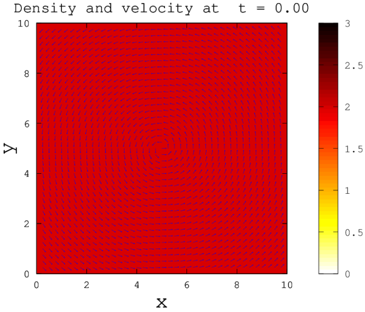
$$\begin{aligned} \partial_t \rho &= 0 \\ \partial_t (\rho \Omega) &= \frac{\rho}{\eta} (1 - |\Omega|^2) |\Omega|^2. \end{aligned} \tag{26}$$

The convergence of the splitting scheme



Problem:

- The density is uniform on the domain
- The orientation is a vortex



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Thank you for listening!